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XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNI KANONIK KO`RINISHGA KELTIRISH VA UNING UMUMIY YECHIMINI TOPISHGA OID MASHQLAR

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Услубий қўлланмада авторлар хусусий ҳосилали дифференциал тенгламаларнинг ва дифференциал тенгламалар системасининг классификацияси масаласини, ҳамда иккинчи тартибли хусусий ҳосилали дифференциал тенгламаларнинг умумий ечимини топиш усулларини баён этганлар.

Ушбу услубий қўлланма университетларнинг “Амалий математика ва информатика” йўналишлари бўйича бакалаврлар тайёрлайдиган факультет талабалари учун мўлжалланган.

В методическом пособии излагаются вопросы классификации дифференциальных уравнений с частными производными и систем дифференциальных уравнений. Описаны также методы нахождения общих решений дифференциальных уравнений с частными производными второго порядка.

Данное методическое пособие предназначено для студентов, обучающихся по специальности “Прикладная математика и информатика”.

Authors expound problems of classification of partial differential equations and systems of differential equations. They also described methods of finding general solutions of second order partial differential equations.

This textbook is designed for the university students, who are enrolled at the faculty of “Applied Mathematics and computer science”

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Uslubiy qo`llanma Mirzo Ulug`bek nomidagi O`zbekiston Milliy universiteti Ilmiy Kengashi tomonidan nashrga tavsiy qilingan.(2011 yil 26 aprel, 9 –sonli bayonnoma)

M u n d a r i j a

Kirish

Ma`lumki, tabiiy fanlarning har xil sohalarida uchraydigan ko`pgina jarayonlarni o`rganishda xususiy hosilali differensial tenglama yoki xususiy hosilali differensial tenglamalar sistemasini oldindan berilgan boshlang`ich va chegaraviy shartlarda yechish masalalarini o`rganishga duch kelinadi. Bunday jarayonni ifodalavchi matematik masalalar ko`pgina umumiylikka ega bo`lib, matematik fizika tenglamalarining predmetini tashkil etadi. Matematik fizika tenglamalari matematikaning asosiy fundamental va tadbiqiy bo`limlaridan bo`lib, u bakalavryatning matematika, mexanika, amaliy matematika va informatika kabi yo`nalishlari o`quv rejasidagi umumkasbiy fanlardan biri hisoblanadi. Xususiy hosilali differensial tenglama yoki xususiy hosilali differensial tenglamalar sistemasini o`rganish uchun ularni sinflarga ajratish maqsadga muavfiqdir.

Uslubiy qo’llanmada xususiy hosilali differensial tenglamalarning va differensial tenglamalar sistemasining klassifikatsiyasi masalasi, hamda ikkinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimini topish usullari bayon etilgan.

Ushbu uslubiy qo’llanma universitetlarning “Amaliy matematika va informatika” yo’nalishlari bo’yicha bakalavrlar tayyorlaydigan fakultet talabalari uchun mo’ljallangan bo’lib, shu yo’nalishning namunaviy dasturida kiritilgan “Matematik fizika tenglamalari” fani rejasiga asosan yozilgan.

1 - §. XUSUSIY HOSILALI DIFFERENSIAL TENGLAMANING XARAKTERISTIKASI HAQIDA TUSHUNCHA

Ω – orqali $x_1, x_2, ..., x_n,\ n \geq 2$, ortogonal dekart koordinatali x nuqtalarning n – o'lchamli R^n evklid fazosidan olingan sohani belgilaymiz.

Ω – sohadan olingan x nuqta va $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\sum_{j=1}^{n} \alpha_j = k,\ \ k = 0,...,m,\ \ m \geq 1$, manfiymas butun indeksli $p_{\alpha_1 \alpha_2 ... \alpha_n}$ haqiqiy o'zgaruvchili $F\left(x,..., p_{\alpha_1 \alpha_2 ... \alpha_n},...\right)$ – haqiqiy qiymatli funksiya berilgan bo'lib, hech bo'lmaganda $\dfrac{\partial F}{\partial p_{\alpha_1 \alpha_2 ... \alpha_n}}$, bunda $\sum_{j=1}^{n} \alpha_j = m$, hosilalardan birortasi noldan farqli bo'lsin. Bu yerda $p_{\alpha_1 \alpha_2 ... \alpha_n} = \dfrac{\partial^m u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$ deb olamiz.

$$F\left(x,..., \frac{\partial^k u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}},...\right) = 0 \qquad (1)$$

shakldagi tenglik $u(x) = u(x_1, x_2, ..., x_n),\ \ x \in \Omega$ noma'lum fungsiyaga nisbatan $m-$tartibli xususiy hosilali differensial tenglama deyiladi va bu tenglikning chap tomoni esa, $m-$tartibli xususiy hosilali differensial operator deyiladi.

Ω sohada aniqlangan $u(x)$ haqiqiy qiymatli funksiya va uning (1) tenglamada qatnashgan barcha xususiy hosilalari uzluksiz bo'lib, bu tenglamani ayniyatga aylantirsa, u holda shu funksiyaga regulyar yechim deyiladi.

Agar F funksiya $p_{\alpha_1\alpha_2...\alpha_n}$, bunda $|\alpha| = \sum_{j=1}^{n} \alpha_j = k,$ $k = 0,...,m$ barcha o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, u holda (1) tenglamaga chiziqli tenglama deb ataladi. Agar F funksiya $p_{\alpha_1\alpha_2...\alpha_n}$, bunda $|\alpha| = \sum_{j=1}^{n} \alpha_j = m$ o'zgaruvchigagina nisbatan chiziqli bo'lsa, u holda (1) tenglamaga kvazichiziqli tenglama deb ataladi.

$Lu = f(x)$ chiziqli tenglama uning o'ng tomonidagi $f(x)$ funksiyaning barcha $x \in \Omega$ uchun nolga teng yoki aynan noldan farqli bo'lishligiga qarab bir jinsli yoki bir jinsli bo'lmagan tenglama deb ataladi.

Osongina ko'rsatish mumkinki, agar $u(x)$ va $v(x)$ funksiyalar bir jinsli bo'lmagan $Lu = f(x)$ chiziqli tenglamaning yechimlari bo'lsa, u holda ularning ayirmasi $w(x) = u(x) - v(x)$ esa $Lw = 0$ bir jinsli tenglamaning yechimi bo'ladi. Bundan tashqari, agar $u_k(x),\ k = 1,...,l$ funksiyalar bir jinsli tenglamaning yechimlari bo'lsa, u holda $u=\sum_{k=1}^{l} c_k u_k(x)$, bunda c_k– haqiqiy o'zgarmaslar, ham shu tenglamaning yechimi bo'ladi.

(1) tenglamaning tipi

$$K(\xi_1,...,\xi_n)=\sum_{|\alpha|=m}\frac{\partial F}{\partial p_{\alpha_1...\alpha_n}}\xi_1^{\alpha_1}...\xi_n^{\alpha_n}$$

xarakteristik forma orqali aniqlanadi.

$$\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u=f(x) \qquad (2)$$

tenglama m – tartibli xususiy hosilali chiziqli differensial tenglamaning umumiy shakli, bunda $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ – multiindeks, $\alpha_j\ge 0$, $j=\overline{1,\ n}$, bundan tashqari α_j – butun sonlar, $|\alpha|=\alpha_1+\alpha_2+...+\alpha_n$ multiindeks moduli, $D^\alpha u=\dfrac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$ bo'lsin. $D^\alpha\to\xi^\alpha$ almashtirish orqali (2) tenglamaning $\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ simvolini hosil qilamiz, bunda $\xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_n^{\alpha_n}$.

$\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ formaga (2) tenglamaning bosh simvoli yoki xarakteristik ko'phadi deb ataladi.

$x \in \Omega$ tayinlangan nuqta bo'lsin. Noldan farqli $\xi = (\xi_1, \xi_2, ..., \xi_n) \neq 0$ vektor uchun $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha = 0$ bo'lsa, u holda bu vektor xarakteristik yo'nalish deb ataladi.

$F(x_1, x_2, ..., x_n) = 0$ formula bilan berilgan gipersirt uchun har bir nuqta xarakteristik yo'nalishga ega bo'lsa, ya'ni

$$\begin{cases} F(x_1, x_2, ..., x_n) = 0 \\ \sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial F}{\partial x_1} \right)^{\alpha_1} ... \left(\frac{\partial F}{\partial x_n} \right)^{\alpha_n} = 0, \quad gradF \neq 0 \end{cases} \tag{3}$$

bo'lsa, u holda xarakteristik sirt deb ataladi. Bu xarakteristika tenglamasidir, bunda

$$grad\, F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, ..., \frac{\partial F}{\partial x_n} \right).$$

Ikkinchi tartibli kvazichiziqli (barcha yuqori tartibli hosilalarga nisbatan chiziqli) uzluksiz $a_{ij}(x)$ koeffitsientli tenglamani qaraymiz:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \Phi(x, u, grad\, u) = 0. \tag{4}$$

$x = (x_1, x_2, ..., x_n),\ n \geq 2$ o'zgaruvchili $F(x)$ funksiya C^1 sinfdan olingan bo'lib, $F(x) = 0$ sirtda $grad\, F(x) \neq 0$ va

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j} = 0 \tag{5}$$

bo'lsin. U holda $F(x)=0$ esa, (4) kvazichiziqli differensial tenglamaning xarakteristik sirti deb, (5) tenglama esa xarakteristik tenglamasi deb aytiladi. $n=2$ uchun xarakteristik sirti xarakteristik chiziq deb ataladi.

$$\frac{\partial^2 u}{\partial t^2}=a^2\Delta u$$

to'lqin tarqalish tenglamasi uchun xarakteristik tenglama

$$F=0 \quad \text{da} \quad \left(\frac{\partial F}{\partial t}\right)^2-a^2\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2=0$$

shaklga egadir.

Uchi (x_0,t_0) nuqtada bo'lgan xarakteristik konus deb ataluvchi

$$a^2(t-t_0)^2-|x-x_0|^2=0$$

sirt xarakteristik sirt bo'ladi.

$$\frac{\partial u}{\partial t}=a^2\Delta u+f$$

issiqlik o'tkazuvchanlik tenglamasi uchun xarakteristik tenglama

$$F=0 \quad \text{da} \quad -a^2\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2=0$$

shaklga egadir. Uning xarakteristikalari osongina ko'rinadiki, $t=C$ tekisliklar oilasidan iborat bo'ladi.

$$\Delta u=f$$

Puasson tenglamasi uchun xarakteristik tenglama

$$F=0 \quad \text{да} \quad \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2=0$$

shaklga egadir. Bundan $F=0$ да $grad\, F=0$ ekanligi kelib chiqadi, bu esa mumkin emas, ya'ni xarakteristik sirtga ega emas.

Endi xususiy hosilali differensial tenglamaning tartibini va uning chiziqli differensial tenglama ekanligini aniqlashga doir misollar keltiramiz:

1–misol. $\sin(u_x-u_y)-\sin u_x \cos u_y+\sin u_y \cos u_x=0$

tenglama xususiy hosilali differensial tenglama bo'ladimi?

Yechish:

$$\frac{\partial F}{\partial u_x}=\frac{\partial\left(\sin(u_x-u_y)-\sin u_x \cos u_y+\sin u_y \cos u_x\right)}{\partial u_x}=$$

$$=\cos(u_x-u_y)-\cos u_x \cos u_y-\sin u_y \sin u_x=$$

$$=\cos u_x \cos u_y+\sin u_y \sin u_x-\cos u_x \cos u_y-\sin u_y \sin u_x=0 \ .$$

Xuddi shunday, $\frac{\partial F}{\partial u_y}=0$ ekanligini ko'rsatish mumkin. F funksiyadan xususiy hosilalar bo'yicha olingan hosilalar nolga teng ekan. Demak, berilgan tenglama xususiy hosilali differensial tenglama bo\`lmas ekan.

2–misol. $u_{xx}^2+u_{yy}^2-\left(u_{xx}-u_{yy}\right)^2=0$ tenglama xususiy hosilali differensial tenglama bo'ladimi?

Yechish: Avval berilgan tenglamani soddalashtirib olaylik.

$$F = u_{xx}^2 + u_{yy}^2 - \left(u_{xx} - u_{yy}\right)^2 = u_{xx}^2 - u_{yy}^2 - u_{xx}^2 +$$

$+2u_{xx}u_{yy} - u_{yy}^2 = 2u_{xx}u_{yy}$. Endi esa, xususiy hosilalar bo'yicha hosilalarni hisoblaymiz:

$$\frac{\partial F}{\partial u_{xx}} = 2u_{yy} \neq 0, \quad \frac{\partial F}{\partial u_{yy}} = 2u_{xx} \neq 0.$$

Demak, berilgan tenglama 2–tartibli xususiy hosilali differensial tenglama ekan.

3–misol. $\cos^2 u_{xx} + \sin^2 u_{xx} + 4u_x^3 - 2u_y + u = 0$

tenglamaning tartibini aniqlang.

Yechish:

$$\frac{\partial\left(\cos^2 u_{xx} + \sin^2 u_{xx} + 4u_x^3 - 2u_y + u\right)}{\partial u_{xx}} =$$

$$= -2\cos u_{xx} \sin u_{xx} + 2\sin u_{xx} \cos u_{xx} = 0$$

$$\frac{\partial\left(\cos^2 u_{xx} + \sin^2 u_{xx} + 4u_x^3 - 2u_y + u\right)}{\partial u_x} = 12u_x^2 \neq 0$$

Demak, berilgan tenglama 1–tartibli xususiy hosilali differensial tenglama ekan.

4–misol. $x^2u_{xx} + xu_{xy} + \sin y \cdot u_x - y^2u = 0$

tenglama xususiy hosilali chiziqli differensial tenglama bo'ladimi?

Yechish:

$$\frac{\partial\left(x^2u_{xx}+xu_{xy}+\sin y\cdot u_x-y^2u\right)}{\partial u_{xx}}=x^2\neq 0$$

bo`lgani uchun berilgan tenglama 2–tartiblidir. Hamda tenglama $Lu=f$ ko`rinishida bo`lib, bu erda

$$Lu=x^2u_{xx}+xu_{xy}+\sin y\cdot u_x-y^2u$$

operator noma’lum funksiya va uning hosilalari u_{xx}, u_{xy}, u_x, u larga nisbatan chiziqlidir, chunki

$$L(\alpha u+\beta v)=x^2(\alpha u+\beta v)_{xx}+x(\alpha u+\beta v)_{xy}+$$

$$+\sin y\cdot(\alpha u+\beta v)_x-y^2(\alpha u+\beta v)=$$

$$=\alpha x^2u_{xx}+\alpha xu_{xy}+\alpha\sin y\cdot u_x-\alpha y^2u+\beta x^2v_{xx}+$$

$$+\beta v_{xy}+\beta\sin y\cdot v_x-\beta y^2v=\alpha Lu+\beta Lv$$

va tenglamaning o`ng tomoni $f(x)=0$ dir. Shuning uchun 2–tartibli xususiy hosilali bir jinsli chiziqli differensial tenglama bo’ladi.

5–misol. $xu_{xx}^2+(x+y)u_{xy}-5y\cdot u_x-y^2u+\sin(x+y)=0$ tenglama xususiy hosilali chiziqli differensial tenglama bo’ladimi?

Yechish: Berilgan tenglama chiziqli differensial tenglama bo’la olmaydi, chunki $Lu=xu_{xx}^2+(x+y)u_{xy}-5y\cdot u_x-y^2u$ operator u_{xx} ga nisbatan birinchi tartibli (chiziqli) emas.

6–misol. $xu_{xxy} + y^2 u_{yyy} - 5y \cdot u_{xy} - \ln y \cdot u_x - y^2 u +$ $+\sin(x+y) + x^2 = 0$ tenglama xususiy hosilali chiziqli differensial tenglama bo'ladimi?

Yechish: Berilgan tenglama bir jinsli bo'lmagan chiziqli differensial tenglama bo'ladi, chunki

$$Lu = xu_{xxy} + y^2 u_{yyy} - 5y \cdot u_{xy} - \ln y \cdot u_x - y^2 u$$

operator $u_{xxy},\ u_{yyy},\ u_{xy},\ u_x,\ u$ larga nisbatan birinchi darajali differensial ifoda va $f(x) = -\sin(x+y) - x^2 \neq 0$.

7–misol. $u_{xxy} - tgy \cdot u_{yy} + 7y \cdot u_{xy} u_{yy} - y \cdot u_x - yu + xy = 0$ tenglama xususiy hosilali chiziqli differensial tenglama bo'ladimi?

Yechish: Berilgan tenglama chiziqli differensial tenglama bo'la olmaydi, chunki

$$Lu = u_{xxy} - tgy \cdot u_{yy} + 7y \cdot u_{xy} u_{yy} - y \cdot u_x - yu$$

operator $u_{xy},\ u_{yy}$ larga nisbatan birinchi darajali differensial ifoda emas. Ammo ushbu tenglama kvazichiziqli differensial tenglama bo'ladi, chunki

$$Lu = u_{xxy} - tgy \cdot u_{yy} + 7y \cdot u_{xy} u_{yy} - y \cdot u_x - yu$$

operator yuqori tartibli hosila u_{xxy} ga nisbatan birinchi darajali differensial ifoda bo'ladi.

Mustaqil yechish uchun misollar

Quyida berilgan tenglamalarning tartibini aniqlang:

1.1. $\cos^2 u_{xy} + \sin^2 u_{yy} + 4u_x^3 - 2u_y + u = 0$.

1.2. $\cos^2 u_{yy} + \sin^2 u_{yy} + 4u_x^3 - 2u_y + u = 0$.

1.3. $\sin(u_{xy} - u_y) - \sin u_{xy} \cos u_y + \sin u_y \cos u_{xy} = 0$.

1.4. $\cos(u_{xy} - u_y) - \cos u_{xy} \cos u_y + \sin u_y \cos u_x = 0$.

1.5. $\left(u_{xx} - u_{yy}\right)^2 - u_{xx}^2 - u_{yy}^2 = 0$.

1.6. $\left(u_{xx} + u_{yy}\right)^2 - 2u_{xx}u_{yy} = 0$.

1.7. $\left(u_{xx} + u_y\right)^2 - u_{xx}^2 - 2u_{xx}u_y + 2u_x^2 + u = 0$.

1.8. $\left(u_{xx} + u_{yy}\right)^3 - u_{xx}^3 - 3u_{xx}^2 u_{yy} - 3u_{xx}u_{yy}^2 - u_{yy}^3 + u_x^3 + u = 0$.

1.9. $\left(u_{xx} + u_{yy}\right)^3 - u_{xx}^3 - u_{yy}^3 + u_y^2 + 5u = 0$.

1.10. $\left(u_{xx} - u_{yy}\right)^3 - u_{xx}^3 - 3u_{xx}^2 u_{yy} - 3u_{xx}u_{yy}^2 -$

$-u_{yy}^3 + u_y^3 + u_x = 0$

1.11. $\cos^2(u_{xy} + u_{yy}) + \sin^2(u_{xy} + u_{yy}) + u_{yy} -$

$-(u_y + 4)^2 + 8u_y + u = 0.$

1.12. $\lg|u_{xx}u_{yy}| - \lg|u_{xx}| - \lg|u_{yy}| + 6u_y^3 + u_x = 0$.

1.13. $\left(u_{xx}-u_{yy}\right)\left(u_{xx}+u_{yy}\right)-\frac{\partial}{\partial x}(u_x-u_y)-$

$-\frac{\partial}{\partial y}(u_x-u_y)+6u_x=0.$

1.14. $\left(u_x-u_y\right)\left(u_x+u_y\right)-\frac{\partial}{\partial x}(u_x-u_y)-$

$-\frac{\partial}{\partial y}(u_x-u_y)+6u_x=0.$

1.15. $2\left(u_x-2u\right)u_{xy}-\frac{\partial}{\partial y}\left(u_x-2u\right)^2-xy=0.$

1.16. $\frac{\partial}{\partial x}\left(u_{yy}^2-u_y\right)-2u_{yy}\frac{\partial}{\partial y}(u_{xx}-u_x)-2u_x+2=0$.

1.17. $2u_{xx}u_{xxy}-\frac{\partial}{\partial y}(u_{xx}-u_y)^2-2u_yu_{xxy}+u_x=0$.

1.18. $u_{xxy}-\frac{\partial}{\partial y}(u_{xx}-u_y)-2u_yu_{xx}+u_x=0$.

1.19. $\frac{\partial}{\partial x}\left(u_{yy}^2-u_y\right)-2u_{yy}\frac{\partial}{\partial x}(u_{yy}-u_x)-5u_x=0$.

1.20. $\left(tgu_{xx}+ctgu_{xx}\right)^2-tg^2u_{xx}-ctg^2u_{xx}+6u_x=0$.

1.21. $\cos 2u_{xx}-\cos^2 u_{xx}+\sin^2 u_{xx}-\sin u_{xx}+8u_x+u=0$.

1.22. $\frac{\partial}{\partial y}\cos u_{xx}+\cos^2 u_{xx}+\sin^2 u_{xx}+$

$+u_{xxy}\sin u_{xx}+8u_y+5u=0.$

1.23. $2u_{xy}-6\frac{\partial}{\partial x}(u^2-xy)+u_{yy}=0$.

1.24. $\left(tgu_{xx}+ctgu_{yy}\right)^2-tg^2u_{xx}-ctg^2u_{yy}+6u_x=0$.

1.25. $\frac{\partial}{\partial y}\left(yu_y+u_x^2\right)-2u_xu_{xy}+u_x-6u=0$.

Quyida berilgan xususiy hosilali tenglamalarning chiziqli, kvazichiziqli, bir jinsli, yoki bir jinsli emasligini aniqlang:

2.1. $2u_{xy}-6\frac{\partial}{\partial x}(u^2-xy)+u_{yy}=0$.

2.2. $u_{xxy}-\frac{\partial}{\partial y}(u_{xx}-u_y)-2u_yu_{xx}+u_x+x-3y=0$.

2.3. $u_yu_{xxy}-\frac{\partial}{\partial y}(u_{xx}-u_y)-2u_yu_{xx}+u_x+4y^2=0$.

2.4. $y\frac{\partial}{\partial y}\left(yu_y+u_x^2\right)-2yu_xu_{xy}+u_x-6u+xe^y=0$.

2.5. $2u_{xx}u_{xxy}-\frac{\partial}{\partial y}(u_{xx}-u_y)^2-2u_yu_{xxy}+u_x=0$.

2.6. $\frac{\partial}{\partial x}\left(u_{yy}^2-u_y\right)-2u_{yy}\frac{\partial}{\partial y}(u_{xx}-u_x)-2u_x+2=0$.

2.7. $2xu_{xy}-8\frac{\partial}{\partial x}(u^2-xy)-2u_{yy}=0$.

2.8. $\frac{\partial}{\partial y}\left(yu_y+u_x^2\right)-2u_xu_{xy}+u_{yy}-6u_x+u+xe^y=0$.

2.9. $yu_{xy} + u_{yy} - 6u_x + u^2 + 4xy = 0$.

2.10. $yu_{xy} + u_{yy} - yu_x + x^2u - 8xy = 0$.

2.11. $yu_{xyy} + u_{yy} - yu_{xy} + x^2u = 0$.

2.12. $2xu_{xy} - 2\frac{\partial}{\partial y}(y^2 - xyu) - 2u_{yy} + 4y = 0$.

2.13. $u_{xyy} + xu_{yy} - \cos y \cdot u_{xy} + \sin x^2 u = 0$.

2.14. $a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} +$

$+d(x,y)u_x + e(x,y)u_y + h(x,y) = 0.$

2.15. $(3x+4y)u_{xx} + (4x-y)u_{xy} + 4(x-3y)u_{yy} +$

$+(x+y)u_x + yu_y + 2x + 5y = 0.$

2.16. $\cos y \cdot u_{xyy} + u_{xxyy} - yu_{xy} + x^2u^3 + yu = 0$.

2.17. $\left(tgu_{xx} + ctgu_{xx}\right)^2 - tg^2u_{xx} - ctg^2u_{xx} +$

$+4u_{xx} + u_{xy} - 3u = 0.$

2.18. $xu_{xxy} - \frac{\partial}{\partial y}(xu_{xx} - 3xy) - 2u_{xx} + u_x + 3x = 0$.

2.19. $x^2yu_{xyy} + 2e^x y^2 u_{yy} - \left(x^2 + y^2\right)u_{xy} + x^2u = 0$.

2.20. $2u_xu_{xy} - \frac{\partial}{\partial y}\left(yu_y + u_x^2\right) + yu_{yy} - 6u_x + u + ye^{x+y} = 0$.

2.21. $\left(u_x - u_y\right)\left(u_x + u_y\right) - \frac{\partial}{\partial x}(u_x - u_y) -$

$$-\frac{\partial}{\partial y}(u_x - u_y) + 6u_x = 0.$$

2.22. $\frac{\partial}{\partial x}(u_x - u_y) + \frac{\partial}{\partial y}(u_x - u_y) + (x+y)u_x + xe^y = 0$.

2.23. $u_x u_{xyy} + 8e^{x+y}u_{yy} - 9u_{xy} + f(x,y)u = 0$.

2.24. $u_{xyy} + xye^{xy}u_{xy} - 9xu_{yy} + 7u_{yzz} + f(x,y,z)u = 0$.

2.25. $u_t - 6uu_x + u_{xxx} = 0$.

2 - §. XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING KLASSIFIKATSIYASI VA ULARNING KANONIK SHAKLI

Ta'rif. *Agar har qanday* $|\xi| \neq 0$ *учун* $\sum_{|\alpha|=m} a_\alpha(x^0)\xi^\alpha \neq 0$

(boshqacha qilib aytganda, uning haqiqiy xarakteristikasi yo'q) bo'lsa, u holda

$$\sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u = f(x)$$

chiziqli tenglama x^0 *nuqtada elliptik tipdagi tenglama deyiladi.*

Ta'rif. *Agar* $\sum_{j=1}^{n-1}\xi_j^2 \neq 0$ *bo'ladigan har qanday* $\xi_1,...,\xi_{n-1}$ *uchun* $\sum_{|\alpha|=m} a_\alpha(x^0)\,\xi_1^{\alpha_1}...\,\xi_n^{\alpha_n} = 0$ *tenglama* ξ_n *o'zgaruvchiga nisbatan* m *ta haqiqiy va har xil ildizlarga ega bo'lsa, u holda*

$$\sum_{|\alpha|\le m} a_\alpha(x)\,D^\alpha u = f(x)$$

chiziqli tenglama x^0 *nuqtada* x_n *o'q yo'nalishida giperbolik tipdagi tenglama deyiladi.*

Ta'rif. *Agar* x^0 *tayinlangan nuqta uchun shunday bir*

$$\xi_i = \xi_i(\mu_1,...,\mu_n),\ i = 1,2,...,n,$$

o'zgaruvchilarning affin almashtirishini topish mumkin bo'lib, natijada, $\sum_{|\alpha|=m} a_\alpha(x^0)\,\xi^\alpha$ *forma* μ_i *o'zgaruvchilarning faqatgina* l *–tasinigina, bunda* $0 < l < n$ *, saqlasa, u holda* $\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x)$ *tenglama* x^0 *nuqtada parabolik maxsuslikka ega yoki parabolik tipdagi tenglama deyiladi.*

Ikkinchi tartibli xususiy hosilali chiziqli tenglamani quyidagi shaklda yozish mumkin:

$$\sum_{j=1}^{n}\ \sum_{i=1}^{n} a_{ij}(x)\cdot u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\cdot u_{x_i} + c(x)u + f(x) = 0,$$

$(a_{ij}=a_{ji})$, bunda $a,\ b,\ c,\ f$ funksiyalar $x=(x_1,x_2,...,x_n)$ o'zgaruvchiga bog'liqdir. Yangi ξ_k erkli o'zgaruvchilarni $\xi_k=\xi_k(x_1,x_2,...,x_n)$, $k=\overline{1,n}$ shaklda kiritamiz. U holda

$$u_{x_i}=\sum_{k=1}^{n}u_{\xi_k}\cdot\alpha_{ik},$$

$$u_{x_ix_j}=\sum_{k=1}^{n}\sum_{l=1}^{n}u_{\xi_k\xi_l}\cdot\alpha_{ik}\alpha_{jl}+\sum_{k=1}^{n}u_{\xi_k}\cdot(\xi_k)_{x_ix_j},$$

bunda $\alpha_{ik}=\dfrac{\partial\xi_k}{\partial x_i}$. Hosila uchun olingan ifodalarni berilgan tenglamaga qo'ysak, quyidagini hosil qilamiz:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\overline{a}_{kl}\cdot u_{\xi_k\xi_l}+\sum_{k=1}^{n}\overline{b}_k\cdot u_{\xi_k}+cu+f=0,$$

bunda

$$\overline{a}_{kl}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\alpha_{ik}\alpha_{jl},$$

$$\overline{b}_k=\sum_{i=1}^{n}b_i\cdot\alpha_{ik}+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\cdot(\xi_k)_{x_ix_j}.$$

Endi

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(x^0)y_iy_j$$

kvadratik formani qaraymiz. y o'zgaruvchi ustida

$$y_i = \sum_{k=1}^{n} \alpha_{ik}\eta_k$$

chiziqli almashtirish bajarib,

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \bar{a}_{kl}(x^0)\,\eta_k\,\eta_l$$

kvadratik formaga ega bo'lamiz, bunda

$$\bar{a}_{kl}(x^0) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x^0)\,\alpha_{ik}\,\alpha_{jl}$$

bo'ladi.

Ma'lumki, chiziqli almashtirishni mos tanlash yo'li bilan kvadratik formaning $a_{ij}(x^0)$ matritsasini diagonal shaklga, ya'ni $\sum_{i=1}^{n} \alpha_i \cdot \eta_i^2$ kanonik shaklga keltirish mumkin bo'lib, bunda $\alpha_i,\ i = \overline{1, n}$ koeffitsientlar $1,\ -1,\ 0$ qiymatlarni qabul qiladi, bundan tashqari inertsiya qonuniga ko'ra, musbat, manfiy va nolga teng koeffitsientlar soni kvadratik formani kanonik shaklga keltirishdagi chiziqli almashtirishga nisbatan invariantdir.

Agar barcha n ta α_i koeffitsientlar bir xil ishorali bo'lsa, u holda tenglama x^0 nuqtada elliptik tipdagi tenglama deb, agar $n-1$ ta α_i koeffitsientlar bir xil ishorali va bitta koeffitsient unga qarama–qarshi ishorali bo'lsa, u holda tenglama x^0 nuqtada giperbolik tipdagi (yoki normal giperbolik tipdagi) tenglama deb, agar α_i koeffitsientlarning m tasi bir xil ishorali va $n-m$ tasi unga qarama–qarshi ishorali $\left(m>1,\ n-m>1\right)$ bo'lsa, u holda tenglama x^0 nuqtada ultragiperbolik tipdagi tenglama deb, agar α_i koeffitsientlarning hech bo'lmaganda bittasi nolga teng bo'lsa, u holda tenglama x^0 nuqtada parabolik tipdagi tenglama deb ataladi.

Kanonik formalar:

$$\Delta u + \Phi = 0 \qquad \text{(elliptik tip),}$$

$$u_{x_1x_1} = \sum_{i=2}^{n} u_{x_ix_i} + \Phi \qquad \text{(giperbolik tip),}$$

$$\sum_{i=1}^{m} u_{x_ix_i} = \sum_{i=m+1}^{n} u_{x_ix_i} + \Phi \quad \left(m>1,\ n-m>1\right)$$

(ultragiperbolik tip),

$$\sum_{i=1}^{n-m} \left(\pm\, u_{x_ix_i}\right) + \Phi = 0 \quad \left(m>0\right) \qquad \text{(parabolik tip).}$$

O'zgarmas koeffitsientli bo'lgan holda

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{x_ix_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + f = 0$$

tenglama uning aniqlanish sohasining barcha nuqtalari uchun bir vaqtda o'zgaruvchilarni chiziqli almashtirish yordamida kanonik shaklga keltiriladi. u funksiya o'rniga $u = v\cdot e^{\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n}$

tenglik yordamida v yangi funksiyani kiritib va λ_i o'zgarmaslarni tegishli tanlash yordamida biz tenglamani yanada sodda shakldagi kanonik formaga keltirishimiz mumkin bo'ladi.

$n = 2$ uchun

$$v_{\xi\xi} + v_{\eta\eta} + cv + f_1 = 0 \quad \text{(elliptik tip)},$$

$$\left.\begin{array}{l} v_{\xi\eta} + cv + f_1 = 0 \\ yoki \\ v_{\xi\xi} - v_{\eta\eta} + cv + f_1 = 0 \end{array}\right\} \quad \text{(giperbolik tip)},$$

$$v_{\xi\xi} + b_2 v_\eta + f_1 = 0 \qquad \text{(parabolik tip)}.$$

Ikkinchi tartibli ikki o'zgaruvchili xususiy hosilali chiziqli differensial tenglamalarni kanonik shaklga keltirish.

Quyidagi kvazichiziqli tenglamani qaraylik:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0\ , \quad (6)$$

bunda $A, B, C \in C^2(\Omega)$.

Bu differensial tenglama

1) Agar $B^2 - AC > 0$ bo'lsa, u holda giperbolik tipga,

2) Agar $B^2 - AC = 0$ bo'lsa, u holda parabolik tipga,

3) Agar $B^2 - AC < 0$ bo’lsa, u holda elliptik tipga tegishli bo’ladi.

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

funksiyalar ikki marta uzluksiz differensiallanuvchi funksiyalar bo’lib, bundan tashqari Ω sohada yakobian noldan farqli, ya'ni

$$\frac{D(\xi,\eta)}{D(x,y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$

bo’lsin. ξ va η yangi o’zgaruvchilarga nisbatan tenglama quyidagi shaklda yoziladi:

$$\overline{A}\frac{\partial^2 u}{\partial \xi^2} + 2\overline{B}\frac{\partial^2 u}{\partial \xi \partial \eta} + \overline{C}\frac{\partial^2 u}{\partial \eta^2} + \overline{F}\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0, \quad (7)$$

bunda

$$\overline{A}(\xi,\eta) = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$\overline{C}(\xi,\eta) = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2,$$

$$\overline{B}(\xi,\eta) = A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}$$

va

$$\overline{B}^2 - \overline{A}\overline{C} = \left(B^2 - AC\right)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2.$$

$\xi(x,y)$ va $\eta(x,y)$ funksiyalarni shunday tanlash mumkinki, bunda quyidagi shartlardan faqat biri bajariladi:

1) $\bar{A}=0,\ \bar{C}=0$; 2) $\bar{A}=0,\ \bar{B}=0$; 3) $\bar{A}=\bar{C},\ \bar{B}=0.$

1) $B^2-AC>0$ bo'lsin. $A\neq 0$ yoki $C\neq 0$ deb olamiz. Masalan, $A\neq 0$.

$$A\left(\frac{\partial\varphi}{\partial x}\right)^2+2B\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}+C\left(\frac{\partial\varphi}{\partial y}\right)^2=0 \qquad (8)$$

tenglamani qaraylik. Bu tenglamani

$$\left[A\frac{\partial\varphi}{\partial x}+\left(B+\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}\right]\times$$
$$\times\left[A\frac{\partial\varphi}{\partial x}+\left(B-\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}\right]=0$$

shakldda ham yozish mumkin. Bundan, esa

$$A\frac{\partial\varphi}{\partial x}+\left(B+\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}=0 \qquad (9)$$

$$A\frac{\partial\varphi}{\partial x}+\left(B-\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}=0 \qquad (10)$$

tenglamalar hosil bo'ladi. (9) va (10) tenglamalarni integrallash uchun ularga mos oddiy xarakteristik differensial tenglamalarni tuzamiz.

$$\frac{dx}{A}=\frac{dy}{B+\sqrt{B^2-AC}},\quad \frac{dx}{A}=\frac{dy}{B-\sqrt{B^2-AC}}\ ,$$

yoki

$$Ady - \left(B + \sqrt{B^2 - AC} \right) dx = 0,$$

$$Ady - \left(B - \sqrt{B^2 - AC} \right) dx = 0,$$

yoki bitta tenglama ko'rinishida

$$A(dy)^2 - 2Bdydx + C(dx)^2 = 0$$

tenglama hosil bo'ladi. Bundan, $A(x_0, y_0) \neq 0$ bo'lgani uchun

$$\varphi_1(x, y) = const, \quad \varphi_2(x, y) = const$$

integrallar mavjudligi kelib chiqadi. (Haqiqatdan ham, o'zgarmas koeffitsientli bo'lgan holda

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A}, \quad \frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A},$$

$$y = \frac{B + \sqrt{B^2 - AC}}{A} x + C_1, \quad y = \frac{B - \sqrt{B^2 - AC}}{A} x + C_2).$$

$\xi = \varphi_1(x, y), \ \eta = \varphi_2(x, y)$ deb olamiz. U holda (7) tenglamani $2\overline{B}$ ga bo'lib,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta} \right)$$

tenglamani hosil qilamiz, yoki $\xi = \alpha + \beta, \quad \eta = \alpha - \beta$ deb olib,

$$\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \beta^2} = \Phi\left(\alpha, \beta, u, \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta} \right)$$

tenglikga ega bo'lamiz. Bu giperbolik tipdagi tenglamaning kanonik shaklidir.

2) $B^2 - AC = 0$ bo'lsin. $A \neq 0$ deb olamiz. U holda (8) tenglama quyidagi ko'rinishda bo'ladi:

$$A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y} = 0.$$

Bu tenglamaning $\varphi(x,y) = const$ umumiy yechimi yordamida $\xi = \varphi(x,y)$ deb olamiz va $\eta = \eta(x,y)$ sifatida esa, ikki marta uzluksiz differensiallanuvchi ixtiyoriy funksiyani $\frac{D(\xi,\eta)}{D(x,y)} \neq 0$ shart (x_0, y_0) nuqta atrofida bajariladigan qilib olamiz. $B^2 - AC = 0$ shartdan va $A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y} = 0$ tenglikdan $B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y} = 0$ tenglik kelib chiqadi. Shuning uchun $\overline{B} = \left(A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y}\right)\frac{\partial\eta}{\partial x} + \left(B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y}\right)\frac{\partial\eta}{\partial y} = 0$ bo'ladi. $\overline{A} = 0$ tenglik ham o'rinli. $\overline{C}$ koeffitsient esa, $\overline{C} = \frac{1}{A}\left(A\frac{\partial\eta}{\partial x} + B\frac{\partial\eta}{\partial y}\right)^2$ shaklga almashadi, bundan $\overline{C} \neq 0$ ekanligi kelib chiqadi. (7) tenglamada $\overline{C} \neq 0$ ga bo'lib,

$$\frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

tenglamaga ega bo'lamiz. Bu parabolik tipdagi tenglamaning kanonik shaklidir.

3) $B^2 - AC < 0$ bo'lsin. A, B, C koeffitsientlar x va y ga bog'liq analitik funksiyalar deb olamiz. U holda

$$A\frac{\partial \varphi}{\partial x} + \left(B + \sqrt{B^2 - AC}\right)\frac{\partial \varphi}{\partial y} = 0$$

tenglama (x_0, y_0) nuqta atrofida $\varphi(x, y) = \varphi_1(x, y) + i\varphi_2(x, y)$ va shu nuqta atrofida $\left|\frac{\partial \varphi}{\partial x}\right| + \left|\frac{\partial \varphi}{\partial y}\right| \neq 0$ bo'lgan analitik yechimga ega bo'ladi. (Bunday analitik yechimning mavjudligi S.V. Kovalevskaya teoremasidan kelib chiqadi).

$$\xi = \varphi_1(x, y), \quad \eta = \varphi_2(x, y)$$

deb olamiz. $\frac{\partial(\varphi_1, \varphi_2)}{\partial(x, y)} \neq 0$ ekanligini ko'rsatish qiyin emas. Endi

$$A\left(\frac{\partial \varphi}{\partial x}\right)^2 + 2B\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y} + C\left(\frac{\partial \varphi}{\partial y}\right)^2 = 0$$

ayniyatning haqiqiy va mavhum qismlarini ajratib,

$$A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2=$$

$$=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2$$

ekanligini, ya'ni $\overline{A}=\overline{C}$ va

$$A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=0$$

ekanligini, ya'ni $\overline{B}=0$ tengliklarni hosil qilamiz.

$$At_1^2+2Bt_1t_2+Ct_2^2 \quad \left(B^2-AC<0\right)$$

kvadratik formaning aniqlanganligiga ko’ra, faqat va faqat shu holdaki, agar

$$\frac{\partial\xi}{\partial x}=\frac{\partial\xi}{\partial y}=\frac{\partial\eta}{\partial x}=\frac{\partial\eta}{\partial y}=0$$

bo’lsa, u holda $\overline{A}=\overline{C}$ nolga aylanadi. Lekin, biz $\varphi(x,y)$ yechimni bir vaqtda bu tenglikni qanoatlantirmaydigan qilib tanlaganmiz.

Shunday qilib, $\overline{A}$ ga bo’lib,

$$\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=F_3\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)$$

tenglikga ega bo’lamiz. Bu elliptik tipdagi tenglamaning kanonik shaklidir.

O’zgarmas koeffitsiyentli ikki o’zgaruvchili ikkinchi tartibli xususiy hosilali chiziqli differensial tenglamalarni kanonik shaklga keltirishga doir misollar.

1–misol. $u_{xx}+2u_{xy}-3u_{yy}+2u_x+6u_y=0$ tenglamani kanonik ko’rinishga keltiring.

Yechish: A=1, B=1, C= –3, $\Delta=B^2-AC=1+3=4>0$ bo`lgani uchun, yuqoridagi tenglama giperbolik tipdagi tenglama bo’ladi. Endi esa, uning xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$(dy)^2-2dydx-3(dx)^2=0,\quad y'^2-2y'-3=0,$$

$$y'=\frac{2\pm\sqrt{4+12}}{2}=1\pm2,\ y'=3,\quad y'=-1$$

$$y=3x-C_1,\quad y=-x+C_2,\quad C_1=3x-y,\quad C_2=x+y.$$

C_1, C_2 o’zgarmaslarni mos ravishda ξ, η lar bilan almashtiramiz:

$$\begin{cases}\xi=3x-y\\ \eta=x+y\end{cases}.$$

u funksiyani murakkab funksiya deb qarab, birinchi va ikkinchi tartibli xususiy hosilalarni hisoblab, tenglamaga qo’yamiz:

$$u_x=u_\xi\cdot\xi_x+u_\eta\cdot\eta_x=3u_\xi+u_\eta,$$

$$u_y=u_\xi\cdot\xi_y+u_\eta\cdot\eta_y=-u_\xi+u_\eta,$$

$$u_{xx}=u_{\xi\xi}\cdot(\xi_x)^2+2u_{\xi\eta}\cdot\xi_x\cdot\eta_x+u_{\eta\eta}\cdot(\eta_x)^2+$$

$$+u_\xi \cdot \xi_{xx} + u_\eta \cdot \eta_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta},$$

$$u_{xy} = u_{\xi\xi} \cdot \xi_x \cdot \xi_y + u_{\xi\eta} \cdot \left(\xi_x \cdot \eta_y + \xi_y \cdot \eta_x\right) + u_{\eta\eta} \cdot \eta_x \cdot \eta_y +$$

$$+u_\xi \cdot \xi_{xy} + u_\eta \cdot \eta_{xy} = -3u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_{yy} = u_{\xi\xi} \cdot \left(\xi_y\right)^2 + 2u_{\xi\eta} \cdot \xi_y \cdot \eta_y + u_{\eta\eta} \cdot \left(\eta_y\right)^2 +$$

$$+u_\xi \cdot \xi_{yy} + u_\eta \cdot \eta_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

Bu yerda shuni ta'kidlab o'tamizki, agar $\xi,\ \eta$ lar $x,\ y$ larning chiziqli funksiyalari bo'lsa, u holda $\xi_{xx} = 0,$ $\xi_{xy} = 0,$ $\xi_{yy} = 0,$ $\eta_{xx} = 0,$ $\eta_{xy} = 0,$ $\eta_{yy} = 0$ bo'ladi, hamda birinchi va ikkinchi tartibli hosilalar quyidagi formulalarga o'xshab ketadi:

$$u_{xx} = \left(\frac{\partial}{\partial\xi} \cdot \xi_x + \frac{\partial}{\partial\eta} \cdot \eta_x\right)^2 u =$$

$$= u_{\xi\xi} \cdot \left(\xi_x\right)^2 + 2u_{\xi\eta} \cdot \xi_x \cdot \eta_x + u_{\eta\eta} \cdot \left(\eta_x\right)^2,$$

$$u_{xy} = \left(\frac{\partial}{\partial\xi} \cdot \xi_x + \frac{\partial}{\partial\eta} \cdot \eta_x\right) \cdot \left(\frac{\partial}{\partial\xi} \cdot \xi_y + \frac{\partial}{\partial\eta} \cdot \eta_y\right) u =$$

$$= u_{\xi\xi} \cdot \xi_x \cdot \xi_y + u_{\xi\eta} \cdot \left(\xi_x \cdot \eta_y + \xi_y \cdot \eta_x\right) + u_{\eta\eta} \cdot \eta_x \cdot \eta_y,$$

$$u_{yy} = \left(\frac{\partial}{\partial\xi} \cdot \xi_y + \frac{\partial}{\partial\eta} \cdot \eta_y\right)^2 u =$$

$$= u_{\xi\xi} \cdot \left(\xi_y\right)^2 + 2u_{\xi\eta} \cdot \xi_y \cdot \eta_y + u_{\eta\eta} \cdot \left(\eta_y\right)^2.$$

Endi topilgan ifodalarni $u_{xx}+2u_{xy}-3u_{yy}+2u_x+6u_y=0$ tenglamaga olib borib qo'yamiz:

$$9u_{\xi\xi}+6u_{\xi\eta}+u_{\eta\eta}+2\cdot\left(-3u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\right)-3\cdot\left(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}\right)+$$

$$+2\cdot\left(3u_{\xi}+u_{\eta}\right)+6\cdot\left(-u_{\xi}+u_{\eta}\right)=0,$$

ya'ni

$$16u_{\xi\eta}+8u_{\eta}=0\ .$$

Bundan

$$u_{\xi\eta}+\frac{1}{2}u_{\eta}=0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2–misol. $u_{xx}+2u_{xy}+5u_{yy}+2u_x-3u_y=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: A=1, B=1, C=5, $\Delta=B^2-AC=1-5=-4<0$ bo'lgani uchun, yuqoridagi tenglama elliptik tipdagi tenglama bo'ladi. Endi esa, uning xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y'^2-2y'+5=0,\ \ y'=\frac{2+\sqrt{4-20}}{2}=1+2i,$$

$$y=\left(1+2i\right)x-C,\quad x-y+2xi=C.$$

Bu yerda $\begin{cases}\xi=x-y\\ \eta=2x\end{cases}$ almashtirishni bajaramiz. Shunga ko'ra,

$$u_x=u_{\xi}+2u_{\eta}\ ,$$

$$u_y = -u_\xi \ ,$$

$$u_{xx} = u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta} \ ,$$

$$u_{xy} = -u_{\xi\xi} - 2u_{\xi\eta} \ ,$$

$$u_{yy} = u_{\xi\xi}$$

hosil bo'ladi. Topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta} + 2\left(-u_{\xi\xi} - 2u_{\xi\eta}\right) + 5u_{\xi\xi} +$$

$$+2\left(u_\xi + 2u_\eta\right) - 3\left(-u_\xi\right) = 0 \ ,$$

ya'ni

$$4u_{\xi\xi} + 4u_{\eta\eta} + 5u_\xi + 4u_\eta = 0 \ .$$

Bundan

$$u_{\xi\xi} + u_{\eta\eta} + \frac{5}{4}u_\xi + u_\eta = 0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

Endi esa ushbu elliptik tenglamani sodda kanonik shaklga keltiraylik, ya'ni tenglamadagi birinchi tartibli hosilalarni yo'qotamiz. Buning uchun $u(\xi,\eta) = v(\xi,\eta) \cdot e^{\lambda\xi+\mu\eta}$ almashtirish bajaramiz va birinchi va ikkinchi tartibli xususiy hosilalarni hisoblab, tenglamaga qo'yamiz:

$$u_\xi = v_\xi \cdot e^{\lambda\xi+\mu\xi} + v \cdot \lambda e^{\lambda\xi+\mu\eta} \ ,$$

$$u_\eta = v_\eta \cdot e^{\lambda\xi+\mu\eta} + v \cdot \mu e^{\lambda\xi+\mu\eta} \ ,$$

$$u_{\xi\xi} = v_{\xi\xi} \cdot e^{\lambda\xi+\mu\eta} + v_\xi \cdot \lambda e^{\lambda\xi+\mu\eta} +$$

$$+v_\xi \cdot \lambda e^{\lambda\xi+\mu\eta} + v \cdot \lambda^2 e^{\lambda\xi+\mu\eta} =$$

$$= v_{\xi\xi} \cdot e^{\lambda\xi+\mu\eta} + 2v_\xi \cdot \lambda e^{\lambda\xi+\mu\eta} + v \cdot \lambda^2 e^{\lambda\xi+\mu\eta} \ ,$$

$$u_{\eta\eta} = v_{\eta\eta} \cdot e^{\lambda\xi+\mu\eta} + v_\eta \cdot \mu e^{\lambda\xi+\mu\eta} + v_\eta \cdot \mu e^{\lambda\xi+\mu\eta} +$$

$$+v \cdot \mu^2 e^{\lambda\xi+\mu\eta} = v_{\eta\eta} \cdot e^{\lambda\xi+\mu\eta} + 2v_\eta \cdot \mu e^{\lambda\xi+\mu\eta} + v \cdot \mu^2 e^{\lambda\xi+\mu\eta} \ ,$$

$$v_{\xi\xi} \cdot e^{\lambda\xi+\mu\eta} + v_{\eta\eta} \cdot e^{\lambda\xi+\mu\eta} + (2\lambda + \frac{5}{4})v_\xi \cdot e^{\lambda\xi+\mu\eta} +$$

$$+(2\mu+1)v_\eta \cdot e^{\lambda\xi+\mu\eta} + \left(\lambda^2 + \mu^2 + \frac{5}{4}\lambda + \mu\right) \cdot v \cdot e^{\lambda\xi+\mu\eta} = 0 \ .$$

Agar $\lambda = -\frac{5}{8}, \ \mu = -\frac{1}{2}$ deb tanlasak, oxirgi tenglama quyidagi sodda kanonik shaklga keladi:

$$v_{\xi\xi} + v_{\eta\eta} - \frac{41}{64}v = 0 \ .$$

3–misol. $u_{xx} + 4u_{xy} + 4u_{yy} + 2u_x - 3u_y = 0$ tenglamani sodda kanonik ko’rinishga keltiring.

Yechish: A=1, B=2, C=4, $\Delta = B^2 - AC = 4 - 1 \cdot 4 = 0$ bo’lgani uchun, yuqoridagi tenglama parabolik tipdagi tenglama bo’ladi. Endi esa, uning xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y'^2-4y'+4=0, \qquad y'=\frac{4+\sqrt{16-16}}{2}=2,$$

$$y=2x-C, \qquad C=2x-y.$$

Endi esa $\xi=2x-y$ deb, η ni shunday tanlashimiz kerakki, $\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}\neq 0$ shart bajarilishi kerak. Buning uchun $\eta=x$ deb tanlash yetarli, chunki $\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=\begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix}=1\neq 0$ bo`ladi.

Demak,

$$\begin{cases}\xi=2x-y\\ \eta=x\end{cases}$$

almashtirishni bajaramiz.

Shunga ko'ra,

$$u_x=2u_\xi+u_\eta,$$

$$u_y=-u_\xi,$$

$$u_{xx}=4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta},$$

$$u_{xy}=-2u_{\xi\xi}-u_{\xi\eta},$$

$$u_{yy}=u_{\xi\xi}$$

bo'ladi. Topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta}+4\left(-2u_{\xi\xi}-u_{\xi\eta}\right)+4u_{\xi\xi}+$$

$$+2\left(2u_\xi+u_\eta\right)-3\left(-u_\xi\right)=0,$$

ya'ni

$$u_{\eta\eta} + 7u_{\xi} + 2u_{\eta} = 0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

Endi esa ushbu parabolik tipdagi tenglamani sodda kanonik shaklga keltiraylik. Buning uchun $u(\xi,\eta) = v(\xi,\eta) \cdot e^{\lambda\xi + \mu\eta}$ almashtirish bajaramiz va yuqoridagi misoldagi kabi birinchi va ikkinchi tartibli xususiy hosilalarni hisoblab, tenglamaga qo'yamiz:

$$v_{\eta\eta} \cdot e^{\lambda\xi + \mu\eta} + 7v_{\xi} \cdot e^{\lambda\xi + \mu\eta} +$$

$$+(2\mu + 2)v_{\eta} \cdot e^{\lambda\xi + \mu\eta} + \left(\mu^2 + 7\lambda + 2\mu\right) \cdot v \cdot e^{\lambda\xi + \mu\eta} = 0.$$

Agar $\lambda = \frac{1}{7}$, $\mu = -1$ deb tanlasak, oxirgi tenglama quyidagi sodda kanonik shaklga keladi:

$$v_{\eta\eta} + 7v_{\xi} = 0.$$

Mustaqil yechish uchun misollar

Quyida berilgan xususiy hosilali differensial tenglamalarni kanonik shaklga keltiring:

3.1. $4u_{xx} + 2u_{xy} - 12u_{yy} + u_x + u_y + 5x - 3y = 0$.

3.2. $2u_{xx} + 6u_{xy} + 5u_{yy} + u_x + u_y + xy = 0$.

3.3. $u_{xx} + 4u_{xy} + 4u_{yy} + 3u_x + 2u_y + 2x - 3y = 0$.

3.4. $16u_{xx}+8u_{xy}-48u_{yy}+4u_x+4u_y+2y-20=0$.

3.5. $u_{xx}+2u_{xy}+5u_{yy}+2u_x-3u_y+x+2y=0$.

3.6. $9u_{xx}+6u_{xy}+u_{yy}+4u_x-3y=0$.

3.7. $28u_{xx}+14u_{xy}-81u_{yy}+4u_x+u+x+y=0$.

3.8. $u_{xx}+2u_{xy}+10u_{yy}+5u_x+4x+3y=0$.

3.9. $9u_{xx}+12u_{xy}+4u_{yy}+4u_y-3u_x+2xy=0$.

3.10. $3u_{xx}+8u_{xy}+4u_{yy}+u_x+2y+4=0$.

3.11. $4u_{xx}-12u_{xy}+13u_{yy}+2u_x+5u_y+3x-y=0$.

3.12. $4u_{xx}-20u_{xy}+25u_{yy}+3u_x+2x-5y=0$.

3.13. $15u_{xx}+14u_{xy}-32u_{yy}+4u_x+3u_y+2x-17=0$.

3.14. $u_{xx}+4u_{xy}+13u_{yy}+3u_x-2u_y+3x=0$.

3.15. $4u_{xx}-12u_{xy}+9u_{yy}+5u_x+5x+4y=0$.

3.16. $27u_{xx}+20u_{xy}-68u_{yy}+3u_y+3x-2y=0$.

3.17. $u_{xx}-10u_{xy}+26u_{yy}+3u_x-2u_y+3y=0$.

3.18. $4u_{xx}+4u_{xy}+u_{yy}+5u_y+7x+y=0$.

3.19. $39u_{xx}+26u_{xy}-104u_{yy}+2u_x-3u_y-2y=0$.

3.20. $4u_{xx}-4u_{xy}+2u_{yy}-2u_y+3xy=0$.

3.21. $9u_{xx}-30u_{xy}+25u_{yy}+5u_x-u_y+x-2y=0$.

3.22. $14u_{xx}+20u_{xy}-16u_{yy}+4u_x-5u_y+x=0$.

3.23. $4u_{xx} - 20u_{xy} + 29u_{yy} - u_x - 4u_y + 3x = 0$.

3.24. $u_{xx} + 10u_{xy} + 25u_{yy} + 5u_x - 3u_y + 2x - 2y = 0$.

3.25. $26u_{xx} + 26u_{xy} - 52u_{yy} + 4u_x - 5u_y + x + y = 0$.

Quyida berilgan xususiy hosilali differensial tenglamalarni sodda kanonik shaklga keltiring:

4.1. $u_{xx} - 5u_{xy} + 4u_{yy} - 3u_x - 4u_y + 6u = 0$.

4.2. $4u_{xx} + 4u_{xy} + u_{yy} - 2u_x + 5u_y + 4u = 0$.

4.3. $u_{xx} - u_{xy} + 6u_{yy} - 2u_x + 3u_y - 6u = 0$.

4.4. $u_{xx} - 3u_{xy} - 4u_{yy} + 4u_x + 3u_y + 8u = 0$.

4.5. $u_{xx} + 6u_{xy} + 9u_{yy} + 2u_x - u_y - 2u = 0$.

4.6. $u_{xx} + u_{xy} + 12u_{yy} + 4u_x + 2u_y - 5u = 0$.

4.7. $u_{xx} - 2u_{xy} - 8u_{yy} + u_x + 5u_y + 7u = 0$.

4.8. $9u_{xx} - 6u_{xy} + u_{yy} - 4u_x - u_y - 6u = 0$.

4.9. $u_{xx} + 4u_{xy} + 5u_{yy} + 3u_x - 4u_y - u = 0$.

4.10. $2u_{xx} + 5u_{xy} + 3u_{yy} + 2u_x - u_y - 2u = 0$.

4.11. $u_{xx} + 10u_{xy} + 25u_{yy} + 3u_x - 5u_y - 3u = 0$.

4.12. $4u_{xx} + 4u_{xy} + 5u_{yy} - 5u_x - 4u_y - 2u = 0$.

4.13. $3u_{xx} + 5u_{xy} + 2u_{yy} + u_x - 4u_y - 5u = 0$.

4.14. $u_{xx} - 8u_{xy} + 16u_{yy} + 3u_x - 4u_y + 2u = 0$.

4.15. $5u_{xx} + 4u_{xy} + 4u_{yy} - 2u_x - 4u_y - 3u = 0$.

4.16. $2u_{xx} - 7u_{xy} + 5u_{yy} + 2u_x - 5u_y - 10u = 0$.

4.17. $16u_{xx} - 8u_{xy} + u_{yy} + 5u_x - 4u_y + 4u = 0$.

4.18. $5u_{xx} - 4u_{xy} + u_{yy} + 6u_x - 5u_y - 30u = 0$.

4.19. $u_{xx} - 7u_{xy} + 12u_{yy} + 2u_x + 3u_y + 6u = 0$.

4.20. $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x - 6u_y + 4u = 0$.

4.21. $2u_{xx} - 6u_{xy} + 5u_{yy} + 5u_x - 7u_y - u = 0$.

4.22. $5u_{xx} - 4u_{xy} - u_{yy} + 7u_x - 4u_y - 28u = 0$.

4.23. $4u_{xx} - 12u_{xy} + 9u_{yy} + 5u_x - 3u_y - 15u = 0$.

4.24. $3u_{xx} - u_{xy} + 2u_{yy} + 5u_x - 6u_y - 7u = 0$.

4.25. $3u_{xx} - 2u_{xy} - u_{yy} + 5u_x - 6u_y - 7u = 0$.

O'zgaruvchi koeffitsiyentli ikki o'zgaruvchili ikkinchi tartibli chiziqli differensial tenglamalarni kanonik shaklga keltirishga doir misollar.

1–misol. $y^2u_{xx} + 2xyu_{xy} + x^2u_{yy} + 2xu_x + 4u = 0$

tenglamani kanonik ko'rinishga keltiring.

Yechish: $A = y^2$, $B = xy$, $C = x^2$, $\Delta = x^2y^2 - x^2y^2 = 0$

bo`lgani uchun yuqoridagi tenglama koordinata boshidan boshqa barcha joyda parabolik tipdagi tenglama bo`ladi. Endi esa xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y^2 y'^2 - 2xyy' + x^2 = 0,\ y' = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2y^2}}{2y^2} = \frac{x}{y},$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{1}{2}C\ .$$

U holda

$$\begin{cases} \xi = x^2 - y^2 \\ \eta = x \end{cases}$$

almashtirish bajaramiz. Shunga ko`ra,

$$u_x = 2xu_\xi + u_\eta\ ,$$

$$u_y = -2yu_\xi\ ,$$

$$u_{xx} = 4x^2u_{\xi\xi} + 4xu_{\xi\eta} + u_{\eta\eta} + 2u_\xi\ ,$$

$$u_{xy} = -4xyu_{\xi\xi} - 2yu_{\xi\eta}\ ,$$

$$u_{yy} = 4y^2u_{\xi\xi} - 2u_\xi\ .$$

Bu topilgan ifodalarnini tenglamaga olib borib qo`yamiz:

$$y^2\left(4x^2u_{\xi\xi} + 4xu_{\xi\eta} + u_{\eta\eta} + 2u_\xi\right) + 2xy\left(-4xyu_{\xi\xi} - 2yu_{\xi\eta}\right) +$$

$$+x^2\left(4y^2u_{\xi\xi} - 2u_\xi\right) + 2x\left(2xu_\xi + u_\eta\right) + 4u = 0\ .$$

Natijada

$$y^2u_{\eta\eta} + 2\left(x^2 + y^2\right)u_\xi + 2xu_\eta + 4u = 0\ ,$$

yoki

$$\left(\eta^2-\xi\right)u_{\eta\eta}+2\left(2\eta^2-\xi\right)u_{\xi}+2\eta u_{\eta}+4u=0$$

tenglamaga ega bo'lamiz. Bundan esa,

$$u_{\eta\eta}+\frac{2\left(2\eta^2-\xi\right)}{\eta^2-\xi}u_{\xi}+\frac{2\eta}{\eta^2-\xi}u_{\eta}+\frac{4}{\eta^2-\xi}u=0$$

tenglama hosil bo`ladi. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2–misol. $yu_{xx}+u_{yy}=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Bu tenglama Trikomi tenglamasi deb ataladi va unda $A=y,\ B=0,\ \ C=1,\ \ \Delta=-y$.

Agar

1) $y<0$ bo'lsa berilgan tenglama giperbolik tipda,

2) $y=0$ bo'lsa berilgan tenglama parabolik tipda,

3) $y>0$ bo'lsa berilgan tenglama elliptic tipda bo'ladi.

Trikomi tenglamasi gaz dinamikasi uchun muhim bo`lib, giperbolik sohada bu tenglama tovush tezligidan yuqori harakatga mos va elliptik sohada esa, bu tenglama tovush tezligigacha bo`lgan harakatga mos keladi.

1) $y<0$ bo'lsin. U holda xarakteristik tenglama quyidagicha bo`ladi:

$$yy'^2+1=0,\ \sqrt{-y}\cdot y'\pm1=0,\ \sqrt{-y}\cdot dy\pm dx=0\ .$$

Bundan esa,

$$-\frac{2}{3}\sqrt{\left(-y\right)^3}-x=C_1,\quad -\frac{2}{3}\sqrt{\left(-y\right)^3}+x=C_2$$

bo`ladi. U holda quyidagicha almashtirish bajaramiz:

$$\begin{cases}\xi = x+\frac{2}{3}(-y)^{\frac{3}{2}}\\ \eta = x-\frac{2}{3}(-y)^{\frac{3}{2}}\end{cases}.$$

Shunga ko`ra,

$$u_x = u_\xi + u_\eta ,$$

$$u_y = -(-y)^{\frac{1}{2}}u_\xi + (-y)^{\frac{1}{2}}u_\eta ,$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} ,$$

$$u_{yy} = \frac{1}{2\sqrt{-y}}u_\xi - (-y)^{\frac{1}{2}}\left(-(-y)^{\frac{1}{2}}u_{\xi\xi} + (-y)^{\frac{1}{2}}u_{\xi\eta}\right) - \frac{1}{2\sqrt{-y}}u_\eta +$$
$$+(-y)^{\frac{1}{2}}\left(-(-y)^{\frac{1}{2}}u_{\xi\eta} + (-y)^{\frac{1}{2}}u_{\eta\eta}\right) .$$

Bu topilgan ifodalarnini tenglamaga olib borib qo`yamiz :

$$y(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) + \frac{1}{2\sqrt{-y}}(u_\xi - u_\eta) + y(-u_{\xi\xi} + u_{\xi\eta}) -$$
$$-y(-u_{\xi\eta} + u_{\eta\eta}) = 0 ,$$

$$4yu_{\xi\eta}+\frac{1}{2\sqrt{-y}}(u_{\xi}-u_{\eta})=0\ ,$$

$$u_{\xi\eta}+\frac{1}{8y\sqrt{-y}}(u_{\xi}-u_{\eta})=0\ ,$$

$$u_{\xi\eta}-\frac{1}{6(\xi-\eta)}(u_{\xi}-u_{\eta})=0\quad(\xi>\eta)$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2) $y=0$ bo'lsin. U holda xarakteristik tenglama quyidagicha bo`ladi:

$$\left(dx\right)^2=0,\quad dx=0\ .$$

Bundan esa,

$$x=C$$

bo`ladi. Shuning uchun

$$\begin{cases}\xi=x\\ \eta=y\end{cases}$$

almashtirish bajaramiz va $y=0$ bo`lgani uchun $u_{yy}=0$

tenglamaga ega bo'lamiz .

3) $y>0$ bo'lsin. U holda xarakteristik tenglama quyidagicha bo`ladi:

$$y(dy)^2+(dx)^2=0,\ \left(\sqrt{y}dy\right)^2-\left(idx\right)^2=0,\ \sqrt{y}dy\pm idx=0\ .$$

Bundan esa,

$$\frac{2}{3}y^{\frac{3}{2}} \pm ix = C$$

bo`ladi. U holda quyidagicha almashtirish bajaramiz :

$$\begin{cases} \xi = \frac{2}{3}y^{\frac{3}{2}} \\ \eta = x \quad . \end{cases}$$

Shunga ko`ra,

$$u_x = u_\eta \ ,$$

$$u_y = y^{\frac{1}{2}}u_\xi \ ,$$

$$u_{xx} = u_{\eta\eta} \ ,$$

$$u_{yy} = \frac{1}{2\sqrt{y}}u_\xi + y^{\frac{1}{2}}\left(y^{\frac{1}{2}}u_{\xi\xi} \right) = \frac{1}{2\sqrt{y}}u_\xi + yu_{\xi\xi} \ .$$

Bu topilgan ifodalarnini tenglamaga olib borib qo`yamiz . Natijada

$$yu_{\eta\eta} + yu_{\xi\xi} + \frac{1}{2\sqrt{y}}u_\xi = 0,$$

yo`ki

$$u_{\eta\eta} + u_{\xi\xi} + \frac{1}{2y\sqrt{y}}u_\xi = 0,$$

hamda $\xi = \frac{2}{3}y^{\frac{3}{2}}$ ekanligidan

$$u_{\eta\eta} + u_{\xi\xi} + \frac{1}{3\xi} u_\xi = 0$$

kanonik shakildagi tenglamaga ega bo`lamiz.

Mustaqil yechish uchun misollar

Quyida berilgan xususiy hosilali differensial tenglamalarni tipi saqlanadigan sohalarda kanonik shaklga keltiring:

5.1. $(3x+4y)u_{xx} + 2(4x+y)u_{xy} - 4(3y-x)u_{yy} +$

$+(x+y)u_x + yu_y + 2x + 5y = 0$.

5.2. $\left(1+x^2\right)^2 u_{xx} + u_{yy} + 2x\left(1+x^2\right)u_x = 0$.

5.3. $u_{xx} - \left(1+y^2\right)^2 u_{yy} - 2y\left(1+y^2\right)u_y = 0$.

5.4. $\left(1+x^2\right)u_{xx} + \left(1+y^2\right)u_{yy} + xu_x + yu_y - 2u = 0$.

5.5. $x^2 u_{xx} + 6xyu_{xy} + 9y^2 u_{yy} - 2yu_x + ye^{y/x} = 0$.

5.6. $xy^2 u_{xx} - 2x^2 yu_{xy} + x^3 u_{yy} - y^2 u_x = 0$.

5.7. $u_{xx} - 2\sin x \cdot u_{xy} - \cos^2 x \cdot u_{yy} - \cos x \cdot u_y = 0$.

5.8. $e^{2x} u_{xx} + 2e^{x+y} \cdot u_{xy} + e^{2y} \cdot u_{yy} - xu = 0$.

5.9. $u_{xx} - 2xu_{xy} = 0$.

5.10. $xu_{xx} + 2xu_{xy} + (x-1)u_{yy} = 0$.

5.11. $yu_{xx} + xu_{yy} = 0$.

5.12. $xu_{xx} + yu_{yy} + 2u_x + 2u_y = 0$.

5.13. $u_{xx} + 2\sin x \cdot u_{xy} - \cos 2x \cdot u_{yy} - \cos x \cdot u_y = 0$.

5.14. $u_{xx} + xyu_{yy} = 0$.

5.15. $x^2 u_{xx} + y^2 u_{yy} + 2u_x + 2u_y = 0$.

5.16. $xyu_{xx} + 3(x-1)y \cdot u_{xy} + 2\left(x-1{,}5\right) y \cdot u_{yy} + 2u = 0$.

5.17. $e^{2x} u_{xx} + 5e^{x+y} \cdot u_{xy} + 4e^{2y} \cdot u_{yy} - yu_x + xu = 0$.

5.18. $\sin^2 x \cdot u_{xx} - 2\sin^2 x \cdot u_{xy} - \cos^2 x \cdot u_{yy} +$

$+ \sin^2 x \cdot u_x - \cos^2 x \cdot u_y = 0$.

5.19. $u_{xx} - 2(x-1)u_{xy} - 4xu_{yy} + 5u_x - yu_y = 0$.

5.20. $2xu_{xx} - 6(x-2)u_{xy} - 36u_{yy} - xu_x + 4u_y = 0$.

5.21. $e^x u_{xx} - e^{x+y} u_{xy} + \left(e^y - e^{-x}\right) u_{yy} -$

$-e^x u_x + 2e^y u_y = 0$.

5.22. $u_{xx} - 5e^{3x+3y} u_{xy} + 4e^{6x+6y} u_{yy} - u_x + 2u = 0$.

5.23. $u_{xx} - 3\sin x \cdot u_{xy} - 4\sin^2 x \cdot u_{yy} + u_x - u_y + 3u = 0$.

5.24. $2xu_{xx} - xyu_{xy} - xy^2 u_{yy} + yu_x + xu_y - 2u = 0$.

5.25. $u_{xx} - 2\sin x \cdot u_{xy} + 2u_x - u_y = 0$.

Ko'p o'zgaruvchili ikkinchi tartibli chiziqli differensial tenglamalarni kanonik shaklga keltirish.

Bizga ushbu

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x)$$

ikkinchi tartibli xususiy hosilali differensial tenglama berilgan bo'lsin. Bu tenglamani kanonik shaklga keltirish masalasini qaraylik. Berilgan tenglamani har bir tayinlangan $x = x_0$ nuqtada maxsus bo'lmagan $\xi = B^T x$, bunda $x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, chiziqli almashtirish yordamida kanonik shaklga keltirish mumkin. Bu yerda B — shunday matritsaki, $\lambda = B\mu$ chiziqli almashtirish $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_o)\lambda_i\lambda_j$ kvadratik formani kanonik ko'rinishga keltiradi.

Har qanday kvadratik formani kanonik ko'rinishga keltirishning turli usullari mavjud, masalan shunday usullardan biri to'la kvadrat ajratish usulidir.

1–misol. $u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} + 6u_{zz} = 0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Berilgan tenglamaga mos kvadratik formani tuzamiz va uni kanonik shaklga keltiramiz:

$$\lambda_1^2 + 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 + 2\lambda_2^2 + 6\lambda_3^2 = (\lambda_1 + \lambda_2 - \lambda_3)^2 + \lambda_2^2 + 2\lambda_2\lambda_3 + 5\lambda_3^2 =$$

$$= (\lambda_1 + \lambda_2 - \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 + 4\lambda_3^2 = \mu_1^2 + \mu_2^2 + \mu_3^2.$$

Berilgan tenglama elliptik tipda ekanligi ko'rinib turibdi, bu yerda

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2 - \lambda_3 \\ \mu_2 = \lambda_2 + \lambda_3 \\ \mu_3 = 2\lambda_3 \end{cases}.$$

Bundan esa quyidagi tengliklarga ega bo'lamiz:

$$\begin{cases} \lambda_1 = \mu_1 - \mu_2 + \mu_3 \\ \lambda_2 = \mu_2 - \frac{1}{2}\mu_3 \\ \lambda_3 = \frac{1}{2}\mu_3 \end{cases}.$$

Bu almashtirishning

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

matritsasini hosil qilamiz va uni transponirlaymiz. U holda

$$B^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

bo'ladi. Endi quyidagicha almashtirish bajaramiz:

$$\xi = B^T X, \quad \text{bunda} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

ya'ni

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x + y \\ x - \frac{1}{2} y + \frac{1}{2} z \end{pmatrix},$$

yoki

$$\begin{cases} \xi_1 = x \\ \xi_2 = -x + y \\ \xi_3 = x - \frac{1}{2} y + \frac{1}{2} z. \end{cases}$$

Berilgan almashtirish chiziqli ekanligini hisobga olib,

$$u_x = u_{\xi_1} - u_{\xi_2} + u_{\xi_3},$$

$$u_y = u_{\xi_2} - \frac{1}{2} u_{\xi_3},$$

$$u_z = \frac{1}{2} u_{\xi_3},$$

$$u_{xx} = u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} + 2u_{\xi_1\xi_3} - 2u_{\xi_2\xi_3} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3},$$

$$u_{xy} = u_{\xi_1\xi_2} - \frac{1}{2} u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + \frac{3}{2} u_{\xi_2\xi_3} - \frac{1}{2} u_{\xi_3\xi_3},$$

$$u_{yy} = u_{\xi_2\xi_2} - u_{\xi_2\xi_3} + \frac{1}{4} u_{\xi_3\xi_3},$$

$$u_{xz} = \frac{1}{2} u_{\xi_1\xi_3} - \frac{1}{2} u_{\xi_2\xi_3} + \frac{1}{2} u_{\xi_3\xi_3},$$

$$u_{zz} = \frac{1}{4} u_{\xi_3\xi_3}$$

tengliklarga ega bo'lamiz. Bu topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} + 2u_{\xi_1\xi_3} - 2u_{\xi_2\xi_3} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3} +$$
$$+2\left(u_{\xi_1\xi_2} - \frac{1}{2} u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + \frac{3}{2} u_{\xi_2\xi_3} - \frac{1}{2} u_{\xi_3\xi_3} \right) -$$
$$-2\left(\frac{1}{2} u_{\xi_1\xi_3} - \frac{1}{2} u_{\xi_2\xi_3} + \frac{1}{2} u_{\xi_3\xi_3} \right) + 2\left(u_{\xi_2\xi_2} - u_{\xi_2\xi_3} + \frac{1}{4} u_{\xi_3\xi_3} \right) +$$
$$+6 \cdot \frac{1}{4} u_{\xi_3\xi_3} = 0.$$

Natijada $u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3} = 0$ tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2–misol. $4u_{xx} - 4u_{xy} - 2u_{yz} + u_y + u_z = 0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Berilgan tenglamaga mos kvadratik formani tuzamiz va uni kanonik shaklga keltiramiz:

$$4\lambda_1^2 - 4\lambda_1\lambda_2 - 2\lambda_2\lambda_3 = (2\lambda_1 - \lambda_2)^2 - \lambda_2^2 - 2\lambda_2\lambda_3 =$$

$$= (2\lambda_1 - \lambda_2)^2 - (\lambda_2 + \lambda_3)^2 + \lambda_3^2 = \mu_1^2 - \mu_2^2 + \mu_3^2 .$$

Berilgan tenglama giperbolik tipda ekanligi ko'rinib turibdi, bu yerda

$$\begin{cases} \mu_1 = 2\lambda_1 - \lambda_2 \\ \mu_2 = \lambda_2 + \lambda_3 \\ \mu_3 = \lambda_3 \end{cases} .$$

Bundan esa quyidagi tengliklarga ega bo'lamiz:

$$\begin{cases} \lambda_1 = \dfrac{1}{2}\mu_1 + \dfrac{1}{2}\mu_2 - \dfrac{1}{2}\mu_3 \\ \lambda_2 = \mu_2 - \mu_3 \\ \lambda_3 = \mu_3 \end{cases} .$$

Bu almashtirishning B matritsani tuzamiz va uni transponirlaymiz. U holda

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

bo'ladi. Endi quyidagicha almashtirish bajaramiz: $\xi = B^T X$, ya'ni

$$\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ \frac{1}{2} & 1 & 0\\ -\frac{1}{2} & -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}\frac{x}{2}\\ \frac{x}{2}+y\\ -\frac{1}{2}x-y+z\end{pmatrix},$$

yoki

$$\begin{cases}\xi_1=\frac{x}{2}\\ \xi_2=\frac{x}{2}+y\\ \xi_3=-\frac{1}{2}x-y+z.\end{cases}$$

Berilgan almashtirish chiziqli ekanligini hisobga olib,

$$u_x=\frac{1}{2}u_{\xi_1}+\frac{1}{2}u_{\xi_2}-\frac{1}{2}u_{\xi_3},$$

$$u_y=u_{\xi_2}-u_{\xi_3},$$

$$u_z=u_{\xi_3},$$

$$u_{xx}=\frac{1}{4}u_{\xi_1\xi_1}+\frac{1}{2}u_{\xi_1\xi_2}-\frac{1}{2}u_{\xi_1\xi_3}-\frac{1}{2}u_{\xi_2\xi_3}+\frac{1}{4}u_{\xi_2\xi_2}+\frac{1}{4}u_{\xi_3\xi_3},$$

$$u_{xy}=\frac{1}{2}u_{\xi_1\xi_2}-\frac{1}{2}u_{\xi_1\xi_3}+\frac{1}{2}u_{\xi_2\xi_2}-u_{\xi_2\xi_3}+\frac{1}{2}u_{\xi_3\xi_3},$$

$$u_{yz}=u_{\xi_2\xi_3}-u_{\xi_3\xi_3}$$

tengliklarga ega bo’lamiz va topilgan ifodalarni

$$4u_{xx}-4u_{xy}-2u_{yz}+u_y+u_z=0$$

tenglamaga olib borib qo'yamiz. U holda

$$u_{\xi_1\xi_1}+2u_{\xi_1\xi_2}-2u_{\xi_1\xi_3}-2u_{\xi_2\xi_3}+u_{\xi_2\xi_2}+u_{\xi_3\xi_3}-2u_{\xi_1\xi_2}+2u_{\xi_1\xi_3}-$$
$$-2u_{\xi_2\xi_2}+4u_{\xi_2\xi_3}-2u_{\xi_3\xi_3}-2u_{\xi_2\xi_3}+2u_{\xi_3\xi_3}+u_{\xi_2}-u_{\xi_3}+u_{\xi_3}=0.$$

Bundan esa,

$$u_{\xi_1\xi_1}-u_{\xi_2\xi_2}+u_{\xi_3\xi_3}+u_{\xi_2}=0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

3–misol. $u_{xy}-u_{xz}+u_x+u_y-u_z=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Berilgan tenglamaga mos kvadratik formani tuzamiz va uni kanonik shaklga keltiramiz:

$K(\lambda)=\lambda_1\lambda_2-\lambda_1\lambda_3$ kvadratik formani kanonik ko'rinishga keltirish uchun

$$\begin{cases}\lambda_1=\eta_1+\eta_2\\ \lambda_2=\eta_1-\eta_2\\ \lambda_3=\eta_3\end{cases}$$

almashtirish bajaramiz va quyidagi kvadratik formaga ega bo'lamiz:

$$K_1(\eta)=\eta_1^2-\eta_2^2-\eta_1\eta_3-\eta_2\eta_3.$$

Ushbu kvadratik formani kanonik shaklga keltiramiz:

$$K_1(\eta)=\eta_1^2-\eta_2^2-\eta_1\eta_3-\eta_2\eta_3=\left(\eta_1-\frac{1}{2}\eta_3\right)^2-\eta_2^2-\eta_2\eta_3-$$

$$-\frac{1}{4}\eta_3^2=\left(\eta_1-\frac{1}{2}\eta_3\right)^2-\left(\eta_2+\frac{1}{2}\eta_3\right)^2=\mu_1^2-\mu_2^2.$$

Berilgan tenglama parabolik tipda ekanligi ko'rinib turibdi, bu yerda

$$\begin{cases}\mu_1=\eta_1-\frac{1}{2}\eta_3\\ \mu_2=\eta_2+\frac{1}{2}\eta_3\end{cases}$$

va μ_3 o'zgaruvchini shunday tanlaymizki, hosil bo'lgan matritsaning determinanti noldan farqli bo'lsin, masalan, $\mu_3=\eta_3$.

U holda quyidagi almashtirishga ega bo'lamiz:

$$\begin{cases}\mu_1=\eta_1-\frac{1}{2}\eta_3\\ \mu_2=\eta_2+\frac{1}{2}\eta_3\\ \mu_3=\eta_3\end{cases}.$$

Bundan esa quyidagi tengliklarga ega bo'lamiz:

$$\begin{cases}\lambda_1=\mu_1+\mu_2\\ \lambda_2=\mu_1-\mu_2+\mu_3\\ \lambda_3=\mu_3\end{cases}.$$

Bu almashtirishning B matritsasini tuzamiz va uni transponirlaymiz. U holda

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

bo'ladi. Endi quyidagicha almashtirish bajaramiz: $\xi = B^T X$, ya'ni

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ y+z \end{pmatrix},$$

yoki

$$\begin{cases} \xi_1 = x + y \\ \xi_2 = x - y \\ \xi_3 = y + z \end{cases}.$$

Berilgan almashtirish chiziqli ekanligini hisobga olib,

$$u_x = u_{\xi_1} + u_{\xi_2},$$

$$u_y = u_{\xi_1} - u_{\xi_2} + u_{\xi_3},$$

$$u_z = u_{\xi_3},$$

$$u_{xy} = u_{\xi_1\xi_1} - u_{\xi_1\xi_2} + u_{\xi_1\xi_3} + u_{\xi_1\xi_2} - u_{\xi_2\xi_2} + u_{\xi_2\xi_3} =$$

$$= u_{\xi_1\xi_1} + u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + u_{\xi_2\xi_3},$$

$$u_{xz} = u_{\xi_1\xi_3} + u_{\xi_2\xi_3}$$

tengliklarga ega bo'lamiz va topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$u_{\xi_1\xi_1}+u_{\xi_1\xi_3}-u_{\xi_2\xi_2}+u_{\xi_2\xi_3}-u_{\xi_1\xi_3}-u_{\xi_2\xi_3}+$$
$$+u_{\xi_1}+u_{\xi_2}+u_{\xi_1}-u_{\xi_2}+u_{\xi_3}-u_{\xi_3}=0.$$

Bundan esa,

$$u_{\xi_1\xi_1}-u_{\xi_2\xi_2}+2u_{\xi_1}=0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

Mustaqil yechish uchun misollar

Quyida berilgan ko'p o'zgaruvchili xususiy hosilali chiziqli differensial tenglamalarni kanonik shaklga keltiring:

6.1. $u_{xx}+2u_{xy}-2u_{xz}+2u_{yy}+4u_{zz}+2xu_x-2yu_y=0$.

6.2. $u_{xx}+2u_{xy}-2u_{xz}+2u_{yy}+2u_{zz}-2yu_x+2zu_y+2xu_z=0$.

6.3. $u_{xx}+4u_{xy}-2u_{xz}+4u_{yy}+2u_{zz}-xu_x+yu_y=0$.

6.4. $u_{xx}+2u_{xy}-4u_{xz}-6u_{yz}-u_{zz}-2xyu_x+2u=0$.

6.5. $u_{xx}+4u_{xy}-2u_{xz}+4u_{yy}-2u_{yz}+2u_{zz}-2u_x=0$.

6.6. $u_{xx}+2u_{xy}+2u_{yy}+2u_{yz}+2u_{yt}+2u_{zz}+3u_{tt}=0$.

6.7. $u_{xx}+2u_{xy}+2u_{xt}+2u_{zz}-2u_{zt}+2u_{tt}=0$.

6.8. $u_{xy}+u_{yz}+u_{xz}+2u_x-2u_z=0$.

6.9. $u_{xx} + 2u_{xy} - 2u_{xz} - 4u_{yz} + 2u_{yt} + 2u_{zz} = 0$.

6.10. $u_{xx} + 2u_{xz} - 2u_{xt} + u_{yy} + 2u_{yz} + 2u_{yt} + 2u_{zz} + 2u_{tt} = 0$.

6.11. $u_{xx} + u_{xy} + u_{yz} + u_{xz} + 2u_x - 2u_z = 0$.

6.12. $u_{xx} + 2u_{xy} - 4u_{xz} + 2u_{yy} + 4u_{zz} + 2u_x - 2u_z = 0$.

6.13. $u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} - 2u_{yz} + 4u_{zz} + 2u_y - 4u_z = 0$.

6.15. $u_{xy} - 2u_{xz} + u_{yy} - 2u_{yz} + 4u_{zz} + 2u_y - 4u_z = 0$.

6.16. $u_{xy} - 2u_{xz} - 2u_{yz} + u_{zz} + 2u_x + 2u_y - 4u_z = 0$.

6.17. $2u_{xy} + 2u_{yz} - 2u_{xz} + 3u_x - u = 0$.

6.18. $u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} - xu_x + yu_z = 0$.

6.19. $u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} - 2xyu_x + 3xu = 0$.

6.20. $u_{xy} + u_{yz} + u_{xz} - 3x^2u_y + y\sin x \cdot u + xe^{-y} = 0$.

6.21. $u_{xx} + 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_z - u = 0$.

6.22. $u_{xx} - 2u_{xz} - 2u_{xt} + 2u_{yy} - 2u_{yz} - 2u_{yt} + 2u_{zz} + 2u_{tt} = 0$.

6.23. $u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} - 2u_{yz} + 2u_x + 2u_y - 2u_z = 0$.

6.24. $u_{xx} + 4u_{xy} - 2u_{xz} + 5u_{yy} + 3u_{zz} - xu_x + yu_y = 0$.

6.25. $u_{xx} + 2u_{xy} - 2u_{xz} + 3u_{yy} - 2u_{yz} + 3u_{zz} - 3xu_x + 5yu = 0$.

3 - §. CHIZIQLI BO`LMAGAN XUSUSIY HOSILALI DIFFERENSIAL TENGLAMANI UNING BERILGAN YECHIMI BO`YLAB SINFLARGA AJRATISH

Bizga ushbu

$$F\left(x,...,\frac{\partial^k u}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}},...\right)=0 \qquad (1)$$

shakldagi $u(x)=u(x_1,x_2,...,x_n),\quad x\in\Omega$ noma'lum fungsiyaga nisbatan $m-$tartibli xususiy hosilali differensial tenglama berilgan bo`lsin.

Chiziqli bo`lmagan $m-$tartibli xususiy hosilali (1) differensial tenglamani ham

$$K(\xi_1,...,\xi_n)=\sum_{|\alpha|=m}\frac{\partial F}{\partial p_{\alpha_1...\alpha_n}}\xi_1^{\alpha_1}...\xi_n^{\alpha_n}$$

xarakteristik forma orqali sinflarga ajratiladi. Lekin, xarakteristik formaning koeffitsientlari bu holda $x\in\Omega$ nuqtadan tashqari izlanayotgan yechim va uning xususiy hosilalariga ham bog`liq bo`ladi. Bu holda $m-$tartibli xususiy hosilali differensial tenglamani berilgan yechim uchungina sinflarga ajratiladi.

1–misol. Quyidagi tenglamani berilgan yechim bo'ylab tipini aniqlang: $u_{xx}^2+(u_{xx}-2)u_{xy}-u_{yy}^2=0,\quad u=x^2+y^2$.

Yechish: Bu tenglamaga mos,

$$K = F_{u_{xx}}\lambda_1^2 + F_{u_{xy}}\lambda_1\lambda_2 + F_{u_{yy}}\lambda_2^2$$

kvadratik formani qaraylik. Berilgan yechim uchun,

$$K = \left(2u_{xx} + u_{xy}\right)\lambda_1^2 + \left(u_{xx} - 2\right)\lambda_1\lambda_2 - 2u_{yy}\lambda_2^2,$$

$$u_{xx} = 2, \quad u_{xy} = 0, \quad u_{yy} = 2$$

ekanliginni hisobga olsak, $K = 4\lambda_1^2 - 4\lambda_2^2$ bo`ladi. Agar $\xi = 2\lambda_1,\ \eta = 2\lambda_2$ almashtirish bajarsak, $K = \xi^2 - \eta^2$ formaga ega bo'lamiz. Demak, berilgan tenglama uning berilgan yechimi bo`ylab giperbolik tipdagi tenglama bo`ladi.

2–misol. Quyidagi tenglamani berilgan yechim bo'ylab tipini aniqlang: $u_{xx}^2 + \left(u_{xx} - 2\right)u_{xy} + u_{yy}^2 + 4u_{yy} + 4 = 0, \quad u = 2xy.$

Yechish: Bu tenglamaga mos,

$$K = \left(2u_{xx} + u_{xy}\right)\lambda_1^2 + \left(u_{xx} - 2\right)\lambda_1\lambda_2 + (2u_{yy} + 4)\lambda_2^2$$

va $u_{xx} = 0, \quad u_{xy} = 2, \quad u_{yy} = 0$ ekanligini hisobga olsak, u holda $K = 2\lambda_1^2 - 2\lambda_1\lambda_2 + 4\lambda_2^2 = 2\left(\lambda_1 - \frac{1}{2}\lambda_2\right)^2 + 3,5\lambda_2^2$

bo`ladi. Agar $\xi = \sqrt{2}\left(\lambda_1 - \frac{1}{2}\lambda_2\right), \quad \eta = \sqrt{3,5}\lambda_2$ almashtirish bajarsak, $K = \xi^2 + \eta^2$ formaga ega bo'lamiz. Demak, berilgan tenglama uning berilgan yechimi bo`ylab elliptik tipdagi tenglama bo`ladi.

3-misol. Quyidagi tenglamani berilgan yechim bo'ylab tipini aniqlang: $u_{xx}+u_{xy}u_{yy}+u_{yy}^2-4u_{yy}=0$, $u=2y^2$.

Yechish: Bu tenglamaga mos,

$$K=\lambda_1^2+u_{yy}\lambda_1\lambda_2+(u_{xy}+2u_{yy}-4)\lambda_2^2$$

va $u_{xx}=0$, $u_{xy}=0$, $u_{yy}=4$ ekanligini hisobga olsak, u holda $K=\lambda_1^2+4\lambda_1\lambda_2+4\lambda_2^2=\left(\lambda_1+2\lambda_2\right)^2$ bo`ladi. Agar $\xi=\lambda_1+2\lambda_2$ almashtirish bajarsak, $K=\xi^2$ formaga ega bo'lamiz. Demak, berilgan tenglama uning berilgan yechimi bo`ylab parabolik tipdagi tenglama bo`ladi.

Mustaqil yechish uchun misollar

Quyidagi tenglamanlarni berilgan yechimi bo'ylab tipini aniqlang:

7.1. $u_{xy}^2+u_{xx}u_{yy}+u_{yy}^2=8$, $u=2\sqrt{2}xy$.

7.2. $u_{xx}^2-4u_{xy}+u_{yy}^2=0$, $u=\left(x+y\right)^2$.

7.3. $u_{xx}+u_{xy}u_{yy}+u_{yy}^2-4u_{yy}=0$, $u=5xy$.

7.4. $u_{xx}^4-4u_{xy}+u_{yy}^3+u_{yy}+4=0$, $u=xy$.

7.5. $u_{xx}^2-4u_{xy}+u_{yy}^2+u_{yy}-4=0$, $u=x^2$.

7.6. $u_{xx}^2 - 4u_{xy} + u_{yy}^2 = 0$, $u = x^2 + \frac{y^2}{4} + \frac{17}{16}xy$.

7.7. $u_{xx}^3 + u_{xy}u_{yy} + u_{yy}^3 - 8u_{yy} = 0$, $u = x^2 + y^2$.

7.8. $3u_{xx}^3 - 6u_{xy} + u_{yy} - 4 = 0$, $u = \frac{1}{2}\left(x^2 + y^2\right)$.

7.9. $u_{xx}^2 u_{xy} - 5u_{yy} + u_x - 2(x+y) - 8 = 0$, $u = x^2 + 2xy$.

7.10. $u_{xx}^4 + 2u_{xy}^2 - 3u_{yy} + u_y - 2x = 0$, $u = 2xy - 8y$.

7.11. $2u_{xx}^3 + 2u_{xy}^5 + 3u_{yy} - 2u_y + 2x = 0$, $u = xy - \frac{1}{2}x^2$.

7.12. $5u_{xx}^5 - 7u_{xy} + 25u_{yy} - 150y = 0$, $u = \frac{5}{7}xy + \frac{1}{2}x^2 + y^3$.

7.13. $u_{xx}^2 + 5u_{xy}^2 + 6u_{yy}^2 = 12$, $u = \frac{1}{2}(x+y)^2$.

7.14. $u_{xx}^3 - 4u_{xy}^2 + 7u_{yy} - 4u_x + u_y + 3x + 4y + 3 = 0$,

$u = \frac{1}{2}x^2 + xy$.

7.15. $u_{xx}^2 - 2u_{xy}^2 + u_{yy}^2 + 2u_x - 2(x+y) = 0$, $u = \frac{1}{2}(x+y)^2$.

7.16. $u_{xy}^2 + u_{xx}u_{yy} + u_{yy}^2 + 2u_{xx} + 2u_{yy} = 0$, $u = x^2 - y^2$.

7.17. $u_{xx}^2 - 2u_{xy}^2 + u_{yy}^2 + u_x + u_y - 2(x+y) - 8 = 0$, $u = x^2 + y^2$.

7.18. $u_{xx}^4 - 4u_{xy}^5 + 8u_{yy} + 4u_y - 4x + 3 = 0$, $u = \frac{1}{2}x^2 + xy$.

7.19. $u_{xx}^2 u_{xy} - 5u_{xy}^2 + u_{yy}^3 + u_y - 2x - y + 1 = 0$, $u = x^2 + xy$.

7.20. $u_{xx}^2 + u_{xx}u_{yy} + u_{yy}^2 - 4u_{xy} + 2u_{yy} = 0$, $u = x^2 + xy$.

7.21. $5u_{xx}^5 - 3u_{xy} - 2u_{yy} - u_x + u_y = 0$, $u = \frac{1}{2}(x + y)^2$.

7.22. $u_{xx}^3 u_{xy} - 4u_{xy}^2 - u_{yy}^2 + u_y - x - 2y = 0$, $u = x^2 + xy + y^2$.

7.23. $u_{xx}^2 + u_{xx}u_{yy} - u_{yy}^3 - 2u_{xy} + 2u_{yy} = 0$, $u = (x + y)^2$.

7.24. $u_{xx}^4 + u_{xx}u_{yy} - u_{yy}^3 - 3u_{xy} + 2u_{yy} + u_x - x - y + 2 = 0$,

$$u = \frac{1}{2}x^2 + xy \ .$$

7.25. $u_{xx}^5 + u_{xx}u_{yy} - u_{xy}^3 - u_{yy} - 2u_x + 2u_y = 0$,

$$u = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \ .$$

4 - §. XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR SISTEMASINING TIPINI ANIQLASH

Bizga $u_1, u_2, ..., u_N$ noma`lum funksiyalar qatnashgan har biri m – tartibli quyidagi N ta xususiy hosilalali differensial tenglamalar sistemasi berilgan bo'lsin:

$$
\begin{cases}
F_1\left(x, u_1, u_2, \ldots, u_N, \ldots, p^1_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}, \ldots p^N_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}\right) = 0 \\
F_2\left(x, u_1, u_2, \ldots, u_N, \ldots, p^1_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}, \ldots p^N_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}\right) = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
F_N\left(x, u_1, u_2, \ldots, u_N, \ldots, p^1_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}, \ldots p^N_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}}\right) = 0,
\end{cases}
$$

bu yerda $p^j_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}} = \dfrac{\partial^{|i|} u_j}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_n^{i_n}}$, $0 \le |i| \le m$, $0 \le j \le N$.

Ushbu tenglamalar sistemasining tipini aniqlash uchun uning xarakteristik formasini tuzamiz. Buning uchun bizga quyidagi kvadratik matritsalar zarur bo'ladi:

$$
A_{i_1 i_2 \ldots i_n} = \begin{pmatrix}
\dfrac{\partial F_1}{\partial p^1_{i_1 i_2 \ldots i_n}} & \dfrac{\partial F_1}{\partial p^2_{i_1 i_2 \ldots i_n}} & \cdots & \dfrac{\partial F_1}{\partial p^N_{i_1 i_2 \ldots i_n}} \\
\dfrac{\partial F_2}{\partial p^1_{i_1 i_2 \ldots i_n}} & \dfrac{\partial F_2}{\partial p^2_{i_1 i_2 \ldots i_n}} & \cdots & \dfrac{\partial F_2}{\partial p^N_{i_1 i_2 \ldots i_n}} \\
. & . & \cdots & . \\
\dfrac{\partial F_N}{\partial p^1_{i_1 i_2 \ldots i_n}} & \dfrac{\partial F_N}{\partial p^2_{i_1 i_2 \ldots i_n}} & \cdots & \dfrac{\partial F_N}{\partial p^N_{i_1 i_2 \ldots i_n}}
\end{pmatrix}, \quad \sum_{k=1}^{n} i_k = m \ .
$$

Bu matritsalardan foydalanib, $\lambda_1,\ \lambda_2,\ ...,\ \lambda_n$ haqiqiy skalyar parametrlarga nisbatan ushbu Nm – tartibli xarasteristik formani tuzamiz:

$$K(\lambda_1, \lambda_2, \dots, \lambda_n) = \det\left(\sum_{|i|=m} A_{i_1 i_2 \dots i_n} \lambda_1^{i_1} \dots \lambda_n^{i_n}\right).$$

Yuqoridagi sistemaning tipini aniqlash ushbu xaraktristik formaning shakliga qarab, m – tartibli bitta tenglama qaralgani singari tiplarga bo`linadi.

1–misol. $\begin{cases} 2u_x - 4v_x + 3u_y + 8v_y - u = 0 \\ 3u_x - 2v_x + 6u_y + 3v_y + 2u = 0 \end{cases}$ tenglamalar sistemasining tipini aniqlang.

Yechish: Avvalambor, biz

$$A_{i_1 i_2 \dots i_n}, \quad \sum_{k=1}^{n} i_k = m$$

matritsalarni tuzamiz. Bizning misolda N=2, n=2, $\sum_{k=1}^{2} i_k = 1,\ u_1 = u$, $u_2 = v$ bo'lgani uchun $A_{i_1 i_2 \dots i_n}$ matritsalar quyidagicha bo'ladi:

$$A_{10} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x} \\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 3 & -2 \end{pmatrix}, \text{ bu erda } i_1 = 1,\ i_2 = 0,$$

$$A_{01} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y} \\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y} \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 6 & 3 \end{pmatrix}, \text{ bu erda } i_1 = 0,\ i_2 = 1.$$

Endi esa $K(\lambda_1, \lambda_2, \ldots, \lambda_n) = \det\left(\sum_{|i|=m} A_{i_1 i_2 \ldots i_n} \lambda_1^{i_1} \ldots \lambda_n^{i_n} \right)$

xarakteristik ko'phadni tuzamiz:

$$K(\lambda_1, \lambda_2) = \det\left(\begin{pmatrix} 2 & -4 \\ 3 & -2 \end{pmatrix} \lambda_1 + \begin{pmatrix} 3 & 8 \\ 6 & 3 \end{pmatrix} \lambda_2 \right) =$$

$$= \det\begin{pmatrix} 2\lambda_1 + 3\lambda_2 & -4\lambda_1 + 8\lambda_2 \\ 3\lambda_1 + 6\lambda_2 & -2\lambda_1 + 3\lambda_2 \end{pmatrix} = -4\lambda_1^2 + 9\lambda_2^2 + 12\lambda_1^2 - 48\lambda_2^2 =$$

$$= 8\lambda_1^2 - 39\lambda_2^2.$$

$B^2 - AC = 0^2 - 8 \cdot (-39) = 312 > 0$. Demak, berilgan tenglamalar sistemasi tekislikning hamma nuqtalarida giperbolik tipda bo`ladi.

2–misol. $\begin{cases} u_x + u_y + v_y + v_z - xyu = 0 \\ v_x - u_y - v_y + u_z + 2u = 0 \end{cases}$ tenglamalar sistemasining tipini aniqlang.

Yechish: Avvalambor, biz

$$A_{i_1 i_2 \ldots i_n}, \qquad \sum_{k=1}^{n} i_k = m$$

matritsalarni tuzamiz. Bizning misolda N=2, n=3, $\sum_{k=1}^{3} i_k = 1$, $u_1 = u$, $u_2 = v$ bo'lgani uchun $A_{i_1 i_2 \ldots i_n}$ matritsalar quyidagicha bo'ladi:

$$A_{100} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x} \\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ bu erda } i_1 = 1,\ i_2 = 0,\ i_3 = 0,$$

$$A_{010} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y} \\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \text{ bu erda } i_1 = 0,\ i_2 = 1,\ i_3 = 0,$$

$$A_{001} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_z} & \dfrac{\partial F_1}{\partial v_z} \\ \dfrac{\partial F_2}{\partial u_z} & \dfrac{\partial F_2}{\partial v_z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ bu erda } i_1 = 0,\ i_2 = 0,\ i_3 = 1.$$

Endi esa $$K(\lambda_1, \lambda_2, \ldots, \lambda_n) = \det\left(\sum_{|i|=m} A_{i_1 i_2 \ldots i_n} \lambda_1^{i_1} \ldots \lambda_n^{i_n} \right)$$

xarakteristik ko'phadni tuzamiz:

$$K(\lambda_1,\lambda_2,\lambda_3)=$$

$$=\det\left(\begin{pmatrix}\dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x}\\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x}\end{pmatrix}\lambda_1+\begin{pmatrix}\dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y}\\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y}\end{pmatrix}\lambda_2+\begin{pmatrix}\dfrac{\partial F_1}{\partial u_z} & \dfrac{\partial F_1}{\partial v_z}\\ \dfrac{\partial F_2}{\partial u_z} & \dfrac{\partial F_2}{\partial v_z}\end{pmatrix}\lambda_3\right)=$$

$$=\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\lambda_1+\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\lambda_2+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\lambda_3\right)=$$

$$=\det\begin{pmatrix}\lambda_1+\lambda_2 & \lambda_2+\lambda_3\\ -\lambda_2+\lambda_3 & \lambda_1-\lambda_2\end{pmatrix}=\lambda_1^2-\lambda_2^2+\lambda_2^2-\lambda_3^2=\lambda_1^2-\lambda_3^2.$$

Xarakteristik formaning $K(\lambda_1,\lambda_2,\lambda_3)=\lambda_1^2-\lambda_3^2$ kanonik shaklida ikkinchi koeffitsient nolga tengdir. Shunga ko`ra, berilgan tenglamalar sistemasi fazoning hamma nuqtalarida parabolik tipda bo`ladi.

3-misol. $\begin{cases}u_x+u_y+v_y-u=0\\ v_x-2u_y-v_y+xu=0\end{cases}$ tenglamalar sistemasining tipini aniqlang.

Yechish: Avvalambor, biz

$$A_{i_1 i_2 \ldots i_n},\quad \sum_{k=1}^{n} i_k=m$$

matritsalarni tuzamiz. Bizning misolda N=2, n=2, $\sum_{k=1}^{2} i_k=1$, $u_1=u$, $u_2=v$ bo'lgani uchun $A_{i_1 i_2 \ldots i_n}$ matritsalar quyidagicha bo'ladi:

$$A_{10}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x}\\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \text{bu erda}\ i_1=1,\ i_2=0,$$

$$A_{01}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y}\\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y}\end{pmatrix}=\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix},\ \text{bu erda}\ i_1=0,\ i_2=1.$$

Endi esa $K(\lambda_1,\lambda_2,\dots,\lambda_n)=\det\left(\sum\limits_{|i|=m} A_{i_1 i_2 \dots i_n}\lambda_1^{i_1}\dots\lambda_n^{i_n}\right)$

xarakteristik ko’phadni tuzamiz:

$$K(\lambda_1,\lambda_2)=\det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\lambda_1+\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\lambda_2\right)=$$

$$=\det\begin{pmatrix}\lambda_1+\lambda_2 & \lambda_2\\ -2\lambda_2 & \lambda_1-\lambda_2\end{pmatrix}=\lambda_1^2-\lambda_2^2+2\lambda_2^2=\lambda_1^2+\lambda_2^2.$$

Xarakteristik formaning $K(\lambda_1,\lambda_2)=\lambda_1^2+\lambda_2^2$ kanonik shaklida ikkiala koeffitsient ham birga tengdir. Shunga ko`ra, berilgan tenglamalar sistemasi tekislikning hamma nuqtalarida elliptik tipda bo`ladi.

Mustaqil yechish uchun misollar

Quyida berilgan tenglamalar sistemasining tipini aniqlang:

8.1. $\begin{cases} u_x - v_x + u_y + v_y + u = 0 \\ 2u_x - v_x + 4u_y + 2v_y - u = 0 \,. \end{cases}$

8.2. $\begin{cases} -u_x + u_y + v_x + xy = 0 \\ 2u_x + 3u_y - 3v_y + u = 0 \,. \end{cases}$

8.3. $\begin{cases} u_x + 2u_y - 3v_x + v_y + x = 0 \\ 3u_x + 3u_y - 3v_y - 3v_x + u = 0 \,. \end{cases}$

8.4. $\begin{cases} 4u_x + u_y - 3v_x + xu = 0 \\ u_x - 5u_y + 2v_x + u \sin x = 0 \,. \end{cases}$

8.5. $\begin{cases} 4u_x + 5u_y - 3v_x + x^2 u = 0 \\ 2u_x - 5u_y + v_y + u \cos x = 0 \,. \end{cases}$

8.6. $\begin{cases} 5u_x + 3v_x + v_y + x^2 y = 0 \\ u_x + 2u_y + v_x + u \sin y = 0 \,. \end{cases}$

8.7. $\begin{cases} 10u_x + 2v_x + u_y + x^2 \cos y = 0 \\ 4u_x - 3u_y + 5v_x + v_y - 4x \sin y = 0 \,. \end{cases}$

8.8. $\begin{cases} 2v_x + u_y + 6v_y - x^3 y = 0 \\ u_x + 2u_y + 2v_x - 3v_y + 3xy^2 = 0 \,. \end{cases}$

8.9. $\begin{cases} 6u_x - 2u_y - 3v_x + 4v_y + x^4 + y = 0 \\ 3u_x - 2u_y - 8v_y + u + y^2 = 0 \quad . \end{cases}$

8.10. $\begin{cases} u_x + 6u_y + 5v_x + 3v_y + x + \sin u = 0 \\ 3u_y - 5v_x + v_y + u + \cos x = 0 \end{cases}$.

8.11. $\begin{cases} 2u_x + 2v_x + 12u_y - 2u = 0 \\ v_x + 4u_y + v_y + xy = 0 \end{cases}$.

8.12. $\begin{cases} 2u_x + 7u_y + v_x - 2u = 0 \\ 3u_x + 31u_y + v_y + 3v_x - e^y \sin x = 0 \end{cases}$.

8.13. $\begin{cases} 2u_x + 3v_x - v_y - 5u = 0 \\ 3u_x + 3v_x + 3u_y + 4v_y = 0 \end{cases}$.

8.14. $\begin{cases} u_x + u_z - 3v_x + v_z + x = 0 \\ u_x + v_x - 2v_y - 2v_z + u = 0 \end{cases}$.

8.15. $\begin{cases} v_x + u_z - 2v_z + 5u = 0 \\ u_x - v_x + v_y - v_z + xu = 0 \end{cases}$.

8.16. $\begin{cases} u_x - v_y + 2u_z - 3v_z - u = 0 \\ u_y + 2v_x - 2u_z + v_y + 2u = 0 \end{cases}$.

8.17. $\begin{cases} u_x - u_y + 2v_y - 3v_z + 5u = 0 \\ u_x - v_x + 2u_z + v_z + 4u = 0 \end{cases}$.

8.18. $\begin{cases} u_x - 2u_y + v_y - v_z + 6xu = 0 \\ 2u_x - 4v_x + 2u_z + v_z + 4yu = 0 \end{cases}$.

8.19. $\begin{cases} u_x - v_y - v_z + 2xyu = 0 \\ 2u_x + 2u_z + v_z + 4x + u = 0 \end{cases}$.

8.20. $\begin{cases} 2u_x - u_y + v_z + u = 0 \\ u_x - 2v_x + 2u_z + 4u + 6y = 0\ . \end{cases}$

8.21. $\begin{cases} u_y + 2v_y - v_z + xy = 0 \\ u_x - v_x + 2u_z + v_z = 0\ . \end{cases}$

8.22. $\begin{cases} u_x - u_y + v_x - 2v_z + 5ue^{x+y} = 0 \\ u_x + v_y + 2u_z + v_z + 3u = 0\ . \end{cases}$

8.23. $\begin{cases} 2u_x - u_y + v_x + u = 0 \\ v_y + 2u_z + v_z + 2ue^x = 0\ . \end{cases}$

8.24. $\begin{cases} u_x - u_y + v_x - 2u_z + u + e^y = 0 \\ u_x + v_y + 2u_y + u_z + 3u = 0\ . \end{cases}$

8.25. $\begin{cases} u_x + v_x - u_z + u + e^{xy} = 0 \\ u_x + v_x + 2u_y + v_z - ue^{xy} = 0\ . \end{cases}$

5 - §. IKKI O`ZGARUVCHILI IKKINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING UMUMIY YECHIMINI TOPISHGA DOIR MISOLLAR

Bu paragrafda ikki o`zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimini topishga doir misollarni qaraymiz.

1–misol. $u_{xy} = 0$ tenglamaning umumiy yechimini toping.

Yechish: $u_x = v$ deb belgilash kiritamiz. U holda $v_y = 0$ tenglamaga ega bo'lamiz. Ushbu tenglamani yechish uchun uni integrallaymiz va $v = C(x)$ tenglikka ega bo'lamiz. Topilgan ifodani kiritilgan belgilashga olib, borib quyib, $u_x = C(x)$ tenglamaga ega bo'lamiz. Bu tenglamani integrallab $u(x,y) = f(x) + g(y)$ umumiy yechimga ega bo'lamiz, bu yerda $f(x)$ va $g(x)$ funksiyalar ixtiyoriy differesiallanuvchi funksiyalardir.

2–misol. $u_{xx} - 2u_{xy} - 3u_{yy} = 0$ tenglamaning umumiy yechimini toping.

Yechish:

$$A = 1,\ B = -1,\ C = -3\ , \quad \Delta = B^2 - AC = 1 - 1\cdot(-3) = 4 > 0$$

yuqoridagi tenglama giperbolik tipdagi tenglama ekan. Endi esa xarakteristik temglamasini tuzamiz va uni yechamiz:

$$y'^2 + 2y' - 3 = 0, \quad y' = \frac{-2 \pm \sqrt{4+12}}{2} = -1 \pm 2, \ y' = -3, \quad y' = 1$$

$$y = -3x - C_1, \quad y = x + C_2, \quad C_1 = 3x + y, \quad C_2 = x - y,$$

bu erda $C_1,\ C_2$ o'zgarmaslarni mos ravishda ξ *va* η lar bilan almashtiramiz, ya`ni

$$\begin{cases} \xi = 3x + y \\ \eta = x - y\ . \end{cases}$$

U holda u funksiyani murakkab funksiya deb qarab ξ *va* η o`zgaruvchilar $x$ *va* $y$ o`zgaruvchilarning chiziqli funksiyalari ekanligini hisobga olib birinchi va ikkinchi tartibli xususiy hosilalarni hisoblaymiz:

$$u_x = 3u_\xi + u_\eta \ ,$$

$$u_y = u_\xi - u_\eta \ ,$$

$$u_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta} \ ,$$

$$u_{xy} = 3u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} \ ,$$

$$u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \ .$$

Topilgan ifodalarni $u_{xx} - 2u_{xy} - 3u_{yy} = 0$ tenglamaga olib borib quyamiz. Natijada

$$9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta} - 2\cdot\left(3u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}\right) - 3\cdot\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) = 0 \ ,$$

yoki $16u_{\xi\eta} = 0$,

$$u_{\xi\eta} = 0$$

tenglamaga ega bo’lamiz. Ushbu tenglamaning umumiy yechimi yuqoridagi 1-misolga asosan quyidagicha bo’ladi:

$$u(\xi,\eta) = f(\xi) + g(\eta).$$

Bu erda ξ *va* η o`zgaruvchilar o`rniga ularning x *va* y o`zgaruvchilar orqali ifodalarini quyib,

$$u(x, y) = f(3x + y) + g(x - y)$$

umumiy yechimga ega bo’lamiz.

3-misol. $u_{xy} - 2u_x = 0$ tenglamaning umumiy yechimini toping.

Yechish: $u_x = v$ deb belgilash kiritamiz. U holda $v_y - 2v = 0$ tenglamaga ega bo'lamiz. Ushbu chiziqli tenglamani yechamiz va $v = C(x)\, e^{2y}$ tenglikka ega bo'lamiz. Topilgan ifodani kiritilgan belgilashga olib borib quyib, $u_x = C(x)e^{2y}$ tenglamaga ega bo'lamiz. Bu chiziqli bir jinsli tenglamani yechsak, $u(x,y) = f(x)e^{2y} + g(y)$ umumiy yechimni hosil qilamiz, bu yerda $f(x)$ va $g(x)$ funksiyalar ixtiyoriy differesiallanuvchi funksiyalardir.

4-misol. $u_{xy} - 2u_x - 3u_y + 6u = 2e^{x+y}$ tenglamaning umumiy yechimini toping.

Yechish: Ushbu tenglamani yechish uchun avval tenglamadagi birinchi tartibli xususiy xosilalarni yo`qotamiz. Buning uchun $u(x,y) = v(x,y) \cdot e^{\lambda x + \mu y}$ almashtirish bajaramiz, bu yerdagi $\lambda$ va $\mu$ o`zgarmaslarni keyinchalik tanlaymiz. Birinchi va ikkinchi tartibli xususiy hosilalarni hisoblaymiz:

$$u_x = v_x \cdot e^{\lambda x + \mu y} + v \cdot \lambda e^{\lambda x + \mu y}$$

$$u_y = v_y \cdot e^{\lambda x + \mu y} + v \cdot \mu e^{\lambda x + \mu y}$$

$$u_{xy} = v_{xy} \cdot e^{\lambda x + \mu y} + v_x \cdot \mu e^{\lambda x + \mu y} + v_y \cdot \lambda e^{\lambda x + \mu y} + v \cdot \lambda \mu e^{\lambda x + \mu y}$$

Topilgan ifodalarni $u_{xy} - 2u_x - 3u_y + 6u = 2e^{x+y}$ tenglamaga olib borib quyamiz. Natijada

$$v_{xy}\cdot e^{\lambda x+\mu y}+v_x\cdot \mu e^{\lambda x+\mu y}+v_y\cdot \lambda e^{\lambda x+\mu y}+v\cdot \lambda\mu e^{\lambda x+\mu y}-$$

$$-2\cdot\left(v_x\cdot e^{\lambda x+\mu y}+v\cdot \lambda e^{\lambda x+\mu y}\right)-3\cdot\left(v_y\cdot e^{\lambda x+\mu y}+v\cdot \mu e^{\lambda x+\mu y}\right)+$$

$$+6v\cdot e^{\lambda x+\mu y}=2e^{x+y},$$

yoki

$$v_{xy}\cdot e^{\lambda x+\mu y}+(\mu-2)v_x\cdot e^{\lambda x+\mu y}+(\lambda-3)v_y\cdot e^{\lambda x+\mu y}+$$

$$+(\lambda\mu-2\lambda-3\mu+6)v\cdot e^{\lambda x+\mu y}=2e^{x+y}$$

tenglamaga ega bo’lamiz. Ushbu tenglikda λ va μ o`zgarmaslarni shunday tanlaymizki, oxirgi tenglikda birinchi tartibli xususiy hosilalar qatnashmasin. Buning uchun $\lambda=3,\ \mu=2$ deb tanlaymiz va quyidagi tenglamaga ega bo’lamiz:

$$e^{3x+2y}\cdot v_{xy}=2e^{x+y},$$

ya`ni

$$v_{xy}=2e^{-2x-y}.$$

Bu erda $v_x=\varphi$ deb belgilash kiritamiz. U holda ushbu

$$\varphi_y=2e^{-2x-y}$$

chiziqli tenglamaga ega bo’lamiz va uni yechamiz. Natijada

$$\varphi=-2e^{-2x-y}+C(x),$$

ya'ni

$$v_x=-2e^{-2x-y}+C(x)$$

chiziqli tenglamaga ega bo’lamiz. Bu tenglamani integrallab,

$$v = e^{-2x-y} + f(x) + g(y)$$

ekanligini hosil qilamiz.

$$u(x,y) = v(x,y) \cdot e^{3x+2y}$$

almashtirishga asosan,

$$u(x,y) = e^{x+y} + \left[f(x) + g(y) \right] \cdot e^{3x+2y}$$

umumiy yechimni hosil qilamiz, bu yerda $f(x)$ va $g(x)$ funksiyalar ixtiyoriy differesiallanuvchi funksiyalardir.

Mustaqil yechish uchun misollar

Quyida berilgan tenglamalarning umumiy yechimini toping:

9.1. $u_{xy} - 4u_x + 3u_y - 12u = 0$.

9.2. $u_{xx} - 4u_{xy} - 5u_{yy} = 0$.

9.3. $u_{xy} - \sin y \cdot u_x = 0$.

9.4. $u_{xy} - 2u_x - 5u_y + 10u = 2e^{3x+2y}$.

9.5. $3u_{xx} - 5u_{xy} - 2u_{yy} + 3u_x + u_y = 2$.

9.6. $u_{xy} + au_x + bu_y + abu = 0$.

9.7. $u_{xx} - a^2 u_{yy} = 0$.

9.8. $u_{xx} + 2au_{xy} + a^2 u_{yy} + u_x + au_y = 0$.

9.9. $u_{xx} - 4u_{xy} + 4u_{yy} + u_x - 2u_y = 0$.

9.10. $u_{xx} - 9u_{yy} + 6u_x = 0$.

9.11. $u_{xy} - 3u_x - 4u_y = 2e^{3x+2y}$.

9.12. $u_{xx} - x \cdot u_x = 0$.

9.13. $u_{xx} - u_{xy} - 6u_{yy} + 2u_x + 3u_y = e^{2x-y}$.

9.14. $u_{xx} - 2u_{xy} - 8u_{yy} + 2u_x = e^{3x-2y}$.

9.15. $u_{xy} - y \cdot u_x = 0$.

9.16. $u_{xx} - 16u_{yy} + 6u_y = 0$.

9.17. $u_{xx} - u_{yy} + 6u_x + 4u_y = 0$.

9.18. $u_{xy} - \frac{1}{\cos^2 y} \cdot u_x = 0$.

9.19. $u_{xx} - 6u_{xy} + 5u_{yy} + 2u_x - 4u_y = e^{x+2y}$.

9.20. $u_{xy} - 5u_x - 3u_y + 15u = 2e^{x+3y}$.

9.21. $u_{yy} - 6 \cdot u_y = 0$.

9.22. $u_{xx} - 5u_{xy} - 6u_{yy} = e^{3x-4y}$.

9.23. $u_{xx} - 4u_{yy} + 2u_x - 4u_y = e^{2x+y}$.

9.24. $u_{xy} - 2u_x - 3u_y + 6u = 2e^{3x+4y} \sin x$.

9.25. $u_{xy} - 3u_x - 4u_y + 12u = 2e^{4x+y} \cos x$.

6 - §. LAPLASNING KASKAD USULI

Quyidagi

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \qquad (1)$$

tenglamani qaraylik, bu yerda a, b, c koeffitsentlar va f oldindan berilgan, hamda x va y o`zgaruvchilarga bog`liq funksiyalardir.

Agar bu a, b, c koeffitsentlar uchun

$$h \equiv \frac{\partial a}{\partial x} + ab - c \equiv 0 \qquad (2)$$

ayniyat o’rinli bo’lsa, (1) tenglamani quyidagi ko’rinishda yozish mumkin:

$$\frac{\partial v}{\partial x} + bv = f \quad , \qquad (3)$$

bu yerda

$$v = \frac{\partial u}{\partial y} + au \qquad (4)$$

bo`ladi. Bundan, esa berilgan xususiy hosilali differensial tenglamaning umumiy yechimini

$$u = e^{-\int ady}\left\{X + \int\left\{Y + \int f \cdot e^{\int bdx}dx\right\}e^{\int ady - \int bdx}dy\right\} \qquad (5)$$

shaklida hosil qilamiz, bu yerda X va Y – ixtiyoriy funksiyalar bo’lib, mos ravishda x va y ga bog’liq. Xuddi shunga o’xshash, agar

$$k \equiv \frac{\partial b}{\partial y} + ab - c \equiv 0 \qquad (6)$$

ayniyat o’rinli bo’lsa, (1) tenglamani quyidagi ko’rinishda yozish mumkin:

$$\frac{\partial v}{\partial y} + av = f \quad , \tag{7}$$

bu yerda

$$v = \frac{\partial u}{\partial x} + bu \tag{8}$$

bo`ladi. Bundan, esa berilgan xususiy hosilali differensial tenglamaning umumiy yechimini

$$u = e^{-\int bdx}\left\{Y + \int\left\{X + \int f \cdot e^{\int ady} dy\right\} e^{\int bdx - \int ady} dx\right\} \tag{9}$$

shaklida hosil qilamiz, bu yerda X va Y – ixtiyoriy funksiyalar bo’lib, mos ravishda x va y ga bog’liq.

$h \neq 0$ bo’lgan holda (1) tenglamaga o’xshash quyidagi tenglama qaraladi:

$$L_1 u_1 = \frac{\partial^2 u_1}{\partial x \partial y} + a_1 \frac{\partial u_1}{\partial x} + b_1 \frac{\partial u_1}{\partial y} + c_1 u_1 = f_1 \quad , \tag{10}$$

bu yerda

$$a_1 = a - \frac{\partial \ln h}{\partial y}, \quad b_1 = b,$$

$$c_1 = c - \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - b\frac{\partial \ln h}{\partial y}, \quad f_1 = f \cdot \left(a - \frac{\partial \ln h}{\partial y}\right). \tag{11}$$

Agar u_1 funksiyani topish mumkin bo'lsa, u holda qaralayotgan (1) tenglamaning yechimi quyidagi formula bilan topiladi:

$$u = \frac{\frac{\partial u_1}{\partial x} + bu_1 - f}{h} \quad . \tag{12}$$

(10) tenglama uchun

$$h_1 = 2h - k - \frac{\partial^2 \ln h}{\partial x \partial y}, \quad k_1 = h \tag{13}$$

bo`ladi. Agar $h_1 = 0$ bo'lsa, u holda u_1 funksiya yuqorida ifodalangan usul bo'yicha hosil qolinadi. Agar $h_1 \neq 0$ bo'lsa, u holda yuqoridagi jarayonni davom ettiramiz va yuqoridagiga o'xshash $L_2 u_2 = f_2$ tenglamani hosil qilamiz va hokazo.

$k \neq 0$ bo'lgan holda yuqoridagiga o'xshash quyidagi tenglamalar zanjirini hosil qilish mumkin: $L_{-1} u_{-1} = f_{-1}$, $L_{-2} u_{-2} = f_{-2}$ va hokazo.

Agar qandaydir h_i (yoki k_i) lar $i-$ qadamda nolga aylansa, (1) tenglamaning umumiy yechimini topish mumkin bo`ladi.

1-misol. Yuqoridagi usul yordamida

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\beta'}{x-y}\frac{\partial u}{\partial x} + \frac{\beta}{x-y}\frac{\partial u}{\partial y} = 0 \tag{14}$$

Eyler–Darbu tenglamasini β yoki β' koeffitsentlardan birortasi butun son bo'lsa, u holda bu tenglamani yechamiz, bu erda

$$a(x,y) = -\frac{\beta'}{x-y}, \quad b(x,y) = \frac{\beta}{x-y}, \quad c(x,y) = 0, \quad f(x,y) = 0 \ .$$

Endi (2) formula bilan aniqlangan h ni hisoblaymiz va quyidagi tenglikni hosil qilamiz:

$$h = \frac{\beta'(1-\beta)}{(x-y)^2}.$$

Agar $\beta = 1$ bo’lsa, $h = 0$ bo`ladi. Agar $h \neq 0$ bo’lsa, (11) ga ko’ra quyidagilarni topamiz:

$$a_1 = -\frac{2+\beta'}{x-y}, \quad b_1 = \frac{\beta}{x-y}, \ c_1 = -\frac{\beta'+\beta}{(x-y)^2}, \qquad f_1 = 0 \ .$$

Bundan

$$h_1 = \frac{(1+\beta')(2-\beta)}{(x-y)^2}$$

bo`ladi. Agar $\beta = 2$ bo’lsa, u holda $h_1 = 0$ bo`ladi va hokazo. Umuman olganda,

$$E(\alpha,\beta) \equiv \frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0 \qquad (15)$$

Eyler–Darbu tenglamasining umumiy yechimini ixtiyoriy α va β haqiqiy sonlar uchun ham hosil qilish mumkin. Biz bu erda α va β natural sonlar bo`lgan holda umumiy yechimni topamiz. Shu maqsadda (15) tenglamani

$$(x-y)\frac{\partial^2 u}{\partial x \partial y} - \beta\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial y} = 0 \qquad (16)$$

shaklida yozamiz. (16) tenglamani x bo`yicha differensiallab,

$$(x-y)\frac{\partial^3 u}{\partial x^2 \partial y}-\beta\frac{\partial^2 u}{\partial x^2}+(1+\alpha)\frac{\partial^2 u}{\partial x\partial y}=0,$$

yoki

$$(x-y)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}\right)-\beta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+(1+\alpha)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=0$$

tenglamani hosil qilamiz. Bundan ko`rinadiki, $\frac{\partial u}{\partial x}$ funksiya

$$E(1+\alpha,\beta)=0$$

tenglamani qanoatlantiradi. Shunga asosan, $E(\alpha,\beta)=0$ tenglamaning $Z(\alpha,\beta)$ ixtiyoriy yechimi uchun

$$\frac{\partial Z(\alpha,\beta)}{\partial x}=Z(1+\alpha,\beta)$$

ekanligini hosil qilamiz.

Xuddi shunga o`xshash, (16) tenglamani $y$ bo`yicha differensiallab,

$$\frac{\partial Z(\alpha,\beta)}{\partial y}=Z(\alpha,1+\beta)$$

ekanligini hosil qilamiz. Umuman olganda

$$Z(\alpha+m-1,\ \beta+n-1)=\frac{\partial^{m+n-2}Z(\alpha,\beta)}{\partial x^{m-1}\partial y^{n-1}} \qquad (17)$$

ekanligi kelib chiqadi. (17) fo`rmulada $\alpha=\beta=1$ deb olsak,

$$Z(m,\ n) = \frac{\partial^{m+n-2} Z(1,1)}{\partial x^{m-1} \partial y^{n-1}} \tag{18}$$

tenglikka ega bo`lamiz, bu erda $Z(1,1)$ funksiya

$$E(1,1) \equiv \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{x-y}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

tenglamaning umumiy yechimi bo`lib, bu yechimning ko`rinishi

$$Z(1,1) = \frac{\Phi(x) - \Psi(y)}{x-y}$$

shakldа bo`ladi, bu erda $\Phi(x)$ va $\Psi(y)$ – ixtiyoriy funksiyalar. Shunga ko`ra,

$$E(m,n) \equiv \frac{\partial^2 u}{\partial x \partial y} - \frac{n}{x-y}\frac{\partial u}{\partial x} + \frac{m}{x-y}\frac{\partial u}{\partial y} = 0 \tag{19}$$

tenglamaning umumiy yechimi

$$u(x,y) = \frac{\partial^{m+n-2}}{\partial x^{m-1} \partial y^{n-1}}\left[\frac{\Phi(x) - \Psi(y)}{x-y}\right] \tag{20}$$

fo`rmula bilan beriladi, bu erda $\Phi(x)$ va $\Psi(y)$ – ixtiyoriy funksiyalar.

Xuddi shunga o`xshash,

$$E(-m,-n) \equiv \frac{\partial^2 u}{\partial x \partial y} + \frac{n}{x-y}\frac{\partial u}{\partial x} - \frac{m}{x-y}\frac{\partial u}{\partial y} = 0 \tag{21}$$

tenglamaning umumiy yechimi

$$u(x,y)=Z(-m,-n)=(x-y)^{m+n+1}\frac{\partial^{m+n}}{\partial x^n \partial y^m}\left[\frac{\Phi(x)-\Psi(y)}{x-y}\right]$$

fo`rmula bilan beriladi, bu erda $\Phi(x)$ va $\Psi(y)$ – ixtiyoriy funksiyalardir.

2-misol. $u_{xy}+2xyu_y-2xu=0$ tenglamaning umumiy yechimini toping.

Yechish. Tenglamani yechish uchun $u_y=v$ almashtirish bajaramiz va

$$u=\frac{1}{2x}v_x+yv \tag{22}$$

tenglikka ega bo'amiz. Oxirgi tenglikning ikki tomonini y bo'yicha differensiallaymiz va quyidagi tenglikni hosil qilamiz:

$$u_y=\frac{1}{2x}v_{xy}+v+yv_y.$$

$u_y=v$ dan foydalansak, $v=\frac{1}{2x}v_{xy}+v+yv_y$ tenglikni, bundan esa, o'z navbatida $v_{xy}+2xyv_y=0$ tenglamani hosil qilamiz. Oxirgi tenglamada $v_y=w$ almashtirish bajarsak, $w_x+2xyw=0$ tenglamaga ega bo`lamiz. Uni yechib, $w=f(y)e^{-x^2y}$ tenglikka ega bo'lamiz. Olingan ifodani o'rniga qo'yib, $v_y=f(y)e^{-x^2y}$ tenglikni hosil qilamiz. Uni integrallasak, $v=\int_0^y f(\xi)e^{-x^2\xi}d\xi+g(x)$ ni hosil qilamiz. $v_x = -2x\int_0^y f(\xi)\xi e^{-x^2\xi} d\xi + g'(x)$ ekanligini hisobga olib, topilgan ifodalarni (22) ga olib borib qo'yamiz. Natijada

$$u = \frac{1}{2x} v_x + yv = \frac{1}{2x}\left[-2x\int_0^y f(\xi)\xi e^{-x^2\xi} d\xi + g'(x) \right] +$$

$$+ y\left[\int_0^y f(\xi) e^{-x^2\xi} d\xi + g(x) \right] = yg(x) + \frac{1}{2x} g'(x) +$$

$$+ \int_0^y (y-\xi) f(\xi) e^{-x^2\xi} d\xi$$

hosil bo`ladi. Shunday qilib,

$$u = yg(x) + \frac{1}{2x} g'(x) + \int_0^y (y-\xi) f(\xi) e^{-x^2\xi} d\xi$$

umumiy yechimga ega bo`lamiz.

Mustaqil yechish uchun misollar

Quyida berilgan tenglamalarning umumiy yechimini toping:

10.1. $yu_{xx} + (x-y)u_{xy} - xu_{yy} = 0$.

10.2. $x^2 u_{xx} - y^2 u_{yy} = 0$.

10.3. $x^2u_{xx} + 2xyu_{xy} - 3y^2u_{yy} - 2xu_x = 0$.

10.4. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.

10.5. $u_{xy} - xu_x + u = 0$.

10.6. $u_{xy} + yu_x + xu_y + xyu = 0$.

10.7. $u_{xy} + u_x + yu_y + (y-1)u = 0$.

10.8. $u_{xy} + xu_x + 2yu_y + 2xyu = 0$.

10.9. $u_{xx} - 2\sin xu_{xy} - \cos^2 xu_{yy} - \cos xu_y = 0$.

10.10. $u_{xy} + xu_x + 2yu_y + (2xy+1)u = 0$.

10.11. $u_{xy} + u_x + yu_y + (y+1)u = 0$.

10.12. $u_{xy} + xu_x + yu_y + (xy+1)u = 0$.

10.13. $u_{xy} + xu_x + u = 0$.

10.14. $u_{xy} - xu_x + xu_y - (1+x^2)u = 0$.

10.15. $u_{xy} - xu_x + u_y - xu = 0$.

10.16. $u_{xy} + xu_x + 2yu_y + (2xy+2)u = 0$.

10.17. $u_{xy} + xu_x + yu_y + (xy+1)u = 0$.

10.18. $u_{xy} + xyu_x + 2xyu_y + (2x + 2x^2y^2)u = 0$.

10.19. $u_{xy} + u_x + 8xyu_y + (8x + 8xy)u = 0$.

10.20. $u_{xy} + u_x + 8xy^2u_y + (16xy + 8xy^2)u = 0$.

10.21. $u_{xy} + xu_x + yu_y + (xy+1)u = 0$.

10.22. $u_{xy} + x^3 u_x + y^3 u_y + (3x^2 + x^3 y^3)u = 0$.

10.23. $u_{xy} + yu_x + x^4 u_y + x^4 yu = 0$.

10.24. $u_{xy} + 12u_x + xyu_y + 12xyu = 0$.

10.25. $u_{xy} + \sin x u_x + \sin y u_y + (\cos x + \sin x \sin y)u = 0$.

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