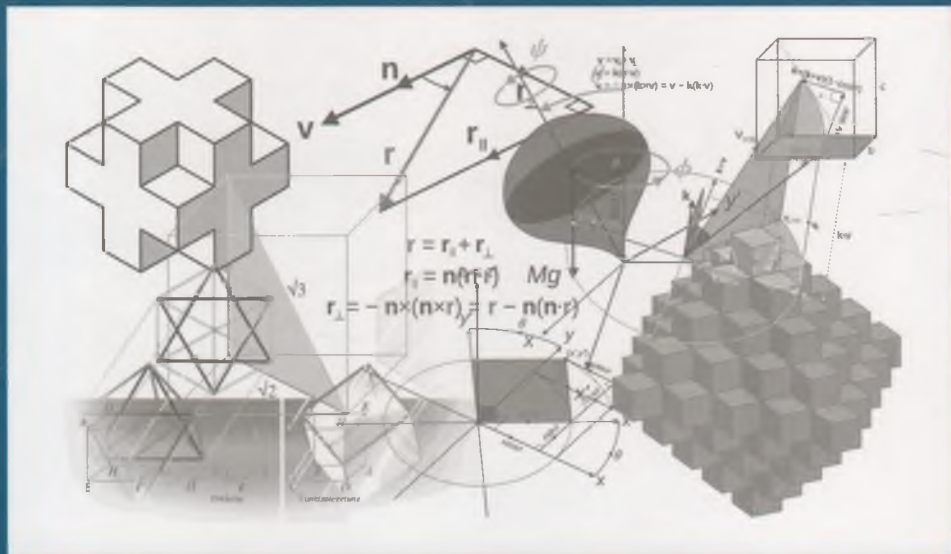


22.1a7  
x-92

# I QISM

# OLIV MATEMATIKA



4  
m



22.107  
x-92

O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS  
TA'LIM VAZIRLIGI  
TOSHKENT ARXITEKTURA QURILISH INSTITUTI

---

SH.R. XURRAMOV

# OLIV MATEMATIKA

*Uch jildlik*

I jild

*O'zbekiston Respublikasi Oliy va o'rtta maxsus ta'lim  
vazirligi barcha texnika yo'nalishlari uchun darslik  
sifatida tavsiya etgan*

*Cho'pon nomidagi nashriyot-matbaa ijodiy uyi  
Toshkent — 2018*

UDK 517(075)  
BBK 22.1a7  
X 92

**Mas'ul muharrirlar:**

*A. Quchqarov* – fizika-matematika fanlari doktori;  
*Sh.A. Rahimova*

**Taqrizchilar:**

*B. Shoimqulov* – fizika-matematika fanlari doktori, QarDU professori;  
*A. Narmanov* – fizika-matematika fanlari doktori, O'zMU professori;  
*Sh. Ismoilov* – fizika-matematika fanlari nomzodi, TAQI dotsenti

**Xurramov, Sh.R.**

X 92 Oliy matematika. Barcha texnika yo'nalishlari uchun darslik.  
I jild. [Matn] darslik/Sh. Xurramov. Oliy va o'rta maxsus ta'lim  
vazirligi. – T.: Cho'lpon nomidagi NMIU, 2018. – 492 b.  
ISBN 978-9943-5379-7-2

Ushbu darslik oliy texnika o'quv yurtlarining oliy matematika fani dasturi asosida yozilgan va bakalavrlar Davlat ta'lim standartlari talablariga mos keladi.

Darslik uch jilddan iborat. Birinchi jild oliy matematikaning chiziqli algebra elementlari, vektorli algebra elementlari, analitik geometriya, matematik analizga kirish, bir o'zgaruvchi funksiyasining differensial hisobi, oliy algebra elementlari va bir o'zgaruvchi funksiyasining integral hisobi bo'limlariga oid materillarni o'z ichiga oladi. Darslikning har bir mavzusi zamonaviy xorijiy adabiyotlar va o'qitish texnologiyalari tahlili asosida yozilgan.

Kitob oliy ta'lim muassasalarining talabalari va o'qituvchilari uchun mo'ljallangan.

UDK 517(075)  
BBK 22.1ya7

ISBN 978-9943-5379-7-2

© Sh. R. Xurramov, 2018  
© Cho'lpon nomidagi NMIU, 2018



# SO‘Z BOSHI

Matematika barcha tabiiy bilimlar asosidir.

David Gilberd

Hozirgi jadal rivojlanish davrida aql-zakovatli, ijodiy fikrlovchi va mustaqil qaror qabul qiluvchi mutaxassislarni tayyorlashda matematik ta'lim asosiy o'rin egallaydi. Talabalarni matematik tayyorlash ularning kasbiy faoliyatida zarur bo'ladigan boshqa tabiiy-ilmiy, umumkasbiy va ixtisoslik fanlarini o'rganishlari uchun nazariy asoslarni ta'minlashi kerak. Bu esa o'z navbatida samarali o'qitishning muhim omillaridan biri bo'lgan zamon talablariga javob beruvchi darsliklar va o'quv qo'llanmalarni yaratishni taqozo etmoqda.

Ushbu darslik oliy texnika o'quv yurtlarining oliy matematika fani dasturi asosida yozilgan va bakalavrlar Davlat ta'lim standartlari talablariga mos keladi.

Darslik uch jildan iborat. Darslikning birinchi jildi yettita bobdan iborat bo'lib, u oliy matematikaning chiziqli algebra elementlari, vektorli algebra elementlari, analitik geometriya, matematik analizga kirish, bir o'zgaruvchi funksiyasining differensial hisobi, oliy algebra elementlari va bir o'zgaruvchi funksiyasining integral hisobi bo'limlariga bag'ishlangan. Bu bo'limlar zamonaviy xorijiy adabiyotlar va o'qitish texnologiyalari tahlili asosida yaratilgan bo'lib, har bir mavzuni yozishda bir qancha xorijiy adabiyotlardan foydalanilgan, mavzular to'liq yoritilgan, tegishli bilimlar talabalar tomonidan mustaqil o'zlashtirilishiga, ularda ko'nikma va malakalarning shakllantirilishiga hamda ijodiy qobiliyatlarni rivojlantirishga yo'naltirgan.

Darslikda talaba asosiy tushunchalarni, ta'riflarni, teoremlarni va tipik masalalarni yechish usullarini ko'rsatuvchi misollarni topadi. Bunda biror tasdiqning isboti keltirilmagan bo'lsa, natijalarning ifodasi uning ma'nosini tushuntiradigan misollar bilan to'ldirilgan.

Darslikning har bir mavzusi ko'p sondagi misol va masalalar yechimlarida tushuntirilgan, ularni o'zlashtirishni mustahkamlashga yo'naltirilgan mashqlar bilan to'ldirilgan. Ayrim misol va masalalarni matematik paketlar yordamida yechish usullari keltirilgan.

№ 403 220

Muallif darslik qo'lyozmasini o'qib, uning sifatini oshirish borasida bildirgan fikr va mulohazalari uchun Qarshi Davlat universitetining professori B.A.Shoimqulovga, O'zbekiston Milliy universitetining o'qituvchilari – professor A.Narmanovga, dotsentlar N.Jabborov va J.Tishaboyevlarga o'z minnatdorchiligini izhor qiladi.

Ayniqsa, darslikning ushbu jildini tuzishda va mazmunini yaxshilashda fizika-matematika fanlari doktori A. Quchqarovning beminnat yordamini muallif e'tirof etishni o'zining burchi deb biladi.

Darslik haqidagi tanqidiy fikr va mulohazalarini bildirgan barcha kitobxonlarga muallif oldindan o'z tashakkurini bildiradi.

# CHIZIQLI ALGEBRA ELEMENTLARI

- Matritsalar
- Determinantlar
- Matritsa ustida almashtirishlar
- Chiziqli tenglamalar sistemasi



*Al-Xorazmiy – Muhammad ibn Musa Xorazmiy (783–850) – xorazmlik matematik, astronom va geograf.*

*Al-Xorazmiy algebra faniga asos soldi, ilmiy ma'lumot va traktatlarini bayon etishning aniq qoidalarini ishlab chiqardi. U o'rtalik pozitsion hisoblash tizimini, nol belgisini va qutblar koordinatalarini birinchilardan bo'lib asoslab berdi va amaliyotga tatbiq etdi.*

*I. Karimov*

Chiziqli algebraning daslabki masalasi chiziqli tenglamalar haqidagi masala hisoblanadi. Bunday tenglamalarni yechish jarayonida determinant tushunchasi paydo bo'ldi.

Chiziqli tenglamalar sistemasi va ularning determinantlarini o'rganish natijasida matritsa tushunchasi kiritildi. G.Frobennus tomonidan matritsaning rangi tushunchasi kiritilishi chiziqli tenglamalar sistemasining birgalikda va aniq bo'lishi shartlarini olish imkonini berdi. Shu zaylda XIX asrning oxirlariga kelib, chiziqli tenglamalar sistemasi nazariyasini barpo qilish jarayoni tugatildi.

## 1.1.MATRITSALAR

Matritsa tushunchasi 1850-yilda James Joseph Sylvester tomonidan kiritilgan. Keling 1858-yilda chop etilgan «Matritsalar nazariyasi haqida memuar» asarida matritsalar nazariyasi mufassal bayon qilingan.

Daslabki vaqtlarda matritsa geometrik obyektlarni almashtirish va chiziqli tenglamalarni yechish bilan bog'liq holda rivojlantirildi. Hozirgi vaqtda matritsalar matematikaning muhim tatbiiy vositalaridan biri hisoblanadi.

Matritsalar matematika, texnika va iqtisodiyotning turli sohalarida keng qo'llaniladi. Masalan, ulardan matematikada algebraik va differensial tenglamalar sistemasini yechishda, kvant nazariyasida fizik kattaliklarni oldindan aytishda, aviatsiyada zamonaviy samolyotlarni yaratishda foydalaniladi.

### 1.1.1. Matritsa va uning turlari

Matritsalar sonlar, algebraik belgilar va matematik funksiyalarning katta massivlarini yagona obyekt sifatida qarash va bunday massivlarni o'z ichiga olgan masalalarni qisqa ko'rinishda yozish va yechish imkonini beradi.

**Matritsa** – bu elementlar (sonlar, algebraik belgilar, matematik funksiyalar) massivining satr hamda ustunlarda berilgan va kichik qavslarga olingan to'g'ri burchakli jadvalidir.

Matritsaning o'lchami uning satrlari soni va ustunlari soni bilan aniqlanadi. Matritsaning o'lchamini ifodalash uchun  $m \times n$  belgi ishlatiladi. Bu belgi matritsaning  $m$  ta satr va  $n$  ta ustundan tashkil topganini bildiradi.

Matritsa lotin alifbosining bosh harflaridan biri bilan belgilanadi.

Masalan,

$3 \times 2$ o'lchamli matritsa	$2 \times 3$ o'lchamli matritsa	$2 \times 2$ o'lchamli matritsa
$A = \begin{pmatrix} 2 & 5 \\ 0 & 7 \\ 3 & 1 \end{pmatrix}$	$B = \begin{pmatrix} -1 & 4 & 7 \\ 2 & 5 & 6 \end{pmatrix}$	$C = \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$

$A$  matritsaning  $i$ -satr va  $j$ -ustunda joylashgan elementi  $a_{ij}$  bilan belgilanadi.

$A = (a_{ij})$ , ( $i = \overline{1, m}, j = \overline{1, n}$ ) yoki  $A = \|a_{ij}\|$ , ( $i = \overline{1, m}, j = \overline{1, n}$ ) yozuv  $A$  matritsa  $a_{ij}$  elementlardan tashkil topganini bildiradi:

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}; \quad A = \|a_{ij}\| = \left\| \begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right\|$$

$1 \times n$  o'lchamli  $A = (a_{11} \ a_{12} \ \dots \ a_{1n})$  matritsaga *satr matritsa* yoki *satr-vektor* deyiladi.

$m \times 1$  o'lchamli  $A = \begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix}$  matritsaga *ustun matritsa* yoki *ustun-vektor*

deyiladi.

$n \times n$  o'lchamli matritsaga  $n$ -*tartibli kvadrat matritsa* deyiladi.

Kvadrat matritsaning chap yuqori burchagidan o'ng quyi burchagiga yo'nalgan  $a_{11}, a_{22}, \dots, a_{nn}$  elementlaridan tuzilgan diagonaliga uning *bosh diagonali*, o'ng yuqori burchagidan chap quyi burchagiga yo'nalgan  $a_{1n}, a_{2(n-1)}, \dots, a_{n1}$  elementlardan tuzilgan diagonaliga uning *yordamchi diagonal* deyiladi.

Bosh diagonalidan yuqorida (pastda) joylashgan barcha elementlari nolga teng bo'lgan

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \quad \left( A' = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \right)$$

matritsaga *yuqoridan uchburchak (quyidan uchburchak) matritsa* deyiladi.

Bosh diagonalda joylashmagan barcha elementlari nolga teng bo'lgan

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

matritsaga *diagonal matritsa* deyiladi.

Barcha elementlari birga teng bo'lgan diagonal matritsaga *birlik matritsa* deyiladi va  $I$  (yoki  $E$ ) harfi bilan belgilanadi.

Barcha elementlari nolga teng bo'lgan ixtiyoriy o'lchamdagi matritsaga *nol matritsa* deyiladi va  $O$  harfi bilan belgilanadi.

$A$  matritsada barcha satrlarni mos ustunlar bilan almashtirish natijasida hosil qilingan  $A^T$  matritsaga  $A$  matritsaning *transponirlangan matritsasi* deyiladi:  $(a_j)^T = (a_{ji})$ .

Agar  $A = A^T$  bo'lsa,  $A$  matritsaga *simmetrik matritsa* deyiladi.

### 1.1.2. Matritsalar ustida arifmetik amallar

#### *Matritsalarining tengligi*

Bir xil o'lchamli  $A = (a_{ij})$  va  $B = (b_{ij})$  matritsalarining barcha mos elementlari teng, ya'ni  $a_{ij} = b_{ij}$  bo'lsa, bu matritsalar *teng matritsalar* deyiladi va  $A = B$  deb yoziladi.

$$\boxed{A = B \Leftrightarrow a_{ij} = b_{ij} \text{ barcha } i = \overline{1, m}, j = \overline{1, n} \text{ uchun}}$$

#### *Matritsani songa ko'paytirish*

**1-ta'rif.**  $A = (a_{ij})$  matritsaning  $\lambda$  songa ko'paytmasi deb, elementlari  $c_{ij} = \lambda a_{ij}$  kabi aniqlanadigan  $C = \lambda A$  matritsaga aytiladi.

$$\boxed{C = \lambda A \Leftrightarrow c_{ij} = \lambda a_{ij}}$$

*1-misol.*  $A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & -1 \end{pmatrix}$  bo'lsin.  $3A$  ni toping.

*Yechish.*

$$3A = 3 \cdot \begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2 & 3 \cdot (-1) & 3 \cdot 0 \\ 3 \cdot 3 & 3 \cdot 4 & 3 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 6 & -3 & 0 \\ 9 & 12 & -3 \end{pmatrix}$$

#### *Matritsalarini qo'shish va ayirish*

Matritsalarini qo'shish va ayirish amallari *bir xil o'lchamli matritsalar* uchun kiritiladi. Bunda yig'indi matritsa qo'shiluvchi matritsalar bilan bir xil o'lchamga ega bo'ladi.

**2-ta'rif.**  $A = (a_{ij})$  va  $B = (b_{ij})$  matritsalarining yig'indisi deb, elementlari  $c_{ij} = a_{ij} + b_{ij}$  kabi aniqlanadigan  $C = A + B$  matritsaga aytiladi.

$$\boxed{C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij}}$$

2-misol.  $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix}$  va  $B = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & -2 \end{pmatrix}$  bo'lsin.  $A+B$  ni toping.

*Yechish.*

$$A+B = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1+2 & -1+3 & 4+2 \\ 3+1 & 0+0 & 1+(-2) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 4 & 0 & -1 \end{pmatrix}$$

$-A = (-1) \cdot A$  matritsa  $A$  matritsaga qarama-qarshi matritsa deb ataladi.

**3-ta'rif.**  $A = (a_{ij})$  va  $B = (b_{ij})$  matritsalarining ayirmasi deb  $C = A - B = A + (-B)$  matritsaga aytiladi. Bunda  $C$  matritsaning elementlari  $c_{ij} = a_{ij} + (-b_{ij}) = a_{ij} - b_{ij}$  kabi topiladi.

$$C = A - B \Leftrightarrow c_{ij} = a_{ij} - b_{ij}.$$

3-misol.  $A = \begin{pmatrix} 2 & -3 & 2 \\ 2 & -1 & 4 \end{pmatrix}$  va  $B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$  bo'lsin.  $A-B$  ni toping.

*Yechish.*

$$A-B = \begin{pmatrix} 2 & -3 & 2 \\ 2 & -1 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2-1 & -3-3 & 2-2 \\ 2-2 & -1-1 & 4-(-1) \end{pmatrix} = \begin{pmatrix} 1 & -6 & 0 \\ 0 & -2 & 5 \end{pmatrix}$$

*Matritsalar ustida chiziqli amallar quyidagi xossalarga ega.*

$A, B, C, O$  matritsalar  $m \times n$  o'lchamli va  $\lambda, \mu$ - skalyar sonlar bo'lsa, u holda:

1°.  $A+B = B+A$ ;

2°.  $(A+B)+C = A+(B+C)$ ;

3°.  $A+O = A$ ;

4°.  $A+(-A) = O$ ;

5°.  $\lambda(A+B) = \lambda A + \lambda B$ ;

6°.  $(\lambda + \mu)A = \lambda A + \mu A$ ;

7°.  $\mu(\lambda A) = \lambda(\mu A) = (\lambda\mu)A$ ;

8°.  $1 \cdot A = A$ ;

9°.  $(A+B)^T = A^T + B^T$ ;

10°.  $(\lambda A)^T = \lambda A^T$ ;

11°.  $A+C = B$  bo'lsa,  $C = B - A$  bo'ladi;

12°.  $\lambda A = O$  bo'lsa,  $\lambda = 0$  yoki  $A = O$  bo'ladi;

13°.  $\lambda A = \lambda B$  va  $\lambda \neq 0$  bo'lsa,  $A = B$  bo'ladi.

Xossalardan ayrimlarini isbotlaymiz.

Birinchi to'rtta xossaning isboti bevosita 2-ta'rifdan kelib chiqadi.

5-xossani qaraymiz.

$A$  va  $B$  bir xil o'lchamli matritsalar bo'lsin.

U holda

$$A + B = (a_{ij} + b_{ij})$$

yoki

$$\lambda(A + B) = \lambda(a_{ij} + b_{ij}) = \lambda(a_{ij}) + \lambda(b_{ij})$$

va ikkinchidan

$$\lambda(a_{ij}) + \lambda(b_{ij}) = \lambda A + \lambda B.$$

bo'ladi.

Bu ikkita tenglikdan  $\lambda(A + B) = \lambda A + \lambda B$  bo'lishi kelib chiqadi.

### *Matritsalar ni ko'paytirish*

Bir xil sondagi elementlarga ega bo'lgan  $A$  satr matritsa va  $B$  ustun matritsa berilgan bo'lsin deylik. Bunda  $A$  satrning  $B$  ustunga ko'paytmasi quyidagicha aniqlanadi:

$$AB = (a_{11} \ a_{12} \ \dots \ a_{1n}) \cdot \begin{pmatrix} b_{11} \\ b_{12} \\ \dots \\ b_{1n} \end{pmatrix} = a_{11}b_{11} + a_{12}b_{12} + \dots + a_{1n}b_{1n},$$

ya'ni ko'paytma berilgan matritsalar mos elementlari ko'paytmalarining yig'indisiga teng bo'ladi.

Matritsalar ni ko'paytirishning bu qoidasi *satrni ustunga ko'paytirish qoidasi* deb yuritiladi.

Ikki matritsani ko'paytirish amali *moslashtirilgan matritsalar* uchun kiritiladi.

$A$  matritsaning ustunlari soni  $B$  matritsaning satrlari soniga teng bo'lsa,  $A$  va  $B$  matritsalar *moslashtirilgan* deyiladi.

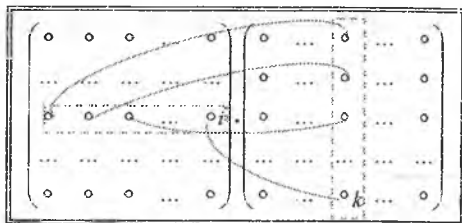
**4-ta'rif.**  $m \times p$  o'lchamli  $A = (a_{ij})$  matritsaning  $p \times n$  o'lchamli  $B = (b_{jk})$  matritsaga ko'paytmasi  $AB$  deb,  $c_{ik}$  elementi  $A$  matritsaning



$i$ -satrini  $B$  matritsaning  $j$ -ustuniga satrni ustunga ko'paytirish qoidasi bilan, ya'ni

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{ip}b_{pk} = \sum_{r=1}^p a_{ir}b_{rk}, \quad i=1, \dots, m, \quad k=1, \dots, n$$

(qo'shiluvchilari quyidagi sxemada keltirilgan) kabi aniqlanadigan  $m \times n$  o'lchamli  $C = (c_{ik})$  matritsaga aytiladi.



4-misol. Berilgan matritsalarini ko'paytiring.

$$1. \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$$2. \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & 4 \\ -1 & 0 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \cdot 3 + 1 \cdot (-1) & 2 \cdot 2 + 1 \cdot 0 & 2 \cdot 4 + 1 \cdot 3 \\ -3 \cdot 3 + 4 \cdot (-1) & -3 \cdot 2 + 4 \cdot 0 & -3 \cdot 4 + 4 \cdot 3 \\ 0 \cdot 3 + 2 \cdot (-1) & 0 \cdot 2 + 2 \cdot 0 & 0 \cdot 4 + 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 11 \\ -13 & -6 & 0 \\ -2 & 0 & 6 \end{pmatrix}$$

Agar  $A$  matritsaning satrlari  $A_1, A_2, \dots, A_m$  bilan va  $B$  matritsaning ustulari  $B_1, B_2, \dots, B_n$  bilan belgilansa, u holda matritsalarini ko'paytirish qoidasini quyidagi ko'rinishda yozish mumkin:

$$C = AB = \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_m \end{pmatrix} \cdot (B_1 \quad B_2 \quad \dots \quad B_n) = \begin{pmatrix} A_1 B_1 & A_1 B_2 & \dots & A_1 B_n \\ A_2 B_1 & A_2 B_2 & \dots & A_2 B_n \\ \dots & \dots & \dots & \dots \\ A_m B_1 & A_m B_2 & \dots & A_m B_n \end{pmatrix}$$

Matritsalarini ko'paytirishda  $A^2$  yozuv ikkita bir xil matritsaning ko'paytmasini bildiradi:

$$A^2 = A \cdot A.$$

Shu kabi

$$A^3 = A \cdot A \cdot A, \dots, A^n = \underbrace{A \cdot A \cdot \dots \cdot A}_{n \text{ marta}}$$

5-misol.  $f(x) = 2x - x^2 + 5$  va  $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$  bo'lsin.  $f(A)$ ni toping.

*Yechish.* Matritsa ko'rinishdagi  $f(A)$  funksiyaga o'tishda  $\lambda$  sonli qo'shiluvchi  $I$  ko'paytma bilan almashtiriladi, bu yerda  $I$ -birlik matritsa.

$$\begin{aligned} f(A) &= 2A - A^2 + 5I = 2 \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 2 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 0 & 5 \end{pmatrix} \end{aligned}$$

### ***Matritsalarini ko'paytirish amali ushbu xossalarga bo'ysunadi***

1°.  $A$  matritsa  $m \times n$  o'lchamli va  $B, C$  matritsalar  $n \times p$  o'lchamli bo'lsa,  $A(B+C) = AB + AC$  bo'ladi;

2°.  $A, B, C$  matritsalar mos ravishda  $m \times n$ ,  $n \times p$ ,  $p \times q$  o'lchamli bo'lsa,  $A(BC) = (AB)C$  bo'ladi;

3°.  $A, B, I, O$  moslashtirilgan matritsalar va  $\lambda, \mu$  skalyar sonlar bo'lsa, u holda:

1)  $(\lambda A)(\mu B) = (\lambda\mu)(AB)$ ;

2)  $A(\lambda B) = (\lambda A)B = \lambda(AB)$ ;

3)  $AI = IA = A$ ;

4)  $AO = OA = O$ ;

5)  $(AB)^T = B^T A^T$ .

4°.  $A, I, O$  -  $n$ - tartibli kvadrat matritsalar va  $p, q$  manfiy bo'lmagan butun sonlar bo'lsa, u holda:

1)  $A^p A^q = A^{p+q}$ ;

2)  $(A^p)^q = (A)^{pq}$ ;

3)  $A^1 = A$ ;

4)  $A^0 = I$ .

Xossalardan ayrimlarini ta'riflar yordamida isbotlaymiz va ayrimlarining to'g'riligiga misollarni yechish orqali ishonch hosil qilamiz

$$(AB)C = (3 \ 7) \cdot \begin{pmatrix} -2 & 4 & 1 \\ 5 & 0 & 2 \end{pmatrix} = (29 \ 12 \ 17).$$

Demak,  $A(BC) = (AB)C$ .

Umuman olganda matritsalarini ko'paytirish nokommutativ, ya'ni  $AB \neq BA$ . Masalan,  $1 \times n$  o'lchamli  $A$  matritsaning  $n \times 1$  o'lchamli  $B$  matritsaga  $AB$  ko'paytmasi sondan, ya'ni  $1 \times 1$  o'lchamli matritsadan iborat bo'lsa,  $BA$  ko'paytmasi  $n$ - tartibli kvadrat matritsa bo'ladi.

Bir xil tartibli  $A$  va  $B$  kvadrat matritsalar uchun  $AB = BA$  bo'lsa,  $A$  va  $B$  matritsalariga *kommutativ matritsalar*,  $AB - BA$  ayirmaga *kommulator* deyiladi.

### 1.1.3. Mashqlar

1.  $A$  kvadrat matritsa bo'lsin.  $A + A^T$  simmetrik matritsa bo'lishini ko'rsating.

2.  $A = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$  matritsani  $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  va  $Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  matritsalarining chiziqli

kombinatsiyasi ko'rinishida ifodalang.

3.  $a \begin{pmatrix} 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$  bo'lsa,  $a$  va  $b$  ni toping.

4. Matritsa 30 ta elementga ega bo'lsa, u qanday tartiblarda berilishi mumkin?

5 - 8.  $A, B$  matritsalar va  $\lambda, \mu$  sonlar berilgan.  $\lambda A + \mu B$  matritsani toping:

5.  $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -3 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 3 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ ,  $\lambda = -1$ ,  $\mu = 2$ .

6.  $A = \begin{pmatrix} 0 & -3 \\ -2 & 1 \\ 1 & 4 \end{pmatrix}$ ,  $B = \begin{pmatrix} -1 & 2 \\ 3 & -1 \\ 2 & -5 \end{pmatrix}$ ,  $\lambda = 2$ ,  $\mu = -3$ .

7.  $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 2 & 3 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -1 & 0 \\ -4 & -3 & 2 \end{pmatrix}$ ,  $\lambda = -3$ ,  $\mu = -2$ .

$$8. A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}, B = I, \lambda = 1, \mu = -v.$$

9.  $A$  va  $B$  moslashtirilgan matritsalar bo'lsin. Quyidagilarni ko'rsating:

(a) agar  $A$  matritsa satr matritsa bo'lsa, u holda  $AB$  satr matritsa bo'ladi;

(b) agar  $B$  matritsa ustun matritsa bo'lsa, u holda  $AB$  ustun matritsa bo'ladi.

$$10. \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 9 & 0 \end{pmatrix} \text{ bo'lsa, } x \text{ va } y \text{ ni toping.}$$

11. Agar  $A$  matritsa  $3 \times 3$  o'lchamli va  $C$  esa  $5 \times 5$  o'lchamli bo'lsa,  $AB$  ko'paytma ma'noga ega bo'lishi uchun  $B$  matritsa qanday o'lchamda bo'lishi kerak?

12.  $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$  matritsa berilgan.  $AB$  ko'paytmani nol matritsaga aylantiruvchi  $B$  matritsani toping.

13-16.  $A$  va  $B$  matritsalar berilgan.  $AB$  matritsani toping:

$$13. A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & 2 & 0 \end{pmatrix}$$

$$14. A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 4 & -2 \\ 2 & 3 \end{pmatrix}$$

$$15. A = \begin{pmatrix} 1 & 1 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 3 \\ 0 & -1 \\ 2 & 1 \end{pmatrix}$$

$$16. A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 3 \\ 1 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 & -2 \\ 2 & -1 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

$$17. A = \begin{pmatrix} 2 & -2 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}, C = B - 3I \text{ bo'lsa, } (AB)C \text{ matritsani toping.}$$

$$18. A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 4 & 5 \\ 2 & 6 \end{pmatrix}, C = \begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix} \text{ bo'lsa, } A(BC) \text{ matritsani toping.}$$

$$19. A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 0 \end{pmatrix} \text{ matritsalar berilgan. } AB, B^T B, A^2$$

matritsalarini toping

20.  $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$  va  $f(x) = 3x^2 + 5x - 4$  bo'lsin.  $f(A)$  ni toping.

21.  $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  bo'lsa,  $A^{20}$  ni toping.

22. Agar  $A^2 = I$  va  $A$  matritsa  $2 \times 2$  o'lchamli bo'lsa,  $A$  ni toping.

## 1.2. DETERMINANTLAR

Determinant tushunchasidan dastlab chiziqli tenglamalar sistemasini yechishda foydalanilgan bo'lib, keyinchalik determinantlar matematikaning bir qancha masalalarini yechishga, jumladan xos sonlarni topishga, differensial tenglamalarni yechishga, vektor hisobiga, keng tatbiq etildi.

Biz avval ikkinchi va uchinchi tartibli determinantlar tushunchalari bilan tanishamiz. Bu tushunchalar keyinchalik yuqori tartibli determinantlarni hisoblash uchun asos bo'ladi.

### 1.2.1. Ikkinchi va uchunchi tartibli determinantlar

*Ikkinchi tartibli determinant*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

kabi belgilanadi va aniqlanadi.

Bunda  $a_{11}, a_{12}, a_{21}, a_{22}$  lar determinantning elementlari deb ataladi.  $a_{ij}$  determinantning  $i$ -satri va  $j$ -ustunda joylashgan elementini ifodalaydi.

$a_{11}, a_{22}$  elementlar joylashgan diagonalga determinantning bosh diagonal,  $a_{21}, a_{12}$  elementlar joylashgan diagonalga determinantning yordamchi diagonal deyiladi.

Shunday qilib, ikkinchi tartibli determinant bosh diagonal elementlari ko'paytmasidan yordamchi diagonal elementlari ko'paytmasining ayirilganiga teng:

$$\boxed{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

1-misol. Berilgan determinantlarni hisoblang.

$$1. \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \cdot 5 - 4 \cdot (-2) = 15 + 8 = 23;$$

$$2. \begin{vmatrix} \operatorname{tg} \alpha & 2 \sin \alpha \\ \sin \alpha & \operatorname{ctg} \alpha \end{vmatrix} = \operatorname{tg} \alpha \cdot \operatorname{ctg} \alpha - \sin \alpha \cdot 2 \sin \alpha = 1 - 2 \sin^2 \alpha = \cos 2\alpha.$$

Matritsaning muhim tavsiflaridan biri determinant hisoblanadi. Determinant faqat kvadrat matritsalar uchun kiritiladi.

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  kvadrat matritsaning determinanti  $\det A$  bilan

belgilanadi va  $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$  kabi aniqlanadi.

Bunda matritsani uning determinanti bilan adashtirmaslik kerak:  $A$  – bu sonlar massivi;  $\det A$  – bu bitta son (yoki ifoda).

*Uchinchi tartibli determinant*

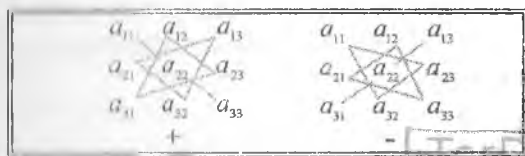
$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

kabi belgilanadi va aniqlanadi.

Uchinchi tartibli determinant uchun satr, ustun, bosh diagonal, yordamchi diagonal tushunchalari ikkinchi tartibli determinantdagi kabi kiritiladi.

Uchinchi tartibli determinantlarni hisoblashda o'ng tomonidagi tushadlarni topishning yodda saqlash uchun oson bo'lgan qoidalaridan foydalaniladi.

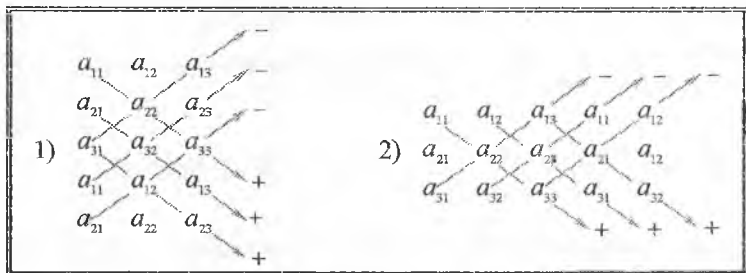
«Uchburchak qoidasi» ushbu sxema bilan tasvirlanadi:



TerDU ARM  
№ 403 220

Bunda diagonallardagi yoki asoslari diagonallarga parallel bo'lgan uchburchaklar uchlaridagi elementlar uchta elementning ko'paytirilishi bilan hosil qiladi. Agar uchburchaklarning asoslari bosh diagonalga parallel bo'lsa, u holda elementlarning ko'paytmasi ishorasini saqlaydi. Agar uchburchaklarning asoslari yordamchi diagonalga parallel bo'lsa, u holda elementlarning ko'paytmasi teskari ishora bilan olinadi.

«Sarryus qoidalari» quyidagi sxemalar bilan ifodalanadi:



1-qoidada avval determinant tagiga uning birinchi ikkita ustuni yoziladi, 2-qoidada esa determinantning o'ng tomoniga uning birinchi ikkita ustuni yoziladi. Bunda diagonallardagi yoki diagonallarga parallel bo'lgan to'g'ri chiziqlardagi elementlar uchta ko'paytuvchini hosil qiladi. Agar to'g'ri chiziqlar bosh diagonalga parallel bo'lsa, u holda elementlarning ko'paytmasi ishorasini saqlaydi. Agar to'g'ri chiziqlar yordamchi diagonalga parallel bo'lsa, u holda elementlarning ko'paytmasi teskari ishora bilan olinadi.

2-misol. 1.  $\det A = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix}$  determinantni uchburchak qoidasi bilan hisoblang.

$\begin{vmatrix} 1 & 5 & 3 \end{vmatrix}$

$$3. \det C = \begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{vmatrix} \text{ determinantni Sarryusning 2-qoidasi bilan}$$

hisoblang.  
Yechish.

$$1. \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix} \Rightarrow -8 + 1 + 27 = 20, \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix} \Rightarrow 6 - 6 + 6 = 6,$$

$$\det A = 20 - 6 = 14.$$

$$2. \begin{vmatrix} 3 & 4 & -1 & 3 \\ 2 & 0 & 3 & 2 \\ 3 & -1 & 2 & 3 \\ & & & -1 \end{vmatrix} \Rightarrow \det B = 0 + 36 + 2 - 0 + 9 - 16 = 31.$$

$$3. \begin{vmatrix} 1 & 5 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 1 \\ 1 & 5 & 3 \\ 3 & 1 & -2 \end{vmatrix} \Rightarrow \det C = 1 - 36 - 20 - 6 - 8 - 15 = -84.$$

### 1.2.2. $n$ - tartibli determinant tushunchasi

$n$  - tartibli determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{vmatrix}$$



oldiga inversiya tushunchasi asosida plus yoki minus ishora qo'yiladi.

$n$ -tartibli determinantni bu qoida asosida ifodalash ancha noqulay. Shu sababli yuqori tartibli determinantlarni hisoblashda bir nechta ekvivalent qoidalardan foydalaniladi. Bunday qoidalardan biri yuqori tartibli determinantlarni quyi tartibli determinantlar asosida hisoblash usuli hisoblanadi. Bu usulda determinant biror satr (yoki ustun) bo'yicha yoyiladi. Bunda quyi (ikkinchi va uchunchi) tartibli determinantlar yuqorida keltirilgan ta'riflar asosida topiladi.

$n$ -tartibli determinantlarni yoyishda minor va algebraik to'ldiruvchi tushunchalaridan foydalaniladi.

$n$ -tartibli determinant  $a_{ij}$  elementining *minori* deb, shu element joylashgan satr va ustunni o'chirishdan hosil bo'lgan  $(n-1)$ -tartibli determinantga aytiladi va  $M_{ij}$  bilan belgilanadi.

Determinant  $a_{ij}$  elementining  $A_{ij}$  algebraik to'ldiruvchisi deb,

$$A_{ij} = (-1)^{i+j} M_{ij}$$

songa aytiladi.

Masalan,  $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix}$  determinantning  $a_{21} = 2$  elementining minori

va algebraik to'ldiruvchisi quyidagicha topiladi:

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix} \Rightarrow M_{21} = \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} = -10, \quad A_{21} = (-1)^{2+1} M_{21} = 10.$$

### 1.2.3. Determinantning xossalari

Determinantning xossalarini uchunchi tartibli determinant uchun keltiramiz. Bu xossalar ixtiyoriy  $n$ -tartibli determinant uchun ham o'rinli bo'ladi.

**1-xossa.** Transponirlash (barcha satrlarni mos ustunlar bilan almashtirish) natijasida determinantning qiymati o'zgarmaydi, ya'ni

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \det A^T.$$

*Isboti.* Xossani isbotlash uchun tenglikning chap va o'ng tomonidagi determinantlarning qiymatlarini uchburchak qoidasi orqali yozib olish va olingan ifodalarning tengligiga ishonch hosil qilish kifoya.

1-xossa satr va ustunlarning teng huquqligini belgilab beradi. Boshqacha aytganda, satrlar uchun isbotlangan xossalar ustunlar uchun ham o'rinti bo'ladi va aksincha. Shu sababli keyingi xossalarni ham satrlar va ham ustunlar uchun ifodalab, ularning isbotini faqat satrlar yoki faqat ustunlar uchun ko'rsatamiz.

2-xossa. Determinant ikkita satrining (ustunining) o'rinlari almashtirilsa, uning qiymati qarama-qarshi ishoraga o'zgaradi. Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Bu xossa ham 1-xossa kabi isbotlanadi.

3-xossa. Agar determinant ikkita bir xil satrga (ustunga) ega bo'lsa, u nolga teng bo'ladi.

*Isboti.* Ikkita bir xil satrning o'rinlari almashtirilsa, determinant (shuningdek, uning qiymati) o'zgarmaydi. Ikkinchi tomondan 2-xossaga ko'ra determinant qiymatining ishorasi o'zgaradi. Demak  $\det A = -\det A$ , yoki  $2\det A = 0$ . Bundan  $\det A = 0$ .

4-xossa. Determinantning biror satri (ustuni) elementlari  $\lambda$  songa ko'paytirilsa, determinant shu songa ko'payadi va aksincha determinant biror satr (ustun) elementlarining umumiy ko'paytuvchisini determinant belgisidan tashqariga chiqarish mumkin.

Masalan,

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

*Isboti.* Tenglikning chap tomondagi determinant hisoblanganida oltita qo'shiluvchining hammasida  $\lambda$  ko'paytuvchi qatnashadi. Bu ko'paytuvchini qavsdan tashqariga chiqarib, qavslar ichidagi qo'shiluvchilardan determinant tuzilsa, tenglikning o'ng tomondagi ifoda hosil bo'ladi.

5-xossa. Agar determinant biror satrining (ustunining) barcha elementlari nolga teng bo'lsa, u nolga teng bo'ladi.

Xossaning *isboti* 4-xossadan  $\lambda=0$  da kelib chiqadi.

**6-xossa.** Agar determinantning ikki satri (ustuni) proporsional bo'lsa, u nolga teng bo'ladi.

Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

*Isboti.* 4-xossaga ko'ra determinant ikkinchi satrining  $\lambda$  ko'paytuvchisini determinant belgisidan chiqarish mumkin. Natijada ikkita bir xil satrli determinant qoladi va u 3-xossaga ko'ra nolga teng bo'ladi.

**7-xossa.** Agar determinant biror satrining (ustunining) har bir elementi ikki qo'shiluvchining yig'indisidan iborat bo'lsa, bu determinant ikki determinant yig'indisiga teng bo'lib, ulardan birinchisining tegishli satri (ustuni) elementlari birinchi qo'shiluvchilardan, ikkinchisining tegishli satri (ustuni) elementlari ikkinchi qo'shiluvchilardan tashkil topadi.

*Isboti.* Determinant birinchi satrining har bir elementi ikkita qo'shiluvchi yig'indisidan iborat bo'lsin.

U holda

$$\begin{vmatrix} a'_{11} + a''_{11} & a'_{12} + a''_{12} & a'_{13} + a''_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a'_{11} + a''_{11})a_{22}a_{33} + (a'_{12} + a''_{12})a_{23}a_{31} + \\ + (a'_{13} + a''_{13})a_{21}a_{32} - (a'_{13} + a''_{13})a_{22}a_{31} - (a'_{12} + a''_{12})a_{21}a_{33} - (a'_{11} + a''_{11})a_{23}a_{32} = \\ = a'_{11}a_{22}a_{33} + a'_{12}a_{23}a_{31} + a'_{13}a_{21}a_{32} - a''_{13}a_{22}a_{31} - a''_{12}a_{21}a_{33} - a''_{11}a_{23}a_{32} + (a'_{11}a_{22}a_{33} + a'_{12}a_{23}a_{31} + \\ + a'_{13}a_{21}a_{32} - a''_{13}a_{22}a_{31} - a''_{12}a_{21}a_{33} - a''_{11}a_{23}a_{32}) = \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a''_{11} & a''_{12} & a''_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

**8-xossa.** Agar determinantning biror satri (ustuni) elementlariga boshqa satrining (ustunining) mos elementlarini biror songa ko'paytirib qo'shilsa, determinantning qiymati o'zgarmaydi.

*Isboti.*  $\det A$  determinantning ikkinchi satri elementlariga  $\lambda$  ga

ko'paytirilgan birinchi satrning mos elementlari qo'shilgan bo'lsin:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \lambda a_{11} & a_{22} + \lambda a_{12} & a_{23} + \lambda a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Qo'shiluvchilardan birinchisi  $\det A$  ga va ikkinchisi esa 3-xossaga ko'ra nolga teng. Demak, yig'indi  $\det A$  ga teng.

**9-xossa.** Determinantning qiymati uning biror satri (ustuni) elementlari bilan shu elementlarga mos algebraik to'ldiruvchilar ko'paytmalarining yig'indisiga teng bo'ladi.

Masalan,  $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

yoki

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

*Isboti.* Tenglikning o'ng tomonida almashtirishlar bajaramiz:

$$a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**10-xossa.** Determinant biror satri (ustuni) elementlari bilan boshqa satri (ustuni) mos elementlari algebraik to'ldiruvchilari ko'paytmalarining yig'indisi nolga teng bo'ladi.

Masalan,  $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = 0$ .

*Isboti.* Determinantni 9-xossani qo'llab, topamiz:

$$a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Bunda  $a_{11}, a_{12}, a_{13}$  mos ravishda  $a_{21}, a_{22}, a_{23}$  bilan bilan almashtirilsa, 3-xossaga ko'ra, determinant nolga teng bo'ladi.

**1-izoh.** Determinantning xossalari asosida quyidagi teorema isbotlangan.

**1-teorema.** Bir xil tartibli  $A$  va  $B$  kvadrat matritsalar ko'paytmasining determinanti bu matritsalar determinantlarining ko'paytmasiga teng, ya'ni

$$\det(A \cdot B) = \det A \cdot \det B.$$

#### 1.2.4. $n$ -tartibli determinantlarni hisoblash

$n$ -tartibli determinantni xossalar yordamida soddalashtirib, keyin tartibini pasaytirish yoki uchburchak ko'rinishga keltirish usullaridan biri bilan hisoblash mumkin.

##### *Tartibini pasaytirish usuli*

$n$ -tartibli determinant, 9-xossaga asosan, biror satr yoki ustun bo'yicha yoyilsa, yoyilmada  $(n-1)$ -tartibli algebraik to'ldiruvchilar hosil bo'ladi, ya'ni  $n$ -tartibli determinantni hisoblash tartibi bittaga past bo'lgan determinantlarni hisoblashga keltiriladi.

Umuman olganda, quyidagi teoremlar o'rinli bo'ladi.

**2-teorema.**  $i$  satrining nomeri qanday bo'lishidan qat'iy nazar,  $n$ -tartibli determinant uchun bu determinantni  $i$ -satr bo'yicha yoyilmasi deb ataluvchi

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}, \quad i = \overline{1, n}$$

formula o'rinli.

**3-teorema.**  $j$  ustunining nomeri qanday bo'lishidan qat'iy nazar,  $n$ -tartibli determinant uchun bu determinantni  $j$ -ustun bo'yicha yoyilmasi deb ataluvchi

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}, \quad j = \overline{1, n}$$

formula o'rinli.

Determinantni biror satr yoki ustun bo'yicha yoyishga *Laplas yoyilmalari usuli* deyiladi.

Laplas yoyilmalari usulida determinantning qaysi bir satrida (ustunida) nollar ko'p bo'lsa, u holda yoyishni shu satr (ustun)

bo'yicha bajarish qulay bo'ladi.

Bundan tashqari, 8-xossani qo'llab, determinantning biror satrida (ustunida) bitta elementdan boshqa elementlarni nollarga keltirish mumkin. Bunda determinantning qiymati shu satrdagi (ustundagi) noldan farqli element bilan uning algebraik to'ldiruvchisining ko'paytmasidan iborat bo'ladi. Shunday qilib,  $n$ -tartibli determinant bitta  $(n-1)$ -tartibli determinantga keltirib, hisoblanadi.

3-misol.

$$\det A = \begin{vmatrix} 2 & -1 & 0 & 4 \\ 4 & 2 & -1 & 3 \\ -2 & 0 & 3 & -4 \\ 1 & 1 & 0 & -2 \end{vmatrix}$$

determinantni tartibini pasaytirish usuli bilan hisoblang.

*Yechish.* Bunda: 1) Ikkita elementi nolga teng bo'lgan uchinchi ustunni tanlaymiz va uning ikkinchi satrida joylashgan elementidan boshqa barcha elementlarini nolga aylantiramiz. Buning uchun ikkinchi satr elementlarini 3 ga ko'paytirib, uchinchi satrning mos elementlariga qo'shamiz va hosil bo'lgan determinantni uchinchi ustun elementlari bo'yicha yoyamiz:

$$\det A = \begin{vmatrix} 2 & -1 & 0 & 4 \\ 4 & 2 & -1 & 3 \\ -2 & 0 & 3 & -4 \\ 1 & 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 4 \\ 4 & 2 & -1 & 3 \\ 10 & 6 & 0 & 5 \\ 1 & 1 & 0 & -2 \end{vmatrix} = (-1) \cdot (-1)^{2+3} \begin{vmatrix} 2 & -1 & 4 \\ 10 & 6 & 5 \\ 1 & 1 & -2 \end{vmatrix}$$

2) Hosil qilingan uchinchi tartibli determinantda birinchi ustunning uchinchi satri elementidan yuqorida joylashgan elementlarini nolga aylantiramiz. Buning uchun avval uchinchi satrni  $(-2)$ ga ko'paytirib, birinchi satrga qo'shamiz, keyin uchinchi satrni  $(-10)$ ga ko'paytirib, ikkinchi satrga qo'shamiz, hosil bo'lgan determinantni birinchi ustun elementlari bo'yicha yoyamiz va hosil bo'lgan ikkinchi tartibli determinantni hisoblaymiz:

$$\det A = \begin{vmatrix} 0 & -3 & 8 \\ 0 & -4 & 25 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 8 \\ -4 & 25 \end{vmatrix} = -75 + 32 = -43.$$

## *Uchburchak ko‘rinishga keltirish usuli*

Bu usulda determinant xossalar yordamida soddalashtiriladi va uchburchak ko‘rinishga keltiriladi, ya’ni diagonalidan pastda (yuqorida) joylashgan barcha elementlari nolga aylantiriladi.

Bunda

$$\det A = (-1)^k \det U$$

bo‘ladi, bu yerda  $k$  – satrlarda va ustunlarda bajarilgan barcha o‘rin almashtirishlar soni;  $\det U$  – berilgan determinantning uchburchak ko‘rinishi va uning qiymati quyidagi xossa asosida hisoblanadi.

**Xossa.** Uchburchak ko‘rinishidagi determinant bosh diagonalda joylashgan elementlarining ko‘paytmasiga teng.

*4-misol.*

$$\det A = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix}$$

determinantni uchburchak ko‘rinishga keltirib, hisoblang.

*Yechish.* Determinant ustida quyidagi soddalashtirishlarni bajaramiz:

– birinchi ustunni o‘zidan o‘ngda joylashgan ustunlar bilan ketma-ket  $k=3$  ta o‘rin almashtirib, to‘rtinchi ustunga o‘tkazamiz;

– birinchi ustunning birinchi satridan pastda joylashgan elementlarini nolga aylantiramiz;

– ikkinchi ustunning ikkinchi satridan pastda joylashgan elementlarini nolga aylantiramiz;

– uchinchi ustunning to‘rtinchi satrida joylashgan elementini nolga aylantiramiz;

–  $(-1)^k = (-1)^3 = -1$  ko‘paytuvchi bilan hosil bo‘lgan uchburchak ko‘rinishdagi determinantning bosh diagonalda joylashgan elementlarini ko‘paytiramiz.

$$\det A = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix} = (-1)^3 \cdot \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{vmatrix} =$$

$$= (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 2 & -2 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 8 \end{vmatrix} = (-1) \cdot 1 \cdot 1 \cdot 1 \cdot 8 = -8.$$

### 1.2.5. Mashqlar

1.  $A$  matritsa  $n \times n$  o'lchamli bo'lsin.  $\det(\lambda A)$  da  $\lambda$  ni determinant belgisidan tashqariga chiqarish uchun formula keltirib chiqaring.

2.  $A$  kvadrat matritsa va  $A^T A = I$  bo'lsin.  $\det A = \pm 1$  bo'lishini ko'rsating.

3.  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  va  $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  bo'lsin.  $\det(A+B) = \det A + \det B$  faqat  $a+d=0$

bo'lganida bajarilishini ko'rsating.

4.  $A = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$  va  $B = \begin{pmatrix} 7 & 1 \\ 3 & 2 \end{pmatrix}$  bo'lsin.  $\det(A \cdot B) = \det A \cdot \det B$  bo'lishiga

shuncha hosil qiling.

5.  $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$  bo'lsin.  $\det A^{1000}$  ni toping.

6.  $A$  va  $B$  matritsalar  $3 \times 3$  o'lchamli,  $\det A = -1$  va  $\det B = 2$  bo'lsin.

Toping:

1)  $\det AB$ ;      2)  $\det 5A$ ;      3)  $\det A^T A$ ;      4)  $\det B^3$ .

7.  $A$  va  $B$  matritsalar  $4 \times 4$  o'lchamli,  $\det A = 4$  va  $\det B = -3$  bo'lsin.

Toping:

1)  $\det AB$ ;      2)  $\det B^5$ ;      3)  $\det 2A$ ;      4)  $\det IA$ .

Ikkinchi tartibli determinantlarni hisoblang.

8.  $\begin{vmatrix} y & x-y \\ x & -x \end{vmatrix}$

9.  $\begin{vmatrix} 1 & a+b \\ b+1 & a+b \end{vmatrix}$

10.  $\begin{vmatrix} \sin^2 \alpha & \cos^2 \alpha \\ \sin^2 \beta & \cos^2 \beta \end{vmatrix}$

11.  $\begin{vmatrix} \operatorname{tg} \alpha + 1 & \operatorname{ctg} \alpha - 1 \\ \sin \alpha & \cos \alpha \end{vmatrix}$

Uchinchi tartibli determinantlarni uchburchak va Sarryus qoidalari bilan hisoblang.

12.  $\begin{vmatrix} 5 & -1 & 1 \\ 4 & 0 & -3 \\ 2 & -3 & 1 \end{vmatrix}$

13.  $\begin{vmatrix} -2 & 0 & -4 \\ 3 & 1 & 1 \\ -1 & 2 & -3 \end{vmatrix}$



Uchinchi tartibli determinantlarni biror satr yoki ustun elementlari bo'yicha yoyib hisoblang:

$$14. \begin{vmatrix} 1 & b & 1 \\ b & b & 0 \\ b & 0 & -b \end{vmatrix}$$

$$15. \begin{vmatrix} x & -1 & x \\ 1 & x & -1 \\ x & 1 & x \end{vmatrix}$$

$$16. \begin{vmatrix} \sin \alpha & \sin \beta & 0 \\ \sin \alpha & 0 & \sin \gamma \\ 0 & \sin \beta & \sin \gamma \end{vmatrix}$$

$$17. \begin{vmatrix} \operatorname{tg} \alpha & \operatorname{ctg} \beta & 0 \\ \operatorname{tg} \alpha & 0 & \operatorname{tg} \beta \\ 0 & \operatorname{ctg} \alpha & \operatorname{tg} \beta \end{vmatrix}$$

Uchinchi tartibli determinantlarni xossalaridan foydalanib hisoblang:

$$18. \begin{vmatrix} 1 & c & ab \\ 1 & b & ca \\ 1 & a & bc \end{vmatrix}$$

$$19. \begin{vmatrix} 1 & 1 & 1 \\ ax & ay & az \\ a^2+x^2 & a^2+y^2 & a^2+z^2 \end{vmatrix}$$

$$20. \begin{vmatrix} a+b & b & b \\ b & a+b & b \\ b & b & a+b \end{vmatrix}$$

$$21. \begin{vmatrix} x & x+y & x-y \\ x & x+z & x-2z \\ x & x & x \end{vmatrix}$$

$$22. \begin{vmatrix} a & a^2+1 & (1-a)^2 \\ b & b^2+1 & (1+b)^2 \\ c & c^2+1 & (1+c)^2 \end{vmatrix}$$

$$23. \begin{vmatrix} 1+\cos \alpha & 1 & 1+\sin \alpha \\ 1-\sin \alpha & 1 & 1-\cos \alpha \\ 1 & 1 & 1 \end{vmatrix}$$

To'rtinchi tartibli determinantlarni hisoblang:

$$24. \begin{vmatrix} 1 & -1 & 2 & 2 \\ 3 & -1 & 5 & -2 \\ -2 & -3 & 0 & 2 \\ 0 & -2 & 4 & 1 \end{vmatrix}$$

$$25. \begin{vmatrix} 1 & 1 & 3 & 2 \\ 2 & 0 & 0 & 8 \\ 3 & 0 & 0 & 2 \\ 4 & 4 & 7 & 5 \end{vmatrix}$$

$$26. \begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix}$$

$$27. \begin{vmatrix} 3 & 2 & 2 & 2 \\ 9 & -8 & 5 & 10 \\ 5 & -8 & 5 & 8 \\ 6 & -5 & 4 & 7 \end{vmatrix}$$

### 1.3. MATRITSA USTIDA ALMASHTIRISHLAR

Matritsa ustida almashtirishlar chiziqli algebra muhim ro'li o'ynaydi. Jumladan, chiziqli algebraik tenglamalar sistemasining

umumiy yechimini topishda, teskari matritsani aniqlashda, matritsaning rangini hisoblashda matritsa ustidagi almashtirishlardan keng foydalaniladi.

*Matritsa satri (ustuni) ustida elementar almashtirishlar uch tipda bo'ladi:*

I. ikkita satrning (ustunning) o'rnini almashtirish;

II. satrni (ustunni) noldan farqli songa ko'paytirish;

III. satrga (ustunga) noldan farqli songa ko'paytirilgan boshqa satrni (ustunni) qo'shish.

Biri ikkinchisidan elementar almashtirishlar natijasida hosil qilingan  $A$  va  $B$  matritsalariga *ekivalent matritsalar* deyiladi va  $A \sim B$  ko'rinishda yoziladi.

### 1.3.1. Teskari matritsa

#### *Asosiy ushunchalar*

Matritsalarini qo'shish, ayirish va ko'paytirish sonlar ustida bajariladigan mos amallarga monand (hamohang) amallar hisoblanadi. Ushbu bandda matritsalar uchun sonlarni bo'lish amaliga monand amal bilan tanishamiz.

Ma'lumki, agar  $k$  soni nolga teng bo'lmasa, u holda har qanday  $m$  soni uchun  $kx = m$  tenglama yagona  $x = \frac{m}{k} = k^{-1}m$  yechimga ega bo'ladi, bu yerda  $k^{-1}$  soni  $k$  soniga teskari son deb ataladi.

Sonlar uchun keltirilgan bu tasdiq matritsali tenglamalarni sonli tenglamalarga monand yechishda muhim ro'l o'ynaydi. Xususan, sonli tenglamalar uchun  $kk^{-1} = 1$  va  $k^{-1}k = 1$  shartlarining bajarilishi hal qiluvchi hisoblansa, matritsali tenglamalar uchun  $AA^{-1} = I$  va  $A^{-1}A = I$  shartlarining bajarilishi muhim hisoblanadi, bu yerda  $A, I$  – bir xil o'lchamli kvadrat matritsalar.

Agar  $A$  va  $A^{-1}$  kvadrat matritsalar uchun  $AA^{-1} = A^{-1}A = I$  tenglik bajarilsa,  $A^{-1}$  matritsa  $A$  matritsaga *teskari matritsa* deyiladi.

Sonlarda,  $k^{-1}$  mavjud bo'lishi uchun  $k \neq 0$  bo'lishi talab etilgani kabi, matritsalarda,  $A^{-1}$  mavjud bo'lishi uchun  $\det A \neq 0$  bo'lishi talab qilinadi.

Agar  $\det A = 0$  bo'lsa,  $A$  matritsaga *singular matritsa* deyiladi. Bunda singular so'ziga sinonim sifatida «xos» yoki «maxsus» terminlaridan ham foydalaniladi. Agar  $\det A \neq 0$  bo'lsa,  $A$  matritsa

*nosingular* (yoki *xosmas* yoki *maxsusmas*) *matritsa* deb ataladi.

Agar  $A$  matritsada avval elementlar mos algebraik to'ldiruvchilar bilan almashtirilsa va keyin transponirlansa, hosil bo'lgan matritsa  $A$  matritsaga *biriktirilgan matritsa* deyiladi va  $\text{adj}A$  bilan belgilanadi:

$$\text{adj}A = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

### *Teskari matritsa haqida teoremlar*

**1- teorema.** Xos matritsa teskari matritsaga ega bo'lmaydi.

*Isboti.*  $A$  matritsa uchun  $A^{-1}$  mavjud bo'lsin deb faraz qilaylik. U holda  $AA^{-1} = I$  bo'ladi. Bundan  $\det(AA^{-1}) = \det I$  yoki  $\det A \cdot \det A^{-1} = \det I$  kelib chiqadi. Bunda  $\det A = 0$  va  $\det I = 1$  ekanini hisobga olsak,  $0 = 1$  ziddiyat hosil bo'ladi. Bu ziddiyat qilingan faraz noto'g'ri ekanini ko'rsatadi, ya'ni teoremani isbotlaydi.

**2- teorema.** Har qanday xosmas  $A$  matritsa uchun teskari matritsa mavjud va yagona bo'ladi.

*Isboti.*  $A$  matritsa xosmas, ya'ni  $\det A \neq 0$  bo'lsin. Avval  $A^{-1}$  mavjud bo'lishini ko'rsatamiz. Buning uchun  $A$  matritsani  $\frac{1}{\det A} \text{adj}A$  matritsaga ko'paytiramiz va ko'paytmaga determinantning 9- va 10- xossalarini qo'llaymiz:

$$A \cdot \left( \frac{1}{\det A} \text{adj}A \right) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} \frac{A_{11}}{\det A} & \frac{A_{21}}{\det A} & \dots & \frac{A_{n1}}{\det A} \\ \frac{A_{12}}{\det A} & \frac{A_{22}}{\det A} & \dots & \frac{A_{n2}}{\det A} \\ \dots & \dots & \dots & \dots \\ \frac{A_{1n}}{\det A} & \frac{A_{2n}}{\det A} & \dots & \frac{A_{nn}}{\det A} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{a_{11}A_{11} + a_{12}A_{21} + \dots + a_{1n}A_{n1}}{\det A} & \frac{a_{11}A_{21} + a_{12}A_{22} + \dots + a_{1n}A_{n2}}{\det A} & \dots & \frac{a_{11}A_{n1} + a_{12}A_{n2} + \dots + a_{1n}A_{nn}}{\det A} \\ \frac{a_{21}A_{11} + a_{22}A_{21} + \dots + a_{2n}A_{n1}}{\det A} & \frac{a_{21}A_{21} + a_{22}A_{22} + \dots + a_{2n}A_{n2}}{\det A} & \dots & \frac{a_{21}A_{n1} + a_{22}A_{n2} + \dots + a_{2n}A_{nn}}{\det A} \\ \dots & \dots & \dots & \dots \\ \frac{a_{n1}A_{11} + a_{n2}A_{21} + \dots + a_{nn}A_{n1}}{\det A} & \frac{a_{n1}A_{21} + a_{n2}A_{22} + \dots + a_{nn}A_{n2}}{\det A} & \dots & \frac{a_{n1}A_{n1} + a_{n2}A_{n2} + \dots + a_{nn}A_{nn}}{\det A} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\det A}{\det A} & 0 & \dots & 0 \\ \frac{\det A}{\det A} & \frac{\det A}{\det A} & \dots & 0 \\ 0 & \frac{\det A}{\det A} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\det A}{\det A} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I = AA^{-1}.$$

Demak,  $A$  matritsaga teskari matritsa mavjud va bu matritsa

$$A^{-1} = \frac{1}{\det A} \text{adj}A$$

formula bilan topiladi. Bunda  $AA^{-1} = I$  tenglik bajariladi.

$A^{-1}A = I$  tenglikning bajarilishi shu kabi ko'rsatiladi.

Endi  $A^{-1}$  yagona ekanini ko'rsatamiz. Buning uchun  $A^{-1}$  dan boshqa  $A^{-1}$  matritsaga teskari  $C$  matritsa mavjud bo'lsin deb faraz qilamiz. U holda ta'rifga ko'ra,  $AC = I$  bo'ladi. Bu tenglikning har ikkala tomonini  $A^{-1}$  ga chapdan ko'paytiramiz:

$$A^{-1}AC = A^{-1}I.$$

$A^{-1}A = I$  bo'lgani uchun  $IC = A^{-1}I$  bo'ladi.

Endi  $IC = C$  va  $A^{-1}I = A^{-1}$  ekanini hisobga olsak,  $C = A^{-1}$  kelib chiqadi. Teorema to'liq isbot qilindi.

**3-teorema.** Teskari matritsa uchun ushbu xossalalar o'rinli bo'ladi:

- 1)  $A$  matritsa  $A^{-1}$  teskari matritsaga ega bo'lsa,  $\det A^{-1} = \frac{1}{\det A}$  bo'ladi;
- 2)  $A$  matritsa  $A^{-1}$  teskari matritsaga ega bo'lsa,  $(A^{-1})^{-1} = A$  bo'ladi;
- 3)  $n \times n$  o'lchamli  $A$  va  $B$  matritsalar  $A^{-1}$  va  $B^{-1}$  teskari matritsalariga ega bo'lsa,  $(AB)^{-1} = B^{-1}A^{-1}$  bo'ladi;
- 4)  $A$  matritsa  $A^{-1}$  teskari matritsaga ega bo'lsa,  $(A^T)^{-1} = (A^{-1})^T$  bo'ladi.

**Isbot.** 1)  $A$  matritsa uchun  $A^{-1}$  mavjud bo'lsin. U holda  $AA^{-1} = I$  yoki  $\det(AA^{-1}) = \det I$  bo'ladi. Bundan  $\det A \cdot \det A^{-1} = 1$  yoki  $\det A^{-1} = \frac{1}{\det A}$  kelib chiqadi.

2)  $A$  matritsa uchun  $A^{-1}$  mavjud bo'lsin. U holda  $AA^{-1} = I = A^{-1}A$  tengliklarga ko'ra  $A^{-1}$  matritsa uchun teskari matritsa mavjud va u  $A$  dan iborat, ya'ni  $(A^{-1})^{-1} = A$  bo'ladi.

3)  $n \times n$  o'lchamli  $A$  va  $B$  matritsalar  $A^{-1}$  va  $B^{-1}$  teskari matritsalariga ega bo'lsin.

U holda  $AB$  va  $B^{-1}A^{-1}$  matritsalar uchun

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

bo'ladi. Demak,  $AB$  uchun teskari matritsa mavjud va  $(AB)^{-1} = B^{-1}A^{-1}$  bo'ladi.

4)  $A$  matritsa uchun  $A^{-1}$  mavjud bo'lsin.

U holda  $A^T$  va  $(A^{-1})^T$  matritsalar uchun

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I,$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

bo'ladi. Demak,  $A^T$  uchun teskari matritsa mavjud va  $(A^T)^{-1} = (A^{-1})^T$  bo'ladi.

**1-izoh.** 3-xossani  $k$  ta  $n \times n$  o'lchamli va teskari matritsalariga ega bo'lgan matritsalar uchun quyidagicha umumlashtirish mumkin:

$$(A_1 A_2 \dots A_{k-1} A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \dots A_2^{-1} A_1^{-1}.$$

Bu formula matematik induksiya metodi bilan isbotlanadi.

*1-misol.*  $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$  matritsaga teskari matritsani toping va natijani

tekshiring.

*Yechish.* Berilgan matritsaning determinantini hisoblaymiz:

$$\det A = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2.$$

$\det A \neq 0$  va  $A$  matritsa uchun teskari matritsa mavjud.

Matritsa elementlarining algebraik to'ldiruvchilarini topamiz:

$$A_{11} = (-1)^{1+1} 2 = 2, \quad A_{12} = (-1)^{1+2} 1 = -1,$$

$$A_{21} = (-1)^{2+1} 4 = -4, \quad A_{22} = (-1)^{2+2} 3 = 3,$$

$A$  matritsaga birlashtirilgan matritsani tuzamiz:

$$\text{adj}A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}.$$

Shunday qilib,

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

2-misol.  $A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}$  matritsaga teskari matritsani toping.

*Yechish.* Bu matritsa uchun:

$$\det A = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{vmatrix} = 0 - 4 + 2 - 0 + 4 + 1 = 3 \neq 0.$$

Matritsa elementlarining algebraik to'ldiruvchilarini topamiz:

$$A_{11} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{21} = - \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = 3, \quad A_{31} = - \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2,$$

$$A_{12} = - \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = 0, \quad A_{22} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3, \quad A_{32} = - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3,$$

$$A_{13} = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2, \quad A_{23} = - \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 3, \quad A_{33} = \begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = 4.$$

$A$  matritsaga biriktirilgan matritsani topamiz:

$$\text{adj}A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

Demak,

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix}.$$

## Teskari matritsani topishning Gauss-Jordan usuli

$A$  xosmas matritsaning  $A^{-1}$  teskari matritsasini topishning qulay usullaridan biri matritsa satrlari ustida elementar almashtirishlarga asoslangan *Gauss-Jordan usuli* hisoblanadi.

$A^{-1}$  matritsani topishning Gauss-Jordan usuli ushbu tartibda amalga oshiriladi.

### Gauss-Jordan usulining algoritmi

- 1°.  $A$  va  $I$  matritsalarini yonma-yon yozib,  $(A|I)$  kengaytirilgan matritsa tuziladi;
- 2°. Elementar almashtirishlar yordamida  $(A|I)$  matritsa  $(I|B)$  ko'rinishga keltiriladi. Bunda  $B$  matritsa  $A$  matritsa uchun teskari matritsa bo'ladi.

3-misol.  $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$  matritsaga teskari matritsani Gauss-Jordan usuli

bilan toping va natijani tekshiring.

*Yechish.*

$$(A|I) = \left( \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) r_1 \rightarrow r_1 + (-2)r_2 \sim$$

$$\sim \left( \begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 1 & 2 & 0 & 1 \end{array} \right) r_2 \rightarrow r_2 + (-1)r_1 \sim \left( \begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 0 & 5 & -1 & 3 \end{array} \right) r_2 \rightarrow r_2 : 5 \sim$$

$$\sim \left( \begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array} \right) r_1 \rightarrow r_1 + 3r_2 \sim \left( \begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array} \right) = (I|A^{-1}).$$

Yuqorida keltirilgan  $r_i \rightarrow r_i + \lambda r_k$  belgilash  $i$ -satr bu satrga  $\lambda$  songa ko'paytirilgan  $k$ -satrni qo'shish natijasida hosil qilinganini,  $r_i \rightarrow r_i : \lambda$  belgi esa  $i$ -satr bu satrni  $\lambda$  songa bo'lish natijasida hosil qilinganini bildiradi.

### Teskari matritsani topishning Gauss-Jordan usuli

$A$  xosmas matritsaning  $A^{-1}$  teskari matritsasini topishning qulay usullaridan biri matritsa satrlari ustida elementar almashtirishlarga asoslangan *Gauss-Jordan usuli* hisoblanadi.

$A^{-1}$  matritsani topishning Gauss-Jordan usuli ushbu tartibda amalga oshiriladi.

#### Gauss-Jordan usulining algoritmi

- 1°.  $A$  va  $I$  matritsalarini yonma-yon yozib,  $(A|I)$  kengaytirilgan matritsa tuziladi;
- 2°. Elementar almashtirishlar yordamida  $(A|I)$  matritsa  $(I|B)$  ko'rinishga keltiriladi. Bunda  $B$  matritsa  $A$  matritsa uchun teskari matritsa bo'ladi.

3-misol.  $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$  matritsaga teskari matritsani Gauss-Jordan usuli

bilan toping va natijani tekshiring.

*Yechish.*

$$(A|I) = \left( \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) r_1 \rightarrow r_1 + (-2)r_2 \sim$$

$$\sim \left( \begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 1 & 2 & 0 & 1 \end{array} \right) r_2 \rightarrow r_2 + (-1)r_1 \sim \left( \begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 0 & 5 & -1 & 3 \end{array} \right) r_2 \rightarrow r_2 : 5 \sim$$

$$\sim \left( \begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array} \right) r_1 \rightarrow r_1 + 3r_2 \sim \left( \begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array} \right) = (I|A^{-1}).$$

Yuqorida keltirilgan  $r_i \rightarrow r_i + \lambda r_k$  belgilash  $i$ -satr bu satrga  $\lambda$  songa ko'paytirilgan  $k$ -satrni qo'shish natijasida hosil qilinganini,  $r_i \rightarrow r_i : \lambda$  belgi esa  $i$ -satr bu satrni  $\lambda$  songa bo'lish natijasida hosil qilinganini bildiradi.



$$\text{Demak, } A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$4\text{-misol. } A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix} \text{ matritsaga teskari matritsani Gauss-Jordan}$$

usuli bilan toping.

*Yechish.*

$$(A|I) = \left( \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 + r_1 \\ r_3 \rightarrow r_3 + (-2)r_1 \end{array} \sim$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 3 & 0 & -2 & 0 & 1 \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 : 2 \\ r_1 \rightarrow r_1 + r_2 \\ r_3 \rightarrow r_3 + (-3)r_2 \end{array} \sim$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -\frac{7}{2} & -\frac{3}{2} & 1 \end{array} \right) \begin{array}{l} r_3 \rightarrow r_3 : (-3) \\ r_1 = r_1 + (-3)r_3 \\ r_2 = r_2 + (-1)r_3 \end{array} \sim$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{array} \right) = (I|A^{-1}).$$

Demak,

$$A^{-1} = \begin{pmatrix} -2 & -1 & 1 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$$

### 1.3.2. Matritsani LU yoyish

Chiziqli algebra da matritsalarining turli yoyilmalari keng qoʻllaniladi.

*Matritsani yoyish* deb, uni biror xossaga (masalan, ortogonallik, simmetriklik, diagonallik xossasiga) ega boʻlgan ikki va ikkidan ortiq matritsalar koʻpaytmasi shaklida ifodalashga aytiladi. Bunday yoyishlardan biri *matritsani LU yoyish* hisoblanadi.

Matritsani LU yoyishda  $m \times n$  oʻlchamli  $A$  matritsa  $A = LU$  shaklida ifodalanadi, bu yerda  $L$  – diagonal elementlari birlardan iborat boʻlgan  $m \times m$  oʻlchamli quyi uchburchak (Lower-triangular) matritsa;  $U$  –  $m \times n$  oʻlchamli yuqori uchburchak ( $m \neq n$  da trapetsiya) (Upper-triangular) matritsa.

Masalan,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matritsaning LU yoyilmasi yana *matritsaning LU faktorizatsiyasi* deb ataladi. Matritsaning LU yoyilmasidan chiziqli algebraik tenglamalar sistemasini yechishda va teskari matritsani topishda foydalaniladi.

$m \times n$  oʻlchamli  $A$  matritsa  $A = LU$  shaklga keltirish (LU yoyish), umuman olganda,  $A$  matritsaning satrlariga noldan farqli songa koʻpaytirilgan boshqa satrni qoʻshish orqali quyidagi tartibda amalga oshiriladi.

#### A matritsani LU yoyish algoritmi

1°.  $A$  matritsa satrlarida ketma-ket elementar almashtirishlar bajariladi va  $U$  shaklga keltiriladi;

2°. Satrlarda bajarilgan elementar almashtirishlar ketma-ketligi asosida  $L$  yozuv hosil qilinadi va bu yozuvda barcha diagonal elementlar ustunlarni boʻlish orqali birlarga aylantiriladi.

$$5\text{-misol. } A = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{pmatrix} \text{ matritsani LU yoying.}$$

*Yechish.* Matritsa satrlarida ketma-ket elementar almashtirishlar bajarimiz:

$$A = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = U.$$

$A$  matritsa 4 ta satrdan tashkil topgani sababli  $L$  matritsa  $4 \times 4$  o'lchamli bo'ladi. Birinchi qadamda belgilangan yozuvlar  $L$  matritsa yozuvining ustunlarini tashkil qiladi. Bu yozuvda barcha diagonal elementlarni birlarga aylantiramiz:

$$\begin{pmatrix} 2 & & & & \\ -4 & 3 & & & \\ 2 & -9 & 2 & & \\ -6 & 12 & 4 & 5 & \\ :2 & :3 & :2 & :5 & \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{matrix}$$

$$\begin{pmatrix} 1 & & & & \\ -2 & 1 & & & \\ 1 & -3 & 1 & & \\ -3 & 4 & 2 & 1 & \end{pmatrix} \cdot \text{Bundan } L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}$$

Demak,

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

$n$ - tartibli kvadrat matritsa berilgan bo'lsin. Bunda  $A$  matritsani  $LU$  yoyish turli algoritmlar bilan amalga oshirilishi mumkin. Shunday algoritmlardan biri bilan tanishamiz.

$A$  matritsa xosmas bo'lsin. U holda ta'rifga ko'ra,

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{pmatrix}}_L \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}}_U = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}}_A.$$

Bundan

$$a_{ij} = \sum_{k=1}^n l_{ik} u_{kj} = \sum_{k=1}^{\min(i,j)} l_{ik} u_{kj}.$$

Bu yig'indidagi oxirgi qo'shiluvchilarni ajratib, topamiz:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \text{ agar } i \leq j \text{ bo'lsa}; \quad (3.1)$$

$$l_{ij} = \frac{1}{u_{jj}} \left( a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right), \text{ agar } i > j \text{ bo'lsa}. \quad (3.2)$$

Shunday qilib,  $L$  va  $U$  matritsalarining noma'lum elementlari  $a_{ij}$  va topilgan  $l_{ik}, u_{kj}$  lar orqali ketma-ket ifodalanadi.

**2-izoh.** (3.1) va (3.2) formulalar shunday tartiblanganki, bunda avval barcha  $u_{ij}$  larni va keyin barcha  $l_{ij}$  larni hisoblab bo'lmaydi, va aksincha. Bu formulalar orqali hisoblashlar quyidagi tartibda bajariladi:

$$u_{1j} = a_{1j}, \quad j = 1, 2, \dots, n;$$

$$l_{in} = \frac{a_{in}}{u_{nn}}, \quad i = 2, 3, \dots, n;$$

$$u_{2j} = a_{2j} - l_{21} u_{1j}, \quad j = 2, 3, \dots, n;$$

$$l_{i2} = \frac{a_{i2} - l_{i1} u_{12}}{u_{22}}, \quad i = 3, 4, \dots, n;$$

va hokazo, ya'ni  $U$  ning satrlari va  $L$  ning ustunlari almashlab hisoblanadi.

5-misol.  $A = \begin{pmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{pmatrix}$  matritsani LU yoying.

*Yechish.* Berilgan matritsa xosmas, chunki  $\det A = 166 \neq 0$ .  
 $LU = A$  yoyilmani tuzamiz:

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{pmatrix}.$$

$L$  va  $U$  matritsalarining noma'lum elementlarini (3.1) va (3.2) formulalar bilan aniqlaymiz:

$$u_{11} = a_{11} = 8, \quad u_{12} = a_{12} = 2, \quad u_{13} = a_{13} = 9,$$

$$l_{21} = \frac{1}{u_{11}} a_{21} = \frac{4}{8} = \frac{1}{2}, \quad u_{22} = a_{22} - l_{21} u_{12} = 9 - \frac{1}{2} \cdot 2 = 8,$$

$$u_{23} = a_{23} - l_{21} u_{13} = 4 - \frac{1}{2} \cdot 9 = -\frac{1}{2}, \quad l_{31} = \frac{1}{u_{11}} a_{31} = \frac{6}{8} = \frac{3}{4},$$

$$l_{32} = \frac{1}{u_{22}} (a_{32} - l_{31} u_{12}) = \frac{1}{8} \left( 7 - \frac{3}{4} \cdot 2 \right) = \frac{11}{16},$$

$$u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = 9 - \frac{3}{4} \cdot 9 - \frac{11}{16} \cdot \left( -\frac{1}{2} \right) = \frac{83}{32}.$$

Demak,

$$\begin{pmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{4} & \frac{11}{16} & 1 \end{pmatrix} \cdot \begin{pmatrix} 8 & 2 & 9 \\ 0 & 8 & -\frac{1}{2} \\ 0 & 0 & \frac{83}{32} \end{pmatrix}.$$

### 1.3.3. Matritsaning rangi

$m \times n$  o'lchamli  $A$  matritsa berilgan bo'lsin. Bu matritsadan biror  $k$  ( $k = \min(m; n)$ ) ta satr va  $k$  ta ustunni ajratamiz. Ajratilgan satr va ustunlarning kesishishida joylashgan elementlardan  $k$ -tartibli kvadrat matritsani tuzamiz. Bu matritsaning determinantiga  $A$  matritsaning  $k$ -tartibli *minori* deyiladi.

$A$  matritsa noldan farqli minorlari tartibining eng kattasiga  $A$  matritsaning rangi deyiladi va  $r(A)$  (yoki  $\text{rang}A$ ) kabi belgilanadi.

Tartibi  $r(A)$ ga teng bo'lgan minorga  $A$  matritsaning bazis minori deyiladi. Matritsa bir nechta bazis minorga ega bo'lishi mumkin.

Matritsa rangining ta'rifidan quyidagi tasdiqlar kelib chiqadi.

1. Matritsaning rangi 0 bilan  $m, n$  sonlarining kichigi orasidagi butun son orqali ifodalanadi, ya'ni  $0 \leq r(A) \leq \min(m, n)$ .

2. Faqat  $A = O$  matritsa uchun  $r(A) = 0$  bo'ladi.

3.  $n$ -tartibli kvadrat matritsa xosmas bo'lganidagina  $r(A) = n$  bo'ladi.

***Matritsanning rangi ushbu xossalarga bo'ysunadi.***

1°. Transponirlash natijasida matritsaning rangi o'zgarmaydi;

2°. Elementar almashtirishlar natijasida matritsaning rangi o'zgarmaydi.

*Isboti.* Bilamizki:

a) transponirlash natijasida determinantning qiymati o'zgarmaydi;

b) ikkita satrning (ustunning) o'zini almashtirilsa, determinantning ishorasi o'zgaradi;

c) satrni (ustunni) noldan farqli songa ko'paytirilsa, determinant shu songa ko'payadi;

d) satrga (ustunga) noldan farqli songa ko'paytirilgan boshqa satrni (ustunni) qo'shilsa determinant o'zgarmaydi.

Demak, transponirlash va elementar almashtirishlar natijasida xos matritsa xosligicha va xosmas matritsa xosmasligicha qoladi, ya'ni uning rangi o'zgarmaydi.

$r(A)$  ni ta'rif asosida topish usuli *minorlar ajratish usuli* deb ataladi.

Bu usulda matritsaning rangi quyidagicha topiladi: agar barcha birinchi tartibli minorlar (matritsa elementlari) nolga teng bo'lsa,  $r(A) = 0$  bo'ladi; agar birinchi tartibli minorlardan hech bo'lmaganda bittasi noldan farqli va barcha ikkinchi tartibli minorlar nolga teng bo'lsa,  $r(A) = 1$  bo'ladi; agar ikkinchi tartibli noldan farqli minor mavjud bo'lsa, uchinchi tartibli minorlar tekshiriladi; bu jarayon yoki barcha  $k$ -tartibli minorlar nolga teng bo'lishi aniq bo'lguncha yoki  $k$ -tartibli minorlar mavjud bo'lmaguncha davom ettiriladi, bunda  $r(A) = k - 1$  bo'ladi.

6-misol.  $A = \begin{pmatrix} 2 & -1 & 3 & -2 \\ 4 & -2 & 5 & 1 \\ 2 & -1 & 1 & 8 \end{pmatrix}$  matritsaning rangini minorlar ajratish

usuli bilan toping.

*Yechish.* Ravshanki,  $1 \leq r(A) \leq \min(3;5) = 3$ .

Ikkinchi tartibli minorlardan biri

$$\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = -5 + 6 = 1 \neq 0.$$

Uchinchi tartibli minorlarni hisoblaymiz:

$$M_1^{(3)} = \begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix} = 0;$$

$$M_2^{(3)} = \begin{vmatrix} 2 & -1 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 8 \end{vmatrix} = 0;$$

$$M_3^{(3)} = \begin{vmatrix} 2 & 3 & -2 \\ 4 & 5 & 1 \\ 2 & 1 & 8 \end{vmatrix} = 0;$$

$$M_4^{(3)} = \begin{vmatrix} -1 & 3 & -2 \\ -2 & 5 & 1 \\ -1 & 1 & 8 \end{vmatrix} = 0.$$

Barcha uchinchi tartibli minorlar nolga teng. Demak,  $r(A) = 2$ .

$r(A)$ ni topishning minorlar ajratish usuli hamma vaqt ham qulay bo'lmaydi, chunki ayrim hollarda bir qancha hisoblashlar bajarishga to'g'ri keladi.

Elementar almashtirishlar orqali har qanday matritsani bosh diagonalning birinchi bir nechta elementlari birlardan va qolgan elementlari nollardan iborat bo'lgan matritsa ko'rinishiga keltirish mumkin. Masalan, ushbu matritsaga

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Bunday matritsaga *kanonik matritsa* deyiladi. Kanonik matritsaning rangi uning bosh diagonalida joylashgan birlar soniga teng bo'ladi.

$r(A)$  ni kanonik matritsaga keltirib topish usuli matritsani *kanonik ko'rinishga keltirish* usuli deb ataladi.

7-misol.  $A = \begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & 0 & 1 & -1 & 2 \\ -1 & 3 & -5 & -10 & 5 \end{pmatrix}$  matritsaning rangini uni

kanonik ko'rishga keltirish usuli bilan toping.

*Yechish.*

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & 0 & 1 & -1 & 2 \\ -1 & 3 & -5 & -10 & 5 \end{pmatrix} \begin{matrix} r_2 \rightarrow r_2 + (-2)r_1 \\ r_3 \rightarrow r_3 + r_1 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 2 & -3 & -7 & 4 \\ 0 & 2 & -3 & -7 & 4 \end{pmatrix} \begin{matrix} r_3 \rightarrow r_3 + (-1)r_2 \end{matrix} \sim \begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 2 & -3 & -7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} A \rightarrow A^r \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -3 & 0 \\ 3 & -7 & 0 \\ -1 & 4 & 0 \end{pmatrix} \begin{matrix} r_2 \rightarrow r_2 + r_1 \\ r_3 \rightarrow r_3 + (-2)r_1 \\ r_4 \rightarrow r_4 + (-3)r_1 \\ r_5 \rightarrow r_5 + r_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & -7 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{matrix} r_2 \rightarrow r_2 : 2 \\ r_3 \rightarrow r_3 : 3 \\ r_4 \rightarrow r_4 : 7 \\ r_5 \rightarrow r_5 : (-4) \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{matrix} r_3 \rightarrow r_3 + r_2 \\ r_4 \rightarrow r_4 + r_2 \\ r_5 \rightarrow r_5 + r_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Demak,  $r(A) = 2$ .

### 1.3.4. Mashqlar

1. Agar  $A$  matritsa xosmas va simmetrik bo'lsa,  $A^{-1}$  matritsa ham xosmas va simmetrik bo'lishini ko'rsating.

2. Agar  $A$  kvadrat matritsa va  $(I - A)$  xosmas matritsa bo'lsa,  $A(I - A)^{-1} = (I - A)^{-1}A$  tenglik bajarilishini ko'rsating.

3.  $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$  matritsaning teskari matritsaga ega bo'lishi shartini toping.



4.  $A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$  va  $C = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$  bo'lsin.  $C = A^{-1}$  ekanini ko'rsating.

5.  $B = \begin{pmatrix} 4 & 0 & -5 \\ -18 & 1 & 24 \\ -3 & 0 & 4 \end{pmatrix}$  matritsa  $A = \begin{pmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{pmatrix}$  matritsaning teskari matritsasi

bo'lishini ko'rsating.

6.  $B = \frac{1}{36} \begin{pmatrix} 11 & -3 & 5 \\ -17 & 21 & -11 \\ -10 & 6 & 2 \end{pmatrix}$  matritsa  $A = \begin{pmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 5 \end{pmatrix}$  matritsaning teskari

matritsasi bo'lishini ko'rsating.

7. Berilgan matritsalaridan qaysi birlari uchun teskari matritsa mavjud bo'ladi?

1)  $A = \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$ ;      2)  $B = \begin{pmatrix} 0 & 5 \\ 7 & 2 \end{pmatrix}$ ;      3)  $C = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 2 & 6 \\ 3 & 5 & 11 \end{pmatrix}$ ;       $D = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 0 & 3 & 10 \end{pmatrix}$

8.  $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$  bo'lsin.  $A^2 = A^{-1}$  va  $A^3 = I$  bo'lishini ko'rsating.

9. Berilgan matritsalaridan qaysi birlari o'zaro teskari matritsalar bo'ladi?

1)  $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$  va  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ ;      2)  $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$  va  $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$

3)  $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$  va  $\frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ ;      4)  $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$  va  $\begin{pmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$

10.  $A = \begin{pmatrix} -3 & 6 \\ 2 & -5 \end{pmatrix}$  matritsa berilgan.  $A^{-1}$  matritsani toping.

11.  $A = \begin{pmatrix} 5 & 2 \\ -1 & 4 \end{pmatrix}$  matritsa berilgan.  $A^{-1}$  matritsani toping.

12. Berilgan shartlarni qanoatlantiruvchi  $A$  matritsani toping:

1)  $(3A)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ ;      2)  $(2A)^T = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}^{-1}$ ;

3)  $(A^T - 2I)^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ ;      4)  $A^{-1} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$ .

13.  $ABC = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  bo'lsin.  $C^{-1}B^{-1}A^{-1}$  ni toping.

14.  $A = \begin{pmatrix} -3 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 5 & -1 \end{pmatrix}$  matritsa berilgan.  $C = A \cdot \text{adj}A$  ko'paytmaning barcha nodiagonal elementlarini toping.

15.  $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 0 \end{pmatrix}$  matritsa berilgan.  $C = A \cdot \text{adj}A$  ko'paytmaning barcha diagonal elementlarini toping.

$A$  matritsa berilgan.  $A^{-1}$  matritsani toping:

16.  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 2 & 4 \end{pmatrix}$

17.  $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 10 & 8 \end{pmatrix}$

$A$  matritsa berilgan.  $A^{-1}$  matritsani Jordan-Gauss usuli bilan toping:

18.  $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix}$

19.  $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$

20.  $A$  matritsa berilgan. Matritsaning  $LU$  yoyilmasini toping:

1)  $A = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$

2)  $A = \begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$

3)  $A = \begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$ ;

4)  $A = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$ ;

5)  $A = \begin{pmatrix} 2 & 0 & 5 & 2 \\ -6 & 3 & -13 & -3 \\ 4 & 6 & 16 & 17 \end{pmatrix}$

6)  $A = \begin{pmatrix} 2 & -3 & 4 \\ -4 & 8 & -7 \\ 6 & -5 & 14 \\ -6 & 9 & -12 \\ 8 & -6 & 10 \end{pmatrix}$

$A$  matritsa berilgan.  $r(A)$ ni minorlar ajratish usuli bilan toping:

21.  $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ -1 & 3 & 0 & 1 \\ 3 & 4 & 1 & 1 \end{pmatrix}$

22.  $A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & -2 \\ 2 & -2 & 7 \end{pmatrix}$

$A$  matritsa berilgan.  $r(A)$  ni elementar almashtirishlar usuli bilan toping:

$$23. A = \begin{pmatrix} 1 & -3 & 2 & -1 \\ 2 & -1 & 4 & -6 \\ -3 & -1 & -6 & 11 \end{pmatrix}$$

$$24. A = \begin{pmatrix} 1 & -1 & 3 & 4 \\ 2 & -1 & 3 & -2 \\ 1 & -4 & 3 & 1 \\ 1 & -3 & 0 & -9 \end{pmatrix}$$

## 1.4. CHIZIQLI TENGLAMALAR SISTEMASI

Chiziqli tenglamalar sistemasini yechish masalasi chiziqli ulgebraning asosiy masalalaridan biri hisoblanadi. Matematika, texnika va iqtisodiyotning ko'pchilik masalalari chiziqli tenglamalar sistemasi orqali ifodalanadi, masalan, geodeziyaviy o'lchash, elektr zanjirlarini loyihalash, chiziqli programmashtirish va iqtisodni rejalashtirish masalalari chiziqli tenglamalar sistemasini yechishga keltiriladi.

### 1.4.1. Asosiy tushunchalar

Ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \quad (4.1)$$

sistemaga  $n$  noma'lumli  $m$  ta chiziqli tenglamalar sistemasi deyiladi.

Bu yerda  $a_{11}, a_{12}, \dots, a_{mn}$  haqiqiy sonlarga *sistemaning koeffitsiyentlari*,  $x_1, x_2, \dots, x_n$  noma'lumlar,  $b_1, b_2, \dots, b_m$  haqiqiy sonlarga *ozod hadlar* deyiladi.

(4.1) sistema koeffitsiyentlaridan tuzilgan

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad (4.2)$$

matritsaga (4.1) sistemaning *matritsasi* (*asosiy matritsasi*) deyiladi.

Bu matritsaga ozod hadlardan tuzilgan ustunni qo'shish orqali

hosil qilingan

$$C = \left( \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right) \quad (4.3)$$

matritsaga (4.1) *sistemaning kengaytirilgan matritsasi* deyiladi.

(4.1) sistemani

$$AX = B \quad (4.4)$$

*matritsa ko'rinishida yozish* mumkin, bu yerda

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} \quad (4.5)$$

Haqiqatdan ham,

$$AX = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots & \dots & \dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} = B.$$

(4.1) sistema tenglamalarini ayniyatga aylantiradigan noma'lumlarning tartiblangan  $x_1^0, x_2^0, \dots, x_n^0$  qiymatlariga (4.1) *sistemaning yechimi* deyiladi.

Kamida bitta yechimga ega sistemaga *birgalikda bo'lgan sistema*, bitta ham yechimga ega bo'lmagan sistemaga *birgalikda bo'lmagan sistema* deyiladi.

Birgalikda bo'lgan va yagona yechimga ega sistemaga *aniq sistema*, cheksiz ko'p yechimga ega sistemaga *aniqmas sistema* deyiladi. Aniqmas sistemaning har bir yechimi *sistemaning xususiy yechimi* deb ataladi. Barcha xususiy yechimlar to'plami *sistemaning umumiy yechimi* deyiladi.

Sistemani *tekshirish* deganda sistemaning birgalikda yoki birgalikda emasligini aniqlash va agar sistema birgalikda bo'lsa, u holda uning aniq yoki aniqmasligini tekshirish tushuniladi. Birgalikda bo'lgan sistemaning umumiy yechimini topishga *sistemani yechish* deyiladi.

Yechimlari to'plami bir xil bo'lgan, ya'ni birinchisining har bir yechimi ikkinchisining yechimi bo'ladigan, va aksincha, ikkinchisining har bir yechimi birinchisining yechimi bo'ladigan ikkita sistemaga

ekvivalent (*teng kuchli*) sistemalar deyiladi.

Ushbu almashtirishlar *sistemada elementar almashtirishlar* deb yuritiladi:

- sistema istalgan ikkita tenglamasining o'rinlarini almashtirish;
- sistemaning istalgan tenglamasini noldan farqli songa ko'paytirish (bo'lish);
- sistemaning istalgan tenglamasiga noldan farqli songa ko'paytirilgan boshqa tenglamasini qo'shish.

Elementar almashtirishlar natijasida ekvivalent sistemalar hosil bo'ladi.

Ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \quad (4.6)$$

$n$  noma'lumli  $n$  ta chiziqli tenglamalar sistemasining

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad (4.7)$$

matritsasi kvadrat matritsa bo'ladi.

$A$  matritsaning

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad (4.8)$$

determinantiga (4.6) *sistemaning determinanti* deyiladi.

Agar  $\det A \neq 0$  bo'lsa, (4.6) sistemaga *xosmas sistema* deyiladi.

Agar  $\det A = 0$  bo'lsa, (4.6) sistemaga *xos sistema* deyiladi.

### 1.4.2. Chiziqli tenglamalar sistemasini yechishning Gauss usuli

$n$  noma'lumli  $m$  ta chiziqli tenglamalar sistemasini yechishning qulay usullaridan biri - *noma'lumlarni ketma-ket yo'qotishga* (chiqarishga) asoslangan *Gauss usulini* ko'rib chiqamiz.

Shu kabi  $a_{22}^{(1)} \neq 0$  deb, sistemaning uchinchi tenglamasi dan boshlab  $x_2$  noma'lumni yo'qotamiz va bu jarayonni mumkin bo'lguniga qadar davom ettiramiz.

Bu bosqichda, agar:

–  $0 = 0$  ko'rinishdagi tengliklar paydo bo'lsa, u holda bu tengliklar tushlab yuboriladi.

–  $0 = b_i^{(k)}$  ( $b_i^k \neq 0$ ) ko'rinishdagi tengliklar paydo bo'lsa, u holda jarayon to'xtatiladi, chunki berilgan sistema birgalikda bo'lmaydi.

*2-bosqich.* Pog'onasimon sistemani yechamiz. Pog'onasimon sistemada  $k$  tenglamalar soni  $n$  noma'lumlar soniga teng yoki no'malumlar sonidan kichik bo'lishi mumkin. Shu sababli bu sistema yagona yoki cheksiz ko'p yechimga ega bo'lishi mumkin. Agar sistema uchburchak ko'rinishga kelsa, ya'ni  $k = n$  bo'lsa, sistema yagona yechimga ega bo'ladi. Agar sistema trapetsiya ko'rinishga kelsa, ya'ni  $k < n$  bo'lsa, sistema cheksiz ko'p yechimga ega bo'ladi.

$$l\text{-misol. } \begin{cases} 2x_1 - 4x_2 - x_3 = -2, \\ 3x_1 + x_2 - 2x_3 = -11, \\ x_1 - 2x_2 + 4x_3 = 8 \end{cases} \text{ tenglamalar sistemasini Gauss usuli}$$

bilan yeching.

*Yechish.*

*1-bosqich.* Sistemada quyidagi almashtirishlarni bajaramiz:

– birinchi va uchinchi tenglamalarning o'rinlarini almashtiramiz;  
 –  $(-3)$  ga ko'paytirilgan birinchi tenglamani ikkinchi tenglamaga va  $(-2)$  ga ko'paytirilgan birinchi tenglamani uchinchi tenglamaga hadma-had qo'shamiz;

– ikkinchi va uchinchi tenglama hadlarini mos ravishda  $7$  ga va  $(-9)$  ga bo'lamiz.

*2-bosqich.*  $x_3$  ning uchinchi tenglamadagi qiymatini birinchi va ikkinchi tenglamalarga qo'yamiz, ikkinchi tenglamadan  $x_2$  ni topamiz va uning qiymatini birinchi tenglamaga qo'yib,  $x_1$  ni topamiz.

Sistemaning yechimlarini  $x_1, x_2, x_3$  ketma-ketlikda yozamiz.

$$\begin{cases} 2x_1 - 4x_2 - x_3 = -2, \\ 3x_1 + x_2 - 2x_3 = -11, \\ x_1 - 2x_2 + 4x_3 = 8 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 + 4x_3 = 8, \\ 3x_1 + x_2 - 2x_3 = -11, \\ 2x_1 - 4x_2 - x_3 = -2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 - 2x_2 + 4x_3 = 8, \\ 7x_2 - 14x_3 = -35, \\ -9x_3 = -18 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 + 4x_3 = 8, \\ x_2 - 2x_3 = -5, \\ x_3 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = 2, \\ x_2 - 2 \cdot 2 = -5, \\ x_1 - 2x_2 + 4 \cdot 2 = 8 \end{cases} \Rightarrow \begin{cases} x_3 = 2, \\ x_2 = -1, \\ x_1 - 2 \cdot (-1) = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2, \\ x_2 = -1, \\ x_3 = 2. \end{cases}$$

Gauss usulining 1-bosqichini sistemaning o'zida emas, balki uning kengaytirilgan matritsasida bajarish qulaylikka ega.

Masalan, yuqoridagi sistemaning 1-bosqichi quyidagicha bajariladi:

$$\left( \begin{array}{cccc|c} 2 & -4 & -1 & -2 & 8 \\ 3 & 1 & -2 & -11 & \\ 1 & -2 & 4 & 8 & \end{array} \right) \begin{array}{l} r_1 \rightarrow r_3 \\ \\ r_3 \rightarrow r_1 \end{array} \sim \left( \begin{array}{cccc|c} 1 & -2 & 4 & 8 & \\ 3 & 1 & -2 & -11 & \\ 2 & -4 & -1 & -2 & \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 + (-3)r_1 \\ r_3 \rightarrow r_3 + (-2)r_1 \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & -2 & 4 & 8 & \\ 0 & 7 & -14 & -35 & \\ 0 & 0 & -9 & -18 & \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 : 7 \\ r_3 \rightarrow r_3 : (-9) \end{array} \sim \left( \begin{array}{cccc|c} 1 & -2 & 4 & 8 & \\ 0 & 1 & -2 & -5 & \\ 0 & 0 & 1 & 2 & \end{array} \right)$$

Xosmas tenglamalar sistemasini Gauss usuli bilan yechishda bu usulning boshqa bir turi *Jordan-Gauss usuli* qo'llaniladi. Bu usulda kengaytirilgan  $(A|B)$  matritsa ustida elementar almashtirishlar bajariladi va  $A$  matritsa o'rnida  $I$  matritsa hosil qilinadi, ya'ni  $u$  ( $I|X$ ) ko'rinishga keltiriladi. Bunda oxirgi kengaytirilgan matritsadagi  $X$  matritsa tenglamalar sistemasining yechimi bo'ladi.

2-misol. 
$$\begin{cases} x_1 - x_2 - x_3 = 0, \\ 5x_1 - x_2 + 4x_3 = 3, \\ x_1 + 2x_2 + 3x_3 = 5 \end{cases}$$
 sistemani Gordan-Gauss usuli bilan

yeching.

*Yechish.*

$$\left( \begin{array}{cccc|c} 1 & -1 & -1 & 0 & \\ 5 & -1 & 4 & 3 & \\ 1 & 2 & 3 & 5 & \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 + (-5)r_1 \\ r_3 \rightarrow r_3 + (-1)r_1 \end{array} \sim \left( \begin{array}{cccc|c} 1 & -1 & -1 & 0 & \\ 0 & 4 & 9 & 3 & \\ 0 & 3 & 4 & 5 & \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 : 4 \sim \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{3}{4} \\ 0 & 3 & 4 & 5 \end{array} \right) r_3 \rightarrow r_3 + (-3)r_1 \sim \left( \begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{3}{4} \\ 0 & 0 & -\frac{11}{4} & \frac{11}{4} \end{array} \right) r_3 \rightarrow r_3 : \left( -\frac{11}{4} \right) \sim$$

$$\left( \begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{3}{4} \\ 0 & 0 & 1 & -1 \end{array} \right) r_1 \rightarrow r_1 + r_3, r_2 \rightarrow r_2 + \left( -\frac{9}{4} \right) r_3 \sim \left( \begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right) r_1 \rightarrow r_1 + r_2 \sim$$

$$\sim \left( \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right)$$

Demak,  $x_1 = 2$ ,  $x_2 = 3$ ,  $x_3 = -1$ .

Shunday qilib, Gaussning noma'lumlarni ketma-ket yo'qotish usulida (4.1) sistema tenglamalarda almashtirishlar bajarish orqali yuqori uchburchak (ayrim hollarda trapetsiya) shaklga keltiriladi va hisob qilingan uchburchak sistema teskari o'rniga qo'yish orqali yechiladi. Bu usul matematik nuqtayi nazardan, sistemani uning matritsasini LU yoyish orqali yechish algoritmgiga ekvivalent.

### Gauss usulining LU yoyishga asoslangan algoritmi.

1. To'g'ri o'rniga qo'yish.  $A$  matritsaning  $A = LU$  yoyilmasi topiladi;

$$2. \text{ Teskari o'rniga qo'yish. } LUX = B \Leftrightarrow \begin{cases} LY = B, \\ UX = Y \end{cases}$$

sistema yechiladi.

$$\text{1-misol} \quad \begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 3, \\ 2x_1 + x_2 - 2x_3 + x_4 = -5, \\ x_1 - x_2 - 3x_3 + 2x_4 = -8 \end{cases} \text{ tenglamalar sistemasini } LU$$

Yechish orqali yeching.



Yechish.

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & -1 & -3 & 2 \end{pmatrix} \begin{matrix} r_2 \rightarrow r_2 + (-2)r_1 \\ r_3 \rightarrow r_3 + (-1)r_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -8 & -1 \\ 0 & -3 & -6 & 1 \end{pmatrix} \begin{matrix} r_3 \rightarrow r_3 + (-1)r_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -8 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix} = U,$$

$$\begin{pmatrix} 1 & & & \\ 2 & -3 & & \\ 1 & -3 & 2 & \\ \vdots & -3 & \vdots & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 1 & 1 & 1 & \end{pmatrix} \quad \text{Bundan } L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Avval  $LY = B$  tenglamani yechamiz:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -8 \end{pmatrix}.$$

Bundan

$$\begin{cases} y_1 = 3, \\ 2y_1 + y_2 = -5, \\ y_1 + y_2 + y_3 = -8 \end{cases} \Rightarrow \begin{cases} y_1 = 3, \\ y_2 = -5 - 2y_1 = -11, \\ y_3 = -8 - y_2 - y_1 = 0. \end{cases}$$

Endi  $UX = Y$  tenglamani yechamiz:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -8 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -11 \\ 0 \end{pmatrix},$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 3, \\ 3x_2 + 8x_3 + x_4 = 11, \\ 2x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = k, \\ x_3 = -k, \\ x_2 = \frac{11 + 7k}{3}, \\ x_1 = -\frac{13 + 8k}{3}. \end{cases}$$

Sistema impetsiyasimon ko'rinishda. Demak, u cheksiz ko'p yechimga ega. Bunda  $k$  ning tayin qiymatida sistema xususiy yechimga ega bo'ladi.

Masalan,  $k=1$  da  $x_1 = -7$ ,  $x_2 = 6$ ,  $x_3 = -1$ ,  $x_4 = 1$ .

Agar sistema  $n$  ta noma'lumdan va  $n$  ta chiziqli tenglamadan iborat bunda xotimas bo'lsa, u holda bu sistemaning yechimi LU yoyish asosida quyidagi formulalar bilan topiladi:

$$y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j, \quad i=1,2,\dots,n, \quad (4.9)$$

$$x_i = \frac{1}{u_{ii}} \left( y_i - \sum_{j=i+1}^n u_{ij} x_j \right), \quad i=n, n-1, \dots, 1. \quad (4.10)$$

*Y-misol* 
$$\begin{cases} x_1 + 3x_2 - x_3 = 0, \\ 3x_1 - x_2 + 2x_3 = 5, \\ 2x_1 - 4x_2 + x_3 = 7 \end{cases}$$
 tenglamalar sistemasini LU yoyish asosida yeching.

*Yechilish* Avval  $L$  va  $U$  matritsalarining elementlarini topamiz.

(1-raq) (3.1) va (3.2) formulalar bilan aniqlaymiz:

$$u_{11} = a_{11} = 1, \quad u_{12} = a_{12} = 3, \quad u_{13} = a_{13} = -1,$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{3}{1} = 3, \quad l_{31} = \frac{a_{31}}{u_{11}} = \frac{2}{1} = 2,$$

$$u_{22} = a_{22} - l_{21} u_{12} = -1 - 3 \cdot 3 = -10, \quad u_{23} = a_{23} - l_{21} u_{13} = 2 - 3 \cdot (-1) = 5,$$

$$l_{32} = \frac{a_{32} - l_{31} u_{12}}{u_{22}} = \frac{-4 - 2 \cdot 3}{-10} = 1,$$

$$u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = 1 - 2 \cdot (-1) - 1 \cdot 5 = -2.$$

(2-raq)  $y$  va  $x$  larni (4.9) va (4.10) formulalar bilan topamiz:

$$y_1 = b_1 = 0, \quad y_2 = b_2 - l_{21} y_1 = 5 - 3 \cdot 0 = 5, \quad y_3 = b_3 - l_{31} y_1 - l_{32} y_2 = 7 - 3 \cdot 0 - 1 \cdot 5 = 2;$$

$$x_1 = \frac{1}{u_{11}} y_1 = \frac{2}{-2} = -1, \quad x_2 = \frac{1}{u_{22}} (y_2 - u_{23} x_3) = \frac{5 - 5 \cdot (-1)}{-10} = -1,$$

$$x_3 = \frac{1}{u_{33}} (y_3 - u_{32} x_2 - u_{33} x_3) = \frac{0 - 3 \cdot (-1) - (-1) \cdot (-1)}{1} = 2.$$

Demak,  $x_1 = 2$ ,  $x_2 = -1$ ,  $x_3 = -1$ .

### 1.4.3. $n$ noma'lumli $m$ ta chiziqli tenglamalar sistemasini tekshirish va yechish

$n$  ta noma'lumdan va  $m$  ta chiziqli tenglamadan iborat (4.1) sistemani qaraymiz. Bu sistemaning matritsasi  $A$ , kengaytirilgan matritsasi  $C$  bo'lsin. Bu matritsalar mos ravishda (4.2) va (4.3) tengliklar bilan ifodalanadi. Quyida (4.1) sistema birgalikda bo'lishining zarur va yetarli shartlarini aniqlovchi teorema bilan tanishamiz. Bu teorema rus va o'zbek tilida yozilgan adabiyotlarda Kroniker-Kapelli teoremasi deb yuritiladi.

**1-teorema.** (4.1) tenglamalar sistemasi birgalikda bo'lishi uchun sistema asosiy va kengaytirilgan matritsalarining ranglari teng, ya'ni  $r(A) = r(C)$  bo'lishi zarur va yetarli.

*Isboti. Zarurligi.* (4.1) sistema birgalikda bo'lsin. Bu noma'lumlarning qandaydir tartiblangan  $x_1^0, x_2^0, \dots, x_n^0$  qiymatlari sistema tenglamalarini ayniyatga aylantirishini anglatadi.  $C$  matritsa ustida quyidagi almashtirishlarni bajaramiz: uning oxirgi ustuniga  $(-x_1^0)$  ga ko'paytirilgan birinchi ustunni qo'shamiz, keyin  $(-x_2^0)$  ga ko'paytirilgan ikkinchi ustunni qo'shamiz va shu kabi davom ettirib, oxirida  $(-x_n^0)$  ga ko'paytirilgan  $n$ -ustunni qo'shamiz. Natijada, sistema yechimining ta'rifiga ko'ra,  $C$  matritsa oxirgi ustun elementlari nollarga aylangan quyidagi  $C_1$  matritsaga o'tadi:

$$C \sim C_1 = \left( \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array} \right)$$

Elementar almashtirishlar matritsaning rangini o'zgartirmaydi, ya'ni  $r(C) = r(C_1)$  bo'ladi.  $C_1$  matritsaning oxirgi ustuni nollardan iborat bo'lgani uchun  $r(C_1) = r(A)$ . Bundan  $r(A) = r(C)$  kelib chiqadi.

*Yetariligi.*  $r(A) = r(C) = r$  bo'lsin. (4.1) sistema birgalikda ekanini ko'rsatamiz.  $A$  matritsaning noldan farqli  $r$ -tartibli minorini sistema tenglamalari va noma'lumlarining o'rinlarini almashtirish orqali chap yuqori burchakda joylashtirish mumkin. Bu minor  $C$  matritsa uchun ham minor bo'ladi. Shu sababli noma'lumlarning biror  $x_1^0, x_2^0, \dots, x_n^0$  qiymatlari dastlabki  $r$  ta tenglamani va shu bilan birga qolgan  $k$ -tenglamani ham qanoatlantiradi, bu yerda  $k > r$ . U holda (4.1) sistemada oxirgi

$m-r$  ta tenglamani tashlab yuborish va uni

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_n = b_r. \end{cases} \quad (4.11)$$

tizim bilan almashtirish mumkin. Bunda ikki hol bo'lishi mumkin:  $r=n$  yoki  $r < n$  ( $r > n$  bo'lmaydi, chunki  $A$  matritsa jami  $n$  ta ustunga ega).

$r=n$  bo'lganda (4.11) sistemaning noma'lumlari soni tenglamalari soniga teng bo'ladi. Bundan tashqari, sistema determinanti nolga teng emas. Shu sababli (4.11) sistema yagona yechimga ega bo'ladi. Bundan bu sistemaga ekvivalent (4.1) sistemaning birgalikda va aniq sistema bo'lishi kelib chiqadi.

$r < n$  da (4.12) sistemani quyidagi ko'rinishda yozish mumkin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1r}x_r = b_1 - a_{1,r+1}x_{r+1} - \dots - a_{1n}x_n, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2r}x_r = b_2 - a_{2,r+1}x_{r+1} - \dots - a_{2n}x_n, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_r = b_r - a_{r,r+1}x_{r+1} - \dots - a_{rn}x_n. \end{cases} \quad (4.12)$$

Bu sistemaning koeffitsiyentlaridan tuzilgan determinant nolga teng emas. U holda  $x_{r+1}, x_{r+2}, \dots, x_n$  noma'lumlarga biror qiymatlar berib, (4.12) sistemaning  $x_1, x_2, \dots, x_r$  yechimini topsa bo'ladi.

$x_{r+1}, x_{r+2}, \dots, x_n$  noma'lumlarga istalgan qiymatlar berish mumkin. Shu sababli (4.12) sistema cheksiz ko'p yechimga ega bo'ladi va bu sistemaga ekvivalent (4.11) sistema birgalikda va aniqmas sistema bo'ladi. Teorema to'liq isbotlandi.

Isbotlangan teoremadan quyidagi natijalar kelib chiqadi.

**1-natija.** Agar birgalikda bo'lgan sistema matritsasining rangi noma'lumlar soniga teng bo'lsa, sistema yagona yechimga ega bo'ladi.

**2-natija.** Agar birgalikda bo'lgan sistema matritsasining rangi noma'lumlar sonidan kichik bo'lsa, sistema cheksiz ko'p yechimga ega bo'ladi.

(4.1) sistema birgalikda bo'lishining zarur va yetarli shartiga va bu sistemani yechishning Gauss usuliga asosan  $n$  noma'lumli  $m$  ta chiziqli tenglamalar sistemasini tekshirish va yechish quyidagi tartibda amalga oshiriladi.

*Tekshirish (Gauss usuli)*: sistemaning kengaytirilgan matritsasi (mos ravishda asosiy matritsasi) elementar almashtirishlar yordamida pog'onasimon ko'rinishga keltiriladi (pog'onasimon matritsa – bu pog'onasimon sistemaning matritsasi).

Bunda:

- agar  $r(A) \neq r(C)$  bo'lsa, sistema birgalikda bo'lmaydi;
- agar  $r(A) = r(C) = n$ , ya'ni sistemaning rangi uning noma'lumlari soniga teng bo'lsa, sistema birgalikda va aniq bo'ladi;
- agar  $r(A) = r(C) < n$ , ya'ni sistemaning rangi uning noma'lumlari sonidan kichik bo'lsa, sistema birgalikda va aniqmas bo'ladi.

*Yechish*: Matritsaning rangi va noma'lumlari soni taqqoslanadi.

Bunda:

- agar  $r(A) = r(C) = n$  bo'lsa, ya'ni sistema bazis minorining tartibi noma'lumlar soni bilan ustma-ust tushsa,  $n$  no'malumli  $n$  ta chiziqli xosmas tenglamalar sistemasi yechiladi;
- agar  $r(A) = r(C) = r < n$  bo'lsa, sistemada  $(n-r)$  ta erkin noma'lumlar hosil qilinadi. Bunda  $x_1, x_2, \dots, x_r$  – bazis (asosiy) noma'lumlar,  $x_{r+1}, x_{r+2}, \dots, x_n$  – erkin noma'lumlar bo'ladi.

Erkin noma'lumlarni sistema tenglamalarining o'ng tomoniga o'tkazamiz:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1r}x_r = b_1 - a_{1,r+1}x_{r+1} - \dots - a_{1n}x_n, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2r}x_r = b_2 - a_{2,r+1}x_{r+1} - \dots - a_{2n}x_n, \\ \dots \\ a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_r = b_r - a_{r,r+1}x_{r+1} - \dots - a_{rn}x_n \end{cases} \quad (4.13)$$

yoki matritsa ko'rinishida

$$\widetilde{A}\widetilde{X}_r = \widetilde{B}_0 + x_{r+1}\widetilde{B}_1 + \dots + x_n\widetilde{B}_{n-r} \quad (4.14)$$

bu yerda

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{pmatrix}, \quad \det \widetilde{A} \neq 0,$$

$$\widetilde{X}_r = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_r \end{pmatrix}, \quad \widetilde{B}_0 = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_r \end{pmatrix}, \quad \widetilde{B}_1 = \begin{pmatrix} a_{1,r+1} \\ a_{2,r+1} \\ \dots \\ a_{r,r+1} \end{pmatrix}, \dots, \quad \widetilde{B}_{n-r} = \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \dots \\ a_{r,n} \end{pmatrix}$$

(4.14) sistema berilgan (4.1) sistemaga ekvivalent va uning yechimi yuqorida qaratilgan usullardan istalgan biri bilan topiladi.

Erkin yechimlar  $x_{r+1} = c_1, x_{r+2} = c_2, \dots, x_n = c_{n-r}$  qiymatlar qabul qilsin.

U holda (4.14) sistema

$$\widetilde{A}\widetilde{X}_r = \widetilde{B}_0 + c_1\widetilde{B}_1 + \dots + c_{n-r}\widetilde{B}_{n-r} \quad (4.15)$$

ko'rinishini oladi va  $x_1, x_2, \dots, x_r$  bazis noma'lumlar  $c_1, c_2, \dots, c_{n-r}$  qiymatlar orqali ma'lum ko'rinishda ifodalanadi:

$$x_i = x_i(c_1, c_2, \dots, c_{n-r}), \quad i = 1, 2, \dots, r.$$

Bu jinsi bo'lmagan  $AX = B$  sistemaning yechimini

$$X = \begin{pmatrix} x_1(c_1, c_2, \dots, c_{n-r}) \\ \dots \\ x_r(c_1, c_2, \dots, c_{n-r}) \\ c_1 \\ \dots \\ c_{n-r} \end{pmatrix} \quad (4.16)$$

ustun matritsa ko'rinishda yozish mumkin.

Erkin noma'lumlar istalgan son qiymatini qabul qilishi mumkin. Shu sababli (4.1) sistema cheksiz ko'p yechimga ega bo'ladi.

(4.16) ifoda (4.1) sistemaning umumiy yechimini aniqlaydi. Hundan o'zgarmaslarning tayin qiymatlariga sistemaning asosiy yechimlari mos keladi.

$$5\text{-misol.} \begin{cases} x_1 + 2x_2 - 4x_3 = 0, \\ 5x_1 + 3x_2 - 7x_3 = 8, \\ 5x_1 - 4x_2 + 6x_3 = -1 \end{cases} \text{ tenglamalar sistemasini tekshiring va}$$

Yeching.

*Yechish. Tekshirish.* Sistemaning kengaytirilgan matritsasi ustida elementar almashtirishlar bajaramiz:

$$C = \left( \begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 5 & 3 & -7 & 8 \\ 5 & -4 & 6 & -1 \end{array} \right) \begin{array}{l} r_2 \rightarrow r_2 + (-5)r_1 \\ r_3 \rightarrow r_3 + (-5)r_1 \end{array} \sim \left( \begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -7 & 13 & 8 \\ 0 & -14 & 26 & -1 \end{array} \right) \begin{array}{l} r_3 \rightarrow r_3 + (-2)r_2 \end{array} \sim$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -7 & 13 & 8 \\ 0 & 0 & 0 & -17 \end{array} \right)$$

$r(A) = 2 \neq 3 = r(C)$ . Demak, sistema birgalikda emas.

6-misol. 
$$\begin{cases} x_1 - 4x_2 + 2x_3 = -1, \\ 2x_1 - 3x_2 - x_3 - 5x_4 = -7, \\ 3x_1 - 7x_2 + x_3 - 5x_4 = -8 \end{cases}$$
 tenglamalar sistemasini tekshiring

va yeching.

*Yechish. Tekshirish.* Sistemaning kengaytirilgan matritsasi ustida elementar almashtirishlar bajaramiz:

$$C = \left( \begin{array}{cccc|c} 1 & -4 & 2 & 0 & -1 \\ 2 & -3 & -1 & -5 & -7 \\ 3 & -7 & 1 & -5 & -8 \end{array} \right) \begin{array}{l} \\ r_2 \rightarrow r_2 + (-2)r_1 \\ r_3 \rightarrow r_3 + (-3)r_1 \end{array}$$

$$\sim \left( \begin{array}{cccc|c} 1 & -4 & 2 & 0 & -1 \\ 0 & 5 & -5 & -5 & -5 \\ 0 & 5 & -5 & -5 & -5 \end{array} \right) \begin{array}{l} \\ \\ r_3 \rightarrow r_3 + (-1)r_2 \end{array} \sim \left( \begin{array}{cccc|c} 1 & -4 & 2 & 0 & -1 \\ 0 & 5 & -5 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \\ r_2 \rightarrow r_2 : 5 \\ \end{array}$$

$$\sim \left( \begin{array}{cccc|c} 1 & -4 & 2 & 0 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} c_2 \rightarrow c_2 + 4c_1 \\ c_3 \rightarrow c_3 + (-2)c_1 \\ c_5 \rightarrow c_5 + c_1 \end{array} \sim \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} c_3 \rightarrow c_3 + c_2 \\ c_4 \rightarrow c_4 + c_2 \\ c_5 \rightarrow c_5 + c_2 \end{array}$$

$$\sim \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Yuqorida keltirilgan  $c_i \rightarrow c_i + \lambda c_k$  belgilash  $i$ -ustun bu ustunga  $\lambda$  soniga ko'paytirilgan  $k$ -ustunni qo'shish natijasida hosil qilinganini bildiradi.

$r(A) = 2 = 2 = r(C) < 4$ . Demak, sistema birgalikda va aniqmas.

*Yechish.*  $x_1$  va  $x_2$  noma'lumlarni bazis deb olamiz va ekvivalent

systemani yozamiz:

$$\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} x_3 - \begin{pmatrix} 0 \\ -5 \end{pmatrix} x_4.$$

$x_1 = c_1$  va  $x_2 = c_2$  deb berilgan sistemaning umumiy yechimini topamiz:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5 + 2x_3 + 4x_4 \\ -1 + x_3 + x_4 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

bu yerda  $c_1, c_2$  - ixtiyoriy o'zgarmlar.

#### 1.4.4. Xosmas tenglamalar sistemasini yechish

$n$  ta noma'lumli va  $n$  ta chiziqli tenglamadan iborat (4.6) xosmas chiziqli tenglamalar sistemasini berilgan bo'lsin.

Xosmas chiziqli tenglamalar sistemasini yechishning ikkita usulini qaraymiz.

##### *Matritsalar usuli*

Bu usul (4.6) sistemaning ushbu

$$AX = B \tag{4.17}$$

matritsa ko'rinishini yechishga asoslanadi.

$A$  matritsa xosmas bo'lgani uchun  $A^{-1}$  mavjud bo'ladi.

(4.17) tenglikning har ikkala qismini chapdan  $A^{-1}$  ga ko'paytiramiz:

$$A^{-1}AX = A^{-1}B, \quad IX = A^{-1}B.$$

Bundan

$$X = A^{-1}B \tag{4.18}$$

kelib chiqadi.

(4.18) tenglamaga *chiziqli tenglamalar sistemasini matritsalar usuli bilan yechish formulasi* deyiladi.

7-misol. 
$$\begin{cases} x_1 - x_2 + x_3 = 5, \\ 2x_1 + x_2 + x_3 = 6, \\ x_1 + x_2 + 2x_3 = 4 \end{cases}$$
 tenglamalar sistemasini matritsalar

usuli bilan yeching.



*Yechish.* Sistemaning determinantini hisoblaymiz:

$$\det A = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2 - 1 + 2 - 1 + 4 - 1 = 5 \neq 0.$$

Demak, sistema – xosmas.

Determinant elementlarining algebraik to'ldiruvchilarini aniqlaymiz

$$A_{11} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \quad A_{12} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3, \quad A_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1,$$

$$A_{21} = -\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = 3, \quad A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \quad A_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2,$$

$$A_{31} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2, \quad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1, \quad A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3.$$

U holda

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 3 & -2 \\ -3 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix}.$$

Sistemaning yechimini (4.18) formula bilan topamiz:

$$X = A^{-1}B = \frac{1}{5} \begin{pmatrix} 1 & 3 & -2 \\ -3 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 + 18 - 8 \\ -15 + 6 + 4 \\ 5 - 12 + 12 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 15 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Demak,  $x_1 = 3$ ,  $x_2 = -1$ ,  $x_3 = 1$ .

### *Determinantlar usuli yoki Kramer formulalari*

Chiziqli tenglamalar sistemasini yechishning bu usuli determinantlar nazariyasiga asoslanadi.

**2-teorema.** Agar (4.6) sistema xosmas bo'lsa, u holda sistema yagona yechimga ega bo'ladi va bu yechim quyidagi formulalar bilan topiladi:

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \dots, \quad x_n = \frac{Dx_n}{D}, \quad (4.19)$$

bu yerda  $D_1, D_2, \dots, D_n$  determinantlar  $D = \det A$  determinantdan mos umumiy nomdagi koeffitsiyentlarni o'zod hadlar bilan almashtirish orqali hosil qilinadi.

Shuning (4.6) sistemaning matritsa ko'rinishidagi yechimini, ya'ni (4.5) formulani quyidagi ko'rinishda yozib olamiz:

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \frac{1}{D} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

ya'ni

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{A_{11}b_1 + A_{21}b_2 + \dots + A_{n1}b_n}{D} \\ \frac{A_{12}b_1 + A_{22}b_2 + \dots + A_{n2}b_n}{D} \\ \dots \\ \frac{A_{1n}b_1 + A_{2n}b_2 + \dots + A_{nn}b_n}{D} \end{pmatrix}$$

Demak

$$\begin{aligned} x_1 &= \frac{A_{11}b_1 + A_{21}b_2 + \dots + A_{n1}b_n}{D}, \\ x_2 &= \frac{A_{12}b_1 + A_{22}b_2 + \dots + A_{n2}b_n}{D}, \\ &\dots \\ x_n &= D \frac{A_{1n}b_1 + A_{2n}b_2 + \dots + A_{nn}b_n}{\Delta} \end{aligned}$$

bu holda shuqali

ifodasi bilan  $A_{11}b_1 + A_{21}b_2 + \dots + A_{n1}b_n$  ifoda

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

det.A matritsaning 1-ustun elementlari bo'yicha yoyilmasiga, ya'ni  $D_1$  ga teng.

Demak

$$x_1 = \frac{D_1}{D}.$$

(4.19) tenglikning qolgan formulari shu kabi isbotlanadi:

$$x_2 = \frac{D_2}{D}, \dots, x_n = \frac{Dx_n}{D}.$$

$x_i = \frac{D_i}{D}$  ( $i = \overline{1, n}$ ) formularga *Kramer formulari* deyiladi.

8-misol. 
$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = 9, \\ x_1 + 2x_2 - 3x_3 = 14, \\ 3x_1 + 4x_2 + x_3 = 16 \end{cases}$$
 tenglamalar sistemasini Kramer

formulari bilan yeching.

*Yechish.*  $D$  va  $D_i$ ,  $i=1,2,3$  determinantlarni hisoblaymiz:

$$D = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & -3 \\ 3 & 4 & 1 \end{vmatrix} = -6 \neq 0, \quad D_1 = \begin{vmatrix} 9 & 3 & 2 \\ 14 & 2 & -3 \\ 16 & 4 & 1 \end{vmatrix} = -12;$$

$$D_2 = \begin{vmatrix} 2 & 9 & 2 \\ 1 & 14 & -3 \\ 3 & 16 & 1 \end{vmatrix} = -18; \quad D_3 = \begin{vmatrix} 2 & 3 & 9 \\ 1 & 2 & 14 \\ 3 & 4 & 16 \end{vmatrix} = 12.$$

Kramer formulari bilan topamiz:

$$x_1 = \frac{D_1}{D} = \frac{-12}{-6} = 2, \quad x_2 = \frac{D_2}{D} = \frac{-18}{-6} = 3, \quad x_3 = \frac{D_3}{D} = \frac{12}{-6} = -2.$$

Shunday qilib,  $n$  noma'lumli  $n$  ta chiziqli xosmas tenglamalar sistemasi yagona yechimga ega bo'ladi va bu yechimni yoki (4.18) matritsalar usuli bilan yoki (4.19) Kramer formulari bilan topish mumkin.

### 1.4.5. Bir jinsli chiziqli tenglamalar sistemasi

Ozod hadlari nolga teng bo'lgan ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0, \\ \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases} \quad (4.20)$$

sistemaga *bir jinsli tenglamalar sistemasi* deyiladi.

(4.20) sistema asosiy va kengaytirilgan matritsalarining ranglari teng bo'ladi. Shu sababli bu sistema hamma vaqt birgalikda va nolga teng bo'lgan (trivial)  $x_1 = x_2 = \dots = x_n = 0$  yechimga ega.

Bir jinsli tenglamalar sistemasi qanday shartlar bajarilganida nolga teng bo'lmagan yechimga ega bo'ladi? Bu savolga quyidagi teoremlar javob beradi.

**3-teorema.** (4.20) tenglamalar sistemasi nolga teng bo'lmagan yechimga ega bo'lishi uchun sistema matrisasining rangi sistema tenglamalari sonidan kichik bo'lishi, ya'ni  $r < n$  bo'lishi zarur va yetarli.

*Isboti. Zarurligi.* Ma'lumki,  $r \leq n$ .  $r = n$  bo'lsin deymiz. U holda Kramer formulasidagi  $D \neq 0$  va barcha  $D_i = 0$  bo'ladi. Shu sababli tenglamalar sistemasi yagona yechimga ega bo'ladi:

$$x_i = \frac{D_i}{D} = 0, \quad D_i = 0, \quad D \neq 0.$$

Demak, (4.20) sistemaning trival yechimlardan boshqa yechimlari yo'q. Agar ular bor bo'lsa, u holda  $r < n$  bo'ladi.

*Yetaliligi.*  $r < n$  bo'lsin. U holda birgalikda bo'lgan bir jinsli sistema muqimmas bo'ladi. Demak, sistema cheksiz ko'p yechimlarga ega bo'ladi, bunda ulardan ayrimlari nolga teng bo'lmaydi.

**4-teorema.**  $n$  noma'lumli  $n$  ta chiziqli bir jinsli tenglamalar sistemasi nolga teng bo'lmagan yechimga ega bo'lishi uchun sistema matrisasining determinanti nolga teng bo'lishi, ya'ni  $\det A = 0$  bo'lishi zarur va yetarli.

*Isboti. Zarurligi.* Agar  $\det A \neq 0$  (yoki  $r = n$ ) bo'lsa, sistema yagona trival yechimga ega bo'ladi. Demak, sistema nolga teng bo'lmagan yechimga ega bo'lsa, sistemaning determinanti  $\det A = 0$  bo'ladi.

*Yetaliligi.* Agar  $\det A = 0$  bo'lsa,  $r < n$  bo'ladi. U holda 3-teoreмага ko'ra, sistema cheksiz ko'p yechimlarga ega bo'ladi, bunda ulardan ayrimlari nolga teng bo'lmaydi.

**5-teorema.** Agar  $X_1$  va  $X_2$  ustun matritsalar (4.20) sistemaning yechimlari bo'lsa, u holda bu yechimlarning chiziqli kombinatsiyasi

$$c_1 X_1 + c_2 X_2$$

ham (4.20) sistemaning yechimi bo'ladi, bu yerda  $c_1, c_2$  – istalgan haqiqiy sonlar.

*Isboti.* Teoremaning shartiga ko'ra,

$$AX_1 = 0 \quad \text{va} \quad AX_2 = 0.$$

U holda istalgan  $c_1$  va  $c_2$  sonlari uchun

$$c_1AX_1 = 0 \text{ dan } A(c_1X_1) = 0,$$

$$c_2AX_2 = 0 \text{ dan } A(c_2X_2) = 0$$

bo'ladi. Bu tengliklarni qo'shamiz:

$$A(c_1X_1) + A(c_2X_2) = 0$$

yoki

$$A(c_1X_1 + c_2X_2) = 0.$$

Demak,  $X_1$  va  $X_2$  yechimlarning  $c_1X_1 + c_2X_2$  chiziqli kombinatsiyasi (4.20) sistemaning yechimi bo'ladi. Bunda  $X_1$  va  $X_2$  yechimlar *fundamental sistema tashkil qiladi* deyiladi.

$$9\text{-misol.} \begin{cases} -x_1 + x_2 - x_3 = 0, \\ 3x_1 - x_2 - x_3 = 0, \\ 2x_1 + x_2 - 2x_3 = 0 \end{cases} \text{ bir jinsli tenglamalar sistemasini}$$

yeching.

*Yechish.* Sistema determinantini hisoblaymiz:

$$\det A = \begin{vmatrix} -1 & 1 & -1 \\ 3 & -1 & -1 \\ 2 & 1 & -2 \end{vmatrix} = -2 - 2 - 3 - 2 - 1 + 6 = -3 \neq 0.$$

Demak, sistema sistema trivial yechimga ega:  $x_1 = x_2 = x_3 = 0$ .

$$10\text{-misol.} \begin{cases} x_1 - x_2 - x_3 + x_4 = 0, \\ 2x_1 - 2x_2 + x_3 + x_4 = 0, \\ 5x_1 - 5x_2 - 2x_3 + 4x_4 = 0 \end{cases} \text{ bir jinsli tenglamalar sistemasini}$$

yeching.

*Yechish.* Sistema matritsasini pog'onasimon ko'rinishga keltiramiz:

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & -2 & 1 & 1 \\ 5 & -5 & -2 & 4 \end{pmatrix} \begin{matrix} r_2 \rightarrow r_2 + (-2)r_1 \\ r_3 \rightarrow r_3 + (-5)r_1 \end{matrix} \sim \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & -1 \end{pmatrix} \begin{matrix} \\ r_3 \rightarrow r_3 + (-1)r_2 \end{matrix} \sim$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} c_2 \rightarrow c_2 + c_1 \\ c_3 \rightarrow c_3 + c_1 \\ c_4 \rightarrow c_4 + (-1)c_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} c_2 \rightarrow c_2 \\ c_4 \rightarrow c_2 \end{matrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} c_3 \rightarrow c_3 + 3c_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} c_2 \rightarrow c_2 : (-1) \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r(A) = 2, \quad n = 4, \quad r < n.$$

Demak, sistema nolga teng bo'lmagan yechimga ega. Bunda yechimlardan ikkitasi bazis o'zgaruvchilar, qolgan ikkita yechim erkin o'zgaruvchilar bo'ladi.

$x_1 = c_1$ ,  $x_3 = c_2$  deymiz va

$$\begin{cases} x_1 - c_1 - c_2 + x_4 = 0, \\ 3c_2 - x_4 = 0 \end{cases}$$

sistemaga ega bo'lamiz.

Bu sistema  $x_4 = 3c_2$ ,  $x_1 = c_1 - 2c_2$  yechimga ega.

Berilgan sistemaning umumiy yechimini topamiz:

$$X = \begin{pmatrix} c_1 - 2c_2 \\ c_1 \\ c_2 \\ 3c_2 \end{pmatrix}$$

Bu yechimni xususiy yechimlarning chiziqli kombinatsiyasi ko'rinishiga keltiramiz:

$$X = \begin{pmatrix} c_1 \\ c_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2c_2 \\ 0 \\ c_2 \\ 3c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix}.$$

Bunda  $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$  va  $X_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix}$  xususiy yechimlar yechimlarning

fundamental sistemasini tashkil qiladi.

### 1.4.6. Chiziqli tenglamalar sistemasini matematik paketlarda yechish

Kompyuterli matematika sistemalari (KMS) yoki matematik paketlar matematikaning turli masalalarini yechishda keng qoʻllaniladi. Hozirgi vaqtda matematik paketlarning turli variantlari yaratilgan: *Maple*, *MathCAD*, *MatLAB*, *Mathematica*, *Dirveve*.

Chiziqli tenglamalar sistemasini bu paketlardan birinchisida, yaʼni *Maple* matematik paketida yechishni qisqa bayon qilamiz. Batafsil bayon maxsus kurslarda beriladi.

$Ax = b$  tenglamalar sistemasini Maple paketida ikki usuldan biri bilan yechilishi mumkin.

**1-usul:** *solve buyrugʻi bilan.*

Bu buyruq bilan (4.1) koʻrinishda berilgan chiziqli tenglamalar sistema yechiladi.

$$11\text{-misol. } \begin{cases} 2x + 6y + 5z = 0, \\ 2x + 5y + 6z = -4, \text{ tenglamalar sistemasini yeching.} \\ 5x + 7y + 8z = -7 \end{cases}$$

*Yechish.*

> eq:={2\*x + 6\*y + 5\*z = 0, 2\*x + 5\*y + 6\*z = -4, 5\*x + 7\*y + 8\*z = -7};

> solve {eq, {x,y,z}};

$$\{x=-1, y=2, z=-2\}$$

$$12\text{-misol. } \begin{cases} x - 7y + 9z = -6, \\ 2x - 3y + 5z = -3, \text{ tenglamalar sistemasining umumiy va} \\ 3x + y + z = 0 \end{cases}$$

bitta xususiy yechimini toping.

*Yechish.*

> eq:={x - 7\*y + 9\*z = -6, 2\*x - 3\*y + 5\*z = -3, 3\*x + y + z = 0};

> s:=solve {eq, {x,y,z}};

$$s := \left\{ x = -\frac{3}{11} - \frac{8}{11}z, y = \frac{9}{11} + \frac{13}{11}z, z = z \right\}.$$

Sistemaning xususiy yechimini topish uchun  $z$  oʻzgaruvchiga subs buyrugʻi bilan tayin qiymat berish kerak:

> subs {z=1,s}

$$\{x=-1, y=2, z=1\}$$

2-usul: `linsolve(A,b)` buyrug'i bilan.

Ushbu buyruq bilan `linalg` paketidan  $Ax=b$  tenglamaning yechimi topiladi. Bunda buyruqning argumenti: **A** -matritsa; **b** - vektor.

$$13-usul. \begin{cases} 2x + 7y + 13z = 0, \\ 3x + 14y + 12z = 18, \\ 5x + 25y + 16z = 39 \end{cases} \text{ tenglamalar sistemasini Gauss usuli}$$

bilan, Kramer formulalari bilan matritsalar usuli bilan yeching.

*Yechish.*

1) Sistemani Gauss usuli bilan yechamiz:

» `with(LinearAlgebra):`

`A := <<2,3,5>|<7,14,25>|<13,12,16>>;`

$$A := \begin{bmatrix} 2 & 7 & 13 \\ 3 & 14 & 12 \\ 5 & 25 & 16 \end{bmatrix}$$

» `b := <0,18,39>;`

$$b = \begin{bmatrix} 0 \\ 18 \\ 39 \end{bmatrix}$$

» `GaussianElimination(A);`

$$\begin{bmatrix} 2 & 7 & 13 \\ 0 & 7 & -15 \\ 0 & 3 & -3 \\ 0 & 0 & -\frac{3}{7} \end{bmatrix}$$

» `GaussianElimination(A,'method'='FractionFree');`

$$\begin{bmatrix} 2 & 7 & 13 \\ 0 & 7 & -15 \\ 0 & 0 & -3 \end{bmatrix}$$

» `ReducedRowEchelonForm(<A|b>);`

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Demak,

$$x := -4 \quad y := 3 \quad z := -1$$



2) Sistemani Kramer formulalari bilan yechamiz:

> with(Student[LinearAlgebra]):

> d := <<2,3,5>|<7,14,25>|<13,12,16>>;

$$d = \begin{vmatrix} 2 & 7 & 13 \\ 3 & 14 & 12 \\ 5 & 25 & 16 \end{vmatrix}$$

> d:=Determinant(d);

$$d := -3$$

> dx1:=<<0,18,39>|<7,14,25>|<13,12,16>>;

$$dx1 = \begin{vmatrix} 0 & 7 & 13 \\ 18 & 14 & 12 \\ 39 & 25 & 16 \end{vmatrix}$$

> d1:=Determinant(dx1);

$$d1 := 12$$

> dx2 := <<2,3,5>|<0,18,39>|<13,12,16>>;

$$dx2 = \begin{vmatrix} 2 & 0 & 13 \\ 3 & 18 & 12 \\ 5 & 39 & 16 \end{vmatrix}$$

> d2:=Determinant(dx2);

$$d2 := -9$$

> dx3 := <<2,3,5>|<7,14,25>|<0,18,39>>;

$$dx3 = \begin{vmatrix} 2 & 7 & 0 \\ 3 & 14 & 18 \\ 5 & 25 & 39 \end{vmatrix}$$

> d3:=Determinant(dx3);

$$d3 := 3$$

> x:=d1/d;y:=d2/d;z:=d3/d;

$$x := -4 \quad y := 3 \quad z := -1$$

3) Sistemani matritsalar usuli bilan yechamiz:

> with(Student[LinearAlgebra]):

> A := <<2,3,5>|<7,14,25>|<13,12,16>>;

$$A = \begin{pmatrix} 2 & 7 & 13 \\ 3 & 14 & 12 \\ 5 & 25 & 16 \end{pmatrix}$$

$$\Rightarrow A^{-1};$$

$$\begin{pmatrix} 76 & 213 & 98 \\ 3 & 3 & 3 \\ 12 & 33 & 15 \\ 3 & 3 & 3 \\ 5 & 15 & 7 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\Rightarrow B := \langle\langle 0, 18, 39 \rangle\rangle;$$

$$B := \begin{pmatrix} 0 \\ 18 \\ 39 \end{pmatrix}$$

$$\Rightarrow X := A^{-1} \cdot B;$$

$$X := \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}$$

`Insolve(A,b)` buyrug'ining argumenti sifatida mos ravishda  $A$  va  $B$  matritsalar olinsa, bu buyruq bilan  $AX=B$  matritsaviy tenglama yechimini ham topish mumkin.

14-misol.  $\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix} X = \begin{pmatrix} 13 & -4 & 6 \\ 2 & 4 & 2 \\ -2 & 5 & 5 \end{pmatrix}$  matritsaviy tenglamani

yeching.

*Yechish.*

$$\Rightarrow \text{with(Student[LinearAlgebra]);}$$

$$\Rightarrow A := \langle\langle 1, 0, 1 \rangle \langle 3, 1, -1 \rangle \langle -1, -2, 0 \rangle\rangle;$$

$$A := \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow A^{-1} := A^{-1};$$

$$A^{-1} := \begin{pmatrix} \frac{2}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{1}{7} & -\frac{2}{7} \\ \frac{1}{7} & -\frac{4}{7} & -\frac{1}{7} \end{pmatrix}$$

$$\Rightarrow B := \langle\langle 13, 2, -2 \rangle \langle -4, -4, 5 \rangle \langle 6, 2, 5 \rangle\rangle;$$

$$B := \begin{pmatrix} 13 & -4 & 6 \\ 2 & 4 & 2 \\ -2 & 5 & 5 \end{pmatrix}$$

$$> X := A^{(-1)} \cdot B;$$

$$X := \begin{pmatrix} 2 & 3 & 5 \\ 4 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

*A matritsaning yadrosi* deb shunday vektorlar to'plami  $x$  ga aytiladiki,  $A$  matritsaning bu vektorlar to'plamiga ko'paytmasi nolga teng, ya'ni  $Ax=0$  bo'ladi. Bunda  $A$  matritsaning yadrosini topish bir jinsli tenglamalar sistemasini yechimlarini topishga ekvivalent bo'ladi.

$A$  matritsaning yadrosini **kernel** ( $A$ ) buyrug'i bilan topish mumkin.

$$4.15\text{-misol.} \begin{cases} x + y = 0, \\ 2y - z = 0, \\ x + 3y - z = 0 \end{cases} \text{ tenglamalar sistemasini yeching.}$$

*Yechish.*

> **with(linalg):**

$$> A := \text{array} ([ [1,1,0], [0,2,-1], [1,3,-1] ])$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 1 & 3 & -1 \end{pmatrix}$$

> **kernel (A) :**

$$\{[-1,1,2]\}.$$

## 1.4.7. Mashqlar

Tenglamalar sistemasini tekshiring:

$$4.1. \begin{cases} x_1 - x_2 + x_3 = 2, \\ x_1 + x_2 - x_3 = 1, \\ 5x_1 - x_2 + x_3 = 7. \end{cases}$$

$$4.2. \begin{cases} x_1 - x_2 - x_3 = -1, \\ 5x_1 - x_2 + 2x_3 = 3, \\ 4x_1 + 3x_3 = 4. \end{cases}$$

$$4.3. \begin{cases} x_1 + x_2 + 5x_3 + 2x_4 = 1, \\ 2x_1 + x_2 + 3x_3 + 2x_4 = -3, \\ 2x_1 + 3x_2 + 11x_3 + 5x_4 = 2, \\ x_1 + x_2 + 3x_3 + 4x_4 = -3. \end{cases}$$

$$4.3. \begin{cases} x_1 + x_2 - x_3 + 2x_4 = 3, \\ 2x_1 - x_2 + x_3 - x_4 = 1, \\ 3x_1 + x_2 + 2x_3 - x_4 = 5, \\ x_1 - x_2 + 4x_3 - 5x_4 = 2. \end{cases}$$

Tenglamalar sistemasini Gauss usuli bilan yeching:

$$4.5. \begin{cases} 2x_1 + x_2 + 3x_3 = -13, \\ x_1 + 2x_2 - x_3 = -2, \\ 3x_1 + x_2 - 4x_3 = 7. \end{cases}$$

$$4.6. \begin{cases} 3x_1 + 2x_2 - 3x_3 = -1, \\ 2x_1 + x_2 + 2x_3 = 4, \\ x_1 - 3x_2 + x_3 = 9. \end{cases}$$

$$4.7. \begin{cases} x_1 + 2x_2 + x_3 - 2x_4 = -4, \\ x_2 + x_3 + 3x_4 = 1, \\ 2x_1 + x_3 - x_4 = 0, \\ 3x_1 + x_3 + 4x_4 = -2. \end{cases}$$

$$4.9. \begin{cases} 2x_1 + 3x_2 - x_3 - x_4 = 8, \\ 3x_2 + x_3 - x_4 = 8, \\ x_1 - x_2 + x_3 - x_4 = 0, \\ 3x_1 + 7x_2 - 3x_3 - x_4 = 16. \end{cases}$$

$$4.8. \begin{cases} 2x_1 + x_2 + x_4 = 4, \\ x_1 - x_2 + 2x_3 + 2x_4 = 1, \\ x_2 + 3x_3 + 2x_4 = -5, \\ 3x_1 - x_2 + 2x_3 = 3. \end{cases}$$

$$4.10. \begin{cases} x_1 - 2x_2 - 3x_3 + 5x_4 = -1, \\ 2x_1 - 3x_2 + 2x_3 + 5x_4 = -3, \\ 5x_1 - 7x_2 + 9x_3 + 10x_4 = -8, \\ x_1 - x_2 + 5x_3 = -2. \end{cases}$$

Tenglamalar sistemasini Jordan-Gauss usuli bilan yeching:

$$4.11. \begin{cases} x_1 + 6x_2 + 3x_3 = 21, \\ -4x_1 + 8x_2 + x_3 = 18, \\ 3x_1 + 5x_2 + 4x_3 = 33. \end{cases}$$

$$4.12. \begin{cases} x_1 + 3x_2 + x_3 = -2, \\ 3x_1 - 2x_2 + 3x_3 = 5, \\ 4x_1 + 3x_2 + 5x_3 = 1. \end{cases}$$

Tenglamalar sistemasini LU yoyish asosida yeching:

$$4.13. \begin{cases} 3x_1 - 7x_2 - 2x_3 = -7, \\ -3x_1 + 5x_2 + x_3 = 5, \\ 6x_1 - 4x_2 = 2. \end{cases}$$

$$4.14. \begin{cases} x_1 - 2x_2 + x_3 = 3, \\ -4x_1 + 5x_2 + 2x_3 = 0, \\ 6x_1 - 9x_2 + x_3 = 6. \end{cases}$$

$$4.15. \begin{cases} x_1 - x_2 + 2x_3 = 0, \\ x_1 - 3x_2 + x_3 = -5, \\ 3x_1 + 7x_2 + 5x_3 = 7. \end{cases}$$

$$4.16. \begin{cases} x_1 - 3x_2 + 2x_3 = 1, \\ x_1 - 2x_2 = 1, \\ 2x_2 - 3x_3 = -3. \end{cases}$$

Tenglamalar sistemasini matritsalar usuli bilan yeching:

$$4.17. \begin{cases} x_1 + 2x_2 - x_3 = 3, \\ 2x_1 - x_2 + 2x_3 = -1, \\ x_1 + 3x_2 - x_3 = 6. \end{cases}$$

$$1.4.18. \begin{cases} 2x_1 + x_2 - x_3 = 2, \\ 2x_1 + 2x_2 - 3x_3 = -3, \\ x_1 + 2x_2 - 2x_3 = -5. \end{cases}$$

$$4.19. \begin{cases} x_1 + 2x_2 + x_3 = 8, \\ x_1 + 2x_2 + 3x_3 = 10, \\ 2x_1 - 3x_2 - 4x_3 = -4. \end{cases}$$

$$1.4.20. \begin{cases} 2x_1 + 7x_2 - x_3 = 10, \\ x_1 + 2x_2 + x_3 = 2, \\ 3x_1 - 5x_2 + 3x_3 = -5. \end{cases}$$

Tenglamalar sistemasini Kramer formulari bilan yeching:

$$4.21. \begin{cases} 3x_1 - 4x_2 = 17, \\ 5x_1 + 2x_2 = 11. \end{cases}$$

$$1.4.22. \begin{cases} 5x_1 + 7x_2 = 1, \\ 6x_1 + 4x_2 = 10. \end{cases}$$

$$4.23. \begin{cases} x_1 + 2x_2 + 3x_3 = 5, \\ 3x_1 - 2x_2 + 3x_3 = -1, \\ 2x_1 + 3x_2 - 2x_3 = 8. \end{cases}$$

$$1.4.24. \begin{cases} 2x_1 - 2x_2 + x_3 = 8, \\ x_1 + 3x_2 + x_3 = -3, \\ 3x_1 + 2x_2 - 2x_3 = -5. \end{cases}$$

$$4.25. \begin{cases} x_1 + 2x_2 + 3x_3 = 6, \\ 4x_1 + 5x_2 + 6x_3 = 9, \\ 7x_1 + 8x_2 = -6. \end{cases}$$

$$1.4.26. \begin{cases} ax_1 + ax_2 + x_3 = 1, \\ x_1 + a^2x_2 + x_3 = a, \\ x_1 + ax_2 + ax_3 = 1. \end{cases}$$

Bir jinsli tenglamalar sistemasini yeching:

$$4.27. \begin{cases} 2x_1 + 3x_2 + 2x_3 = 0, \\ 3x_1 - x_2 + 3x_3 = 0. \end{cases}$$

$$4.29. \begin{cases} 3x_1 + x_2 + 2x_3 = 0, \\ x_1 + 2x_2 - 3x_3 = 0, \\ 5x_1 + 5x_2 - 4x_3 = 0. \end{cases}$$

$$4.31. \begin{cases} x_1 + 3x_2 - 6x_3 + 2x_4 = 0, \\ 2x_1 - x_2 + 2x_3 = 0, \\ 3x_1 - 2x_2 + 2x_3 - 2x_4 = 0, \\ 2x_1 + x_2 + 4x_3 + 8x_4 = 0. \end{cases}$$

$$1.4.28. \begin{cases} 3x_1 - x_2 + 4x_3 = 0, \\ 5x_1 + 3x_2 + 3x_3 = 0. \end{cases}$$

$$1.4.30. \begin{cases} 2x_1 + 3x_2 + x_3 = 0, \\ 3x_1 - 2x_2 + 3x_3 = 0, \\ 4x_1 + 3x_2 + 5x_3 = 0. \end{cases}$$

$$1.4.32. \begin{cases} x_1 - x_2 - 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 - 4x_4 = 0, \\ x_1 - 4x_2 + x_3 + 10x_4 = 0, \\ 2x_1 + x_2 - 2x_3 - x_4 = 0. \end{cases}$$

Bir jinsli tenglamalar sistemasining fundamental yechimlari sistemasini toping:

$$4.33. \begin{cases} x_1 - x_2 + x_3 - x_4 = 0, \\ 2x_1 + x_2 + x_3 + x_4 = 0, \\ 4x_1 - x_2 + 3x_3 - x_4 = 0. \end{cases}$$

$$1.4.34. \begin{cases} x_1 + 3x_2 - x_3 - x_4 = 0, \\ 2x_1 - x_2 + x_3 = 0, \\ x_1 + 10x_2 - 4x_3 - 3x_4 = 0. \end{cases}$$

Tenglamalar sistemasini Gauss usuli bilan, Kramer formulari bilan matritsalar usuli bilan Maple matematik paketida yeching:

$$4.35. \begin{cases} 5x_1 + 8x_2 - x_3 = -2, \\ x_1 + 2x_2 + 3x_3 = -4, \\ 2x_1 - 3x_2 + 2x_3 = 3. \end{cases}$$

$$1.4.36. \begin{cases} x_1 + 2x_2 + x_3 = 0, \\ 3x_1 - 5x_2 + 3x_3 = i1, \\ 2x_1 + 7x_2 - x_3 = -3. \end{cases}$$

- Vektorlar
- Vektorning ko'paymalari



*Uilyam Rouen Gamilton (1805-1865) – Irlandiyalik matematik, mexanik va fizik.*

*Gamiltonning fizika va matematika sohasidagi kasbiyotlari o'z davrida shiddatli bo'lgan va fanning bir nechta o'n yillar oq'ga siljishiga shima qilgan.*

*Gamiltonning ishlari vektorli algebraning rivojlanishiga va fanga vektor maydon tushunchasining kiritilishiga turki bo'lgan.*

*Gamilton tomonidan yaratilgan godograf tushunchasi kinematika va dinamikada tatbiq qillnadi.*

# VEKTORLI ALGEBRA ELEMENTLARI

## 2.1. VEKTORLAR

Vektor nisbatan yangi matematik tushuncha hisoblanadi. «Vektor» terminining o'zi 1845-yilda Uilyam Rouen Gamilton tomonidan kiritilgan.

Vektor tushunchasiga son qiymati va yo'nalishi bilan xarakterlanuvchi obyektlar bilan ish ko'rilganida duch kelinadi. Bunday obyektlarga kuch, tezlik, tezlanish kabi fizik kattaliklar misol bo'ladi.

Vektor matematikaning turli bo'limlarida, masalan, elementar, analitik va differensial geometriya bo'limlarida qo'llaniladi. Vektorli algebra fizika va mexanikaning turli bo'limlariga, kristallografiyaga, geodeziyaga tatbiq qilinadi. Vektorlarsiz nafaqat klassik matematikani, balki boshqa ko'plab fanlarni tasavvur qilib bo'lmaydi.

### 2.1.1. Asosiy tushunchalar

Tayin uzunlikka va yo'nalishga ega bo'lgan kesma *vektor* deb ataladi.

Vektor  $\overline{AB}$  yoki  $\vec{a}$  bilan belgilanadi. Bunda  $A$  nuqtaga vektorning boshi deyilsa,  $B$  nuqtaga uning oxiri deyiladi.

Boshi va oxiri orasidagi masofaga *vektorning uzunligi* yoki *moduli* deyiladi. Vektorning moduli  $|\overline{AB}|$  yoki  $|\vec{a}|$  ko'inishda belgilanadi.

Boshi va oxiri ustma-ust tushadigan vektor *nol vektor* deb ataladi va  $\vec{0}$  bilan belgilanadi. Bunda  $|\vec{0}|=0$  bo'ladi. Nol vektor yo'nalishga ega bo'lmaydi.

Uzunligi birga teng vektorga *birlik vektor* deyiladi va ko'p hollarda  $\vec{e}$  orqali belgilanadi.

$\vec{a}$  vektor bilan bir xil yo'nalgan birlik vektorga  *$\vec{a}$  vektorning orti* deyiladi va  $\vec{a}^0$  bilan belgilanadi.

**1-ta'rif.** Bir to'g'ri chiziqda yoki parallel to'g'ri chiziqalarda yotuvchi vektorlar *kollinear vektorlar* deb ataladi.

$\vec{a}$  va  $\vec{b}$  vektorlarning kollinearligi  $\vec{a}\parallel\vec{b}$  deb yoziladi. Kollinear vektorlar bir tomonga yo'nalgan (yo'nalishdosh, ular  $\vec{a}\uparrow\vec{b}$  kabi belgilanadi) yoki qarama-qarshi tomonlarga yo'nalgan (ular  $\vec{a}\downarrow\vec{b}$  kabi belgilanadi) bo'lishi mumkin. Hol vektor har qanday vektorga kollinear hisoblanadi.

**2-ta'rif.** Bir tekislikda yoki parallel tekisliklarda yotuvchi vektorlar *komplanar vektorlar* deb ataladi.

**3-ta'rif.**  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear, yo'nalishdosh va uzunliklari teng bo'lsa, ularga *teng vektorlar* deyiladi va  $\vec{a}=\vec{b}$  kabi yoziladi.

Vektorlar tengligining bu ta'rifi *erkli vektorlar* deb ataluvchi vektorlarni xarakterlaydi. Bu ta'rifga asosan erkli vektorni fazoning ixtiyoriy nuqtasiga parallel ko'chirish mumkin bo'ladi.

Erkli vektorlar tushunchasidan foydalanib, vektorlarning kollinearligi va komplanarligi uchun boshqa ekvivalent ta'rifi berish mumkin: agar ikkita nol bo'lmagan vektorlar bir nuqtaga ko'chirilganida bir to'g'ri chiziqda yotsa, bu vektorlarga kollinear vektorlar deyiladi; agar uchta nol bo'lmagan vektorlar bir nuqtaga ko'chirilganida bir tekislikda yotsa, bu vektorlarga komplanar vektorlar deyiladi.

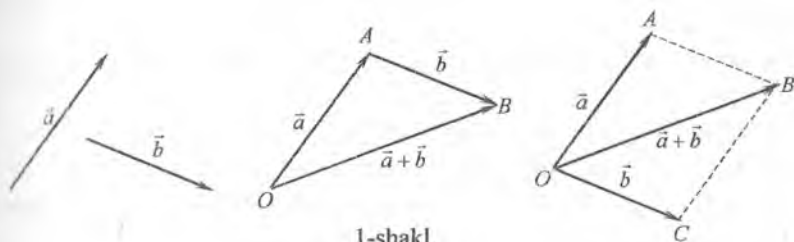
Ayrim hollarda vektorning erkli ko'chirilishi chegaralanishi mumkin. Agar vektorning qo'yilish nuqtasi qat'iy fiksirlangan bo'lsa, bu vektorga *bog'langan vektor* deyiladi. Agar vektorning boshi joylashishi mumkin bo'lgan chiziq berilgan bo'lsa, bu vektorga *sirpanuvchi vektor* deyiladi. Bog'langan va sirpanuvchi vektorlar nazariy mexanikada keng qo'llaniladi. Masalan,  $M$  nuqtaning radius vektori bog'langan vektor bo'ladi; aylanma harakatda aylanish o'qida joylashgan burchak tezlik vektori sirpanuvchi vektor bo'ladi.

## 2.1.2. Vektorlar ustida chiziqli amallar

Vektorlarni qo'shish, ayirish va vektorni songa ko'paytirish amallari *vektorlar ustida chiziqli amallar* hisoblanadi.

Ikkita  $\vec{a}$  va  $\vec{b}$  vektor berilgan bo'lsin. Istalgan  $O$  nuqta olib, bu nuqtaga  $\vec{OA} = \vec{a}$  vektorni parallel ko'chiramiz.  $A$  nuqtaga  $\vec{AB} = \vec{b}$  vektorni qo'yamiz. Bunda birinchi vektorning boshini ikkinchi vektorning oxiri bilan tutashtiruvchi  $\vec{OB}$  vektorga  $\vec{a}$  va  $\vec{b}$  vektorlarning yig'indisi deyiladi, ya'ni  $\vec{OB} = \vec{a} + \vec{b}$  (1-shakl). Vektorlarni qo'shishning bu usuli *uchburchak qoidasi* deb ataladi.

Ikkita vektorni *parallelogramm qoidasi* bilan ham qo'shish mumkin. Buning uchun  $O$  nuqtaga  $\vec{OA} = \vec{a}$  va  $\vec{OC} = \vec{b}$  vektorlarni qo'yamiz va ulardan parallelogramm yasaymiz. Bunda parallelogrammning  $O$  uchidan o'tkazilgan  $\vec{OB}$  diagonal  $\vec{a} + \vec{b}$  vektorni beradi (1-shakl).



1-shakl.

Bir nechta vektorning yig'indisini topish uchun bu vektorlarga teng vektorlardan ko'pburchak (siniq chiziq) hosil qilinadi. Bunda boshi birinchi

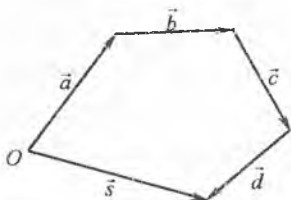
vektorining boshida, oxiri esa oxirgi vektorning oxirida bo'lgan vektor ko'pburchak barcha vektorlarining yig'indisiga teng bo'ladi. Bir nechta vektorni bunday qo'shish usuliga *ko'pburchak qoidasi* deyiladi. 2-shaklda to'rtta  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  vektorlarning yig'indisi  $\vec{s}$  vektor tasvirlangan.

$\vec{a}$  va  $\vec{b}$  vektorlarning ayirmasi deb,  $\vec{a}$  vektor bilan  $\vec{b}$  vektorga qarama-qarshi bo'lgan  $(-\vec{b})$  vektor yig'indisiga aytiladi, ya'ni  $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$ .

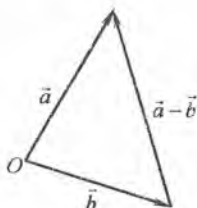
$\vec{a} - \vec{b}$  ayirmani topish uchun  $\vec{a}$  va  $\vec{b}$  vektorni umumiy  $O$  nuqtaga qo'yamiz. Bunda  $\vec{b}$  vektor oxiridan  $\vec{a}$  vektor oxiriga yo'nalgan vektor  $\vec{a} - \vec{b}$  vektorni beradi (3-shakl).



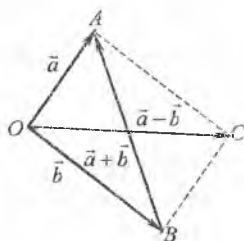
$\overline{OA} = \vec{a}$  va  $\overline{OB} = \vec{b}$  vektorlarga qurilgan  $OACB$  parallelogrammning diagonal vektorlari bu vektorlarning yig'indisidan va ayirmasidan iborat bo'ladi (4-shakl).



2-shakl.



3-shakl.



4-shakl.

$\vec{a}$  vektorning  $\lambda \neq 0$  songa ko'paytmasi deb,  $\vec{a}$  vektorga kollinear, uzunligi  $|\lambda| \cdot |\vec{a}|$  ga teng va yo'nalishi  $\lambda > 0$  bo'lganda  $\vec{a}$  vektor yo'nalishi bilan bir xil,  $\lambda < 0$  bo'lganda  $\vec{a}$  vektor yo'nalishiga qarama-qarshi bo'lgan  $\lambda\vec{a}$  vektorga aytiladi.

Vektorni songa ko'paytirishning bu ta'rifidan quyidagi xossalari kelib chiqadi:

**1-xossa.**  $\vec{a}$  ( $\vec{a} \neq 0$ ) va  $\vec{b}$  vektorlar kollinear bo'lishi uchun  $\vec{b} = \lambda\vec{a}$  bo'lishi zarur va yetarli, bu yerda  $\lambda$ —birorta son;

**2-xossa.**  $\vec{a} = |\vec{a}| \cdot \vec{a}$  ( $\vec{a} \neq 0$ ), ya'ni har bir nol bo'lmagan vektor uzunligi bilan ortining ko'paytmasiga teng bo'ladi.

### Vektorlar ustida chiziqli amallar ushbu xossalarga ega.

1°.  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ ;

2°.  $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ ;

3°.  $\vec{a} + \vec{0} = \vec{a}$ ;

4°.  $\vec{a} + (-\vec{a}) = \vec{0}$ ;

5°.  $\lambda(\mu\vec{a}) = (\lambda \cdot \mu)\vec{a}$ ;

6°.  $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$ ;

7°.  $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$ ;

8°.  $1 \cdot \vec{a} = \vec{a}$ .

Xossalarning isboti vektorlarni qo'shish va ayirish qoidalari hamda vektorning songa ko'paytirish ta'rifidan bevosita kelib chiqadi.

Misol tariqasida xossalardan birinчисini isbotlaymiz.  $\overline{AB}$  vektor  $\vec{a}$  dan va  $\overline{BC}$  vektor  $\vec{b}$  dan iborat bo'lsin (5-shakl). U holda vektorlarni qo'shishning uchburchak xossasiga ko'ra  $ABC$  uchburchakda

$$\overline{AC} = \vec{a} + \vec{b}$$

bo'ladi.

Parallelogrammning xossasiga ko'ra:

$$\overline{AD} = \overline{BC}, \quad \overline{DC} = \overline{AB}.$$

Vektorlarni qo'shishning uchburchak xossasiga ko'ra,  $ADC$  uchburchakda

$$\overline{AC} = \overline{AD} + \overline{DC} = \vec{b} + \vec{a}$$

bo'ladi.

Bundan,  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$  kelib chiqadi.

1-misol.  $ABCD$  to'g'ri to'rtburchakning tomonlari  $AB=3$ ,  $AD=4$ .  $M$  —  $DC$  tomonning o'rtasi,  $N$  —  $CB$  tomonning o'rtasi (6-shakl).  $\overline{AM}$ ,  $\overline{AN}$ ,  $\overline{MN}$  vektorlarni mos ravishda  $\overline{AB}$  va  $\overline{AD}$  tomonlar bo'ylab yo'nalgan  $\vec{i}$  va  $\vec{j}$  birlik vektorlar orqali ifodalang.

Yechish.  $\vec{a} = |\vec{a}| \cdot \vec{a}'$  bo'lishini hisobga olib, topamiz:

$$\overline{AB} = |\overline{AB}| \cdot \vec{i} = 3\vec{i}, \quad \overline{AD} = |\overline{AD}| \cdot \vec{j} = 4\vec{j}.$$

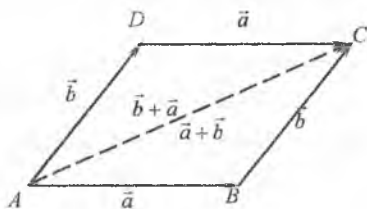
6-shaklga ko'ra,

$$\overline{DM} = \overline{MC} = \frac{1}{2}\overline{DC} = \frac{1}{2}\overline{AB} = \frac{3}{2}\vec{i}, \quad \overline{BN} = \overline{NC} = \frac{1}{2}\overline{BC} = \frac{1}{2}\overline{AD} = 2\vec{j}.$$

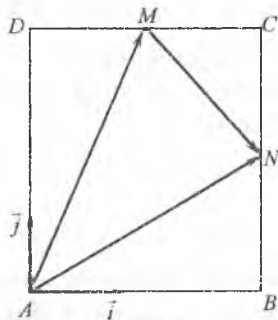
Vektorlarni qo'shish qoidasi bilan topamiz:

$$\overline{AM} = \overline{AD} + \overline{DM} = 4\vec{j} + \frac{3}{2}\vec{i}; \quad \overline{AN} = \overline{AB} + \overline{BN} = 3\vec{i} + 2\vec{j};$$

$$\overline{MN} = \overline{MC} + \overline{CN} = \overline{MC} - \overline{NC} = \frac{3}{2}\vec{i} - 2\vec{j}.$$



5-shakl.



6-shakl.

### 2.1.3. Vektorning o'qdagi proeksiyasi

Sanoq boshini aniqlovchi nuqtasi va birlik vektori berilgan to'g'ri chiziqqa *o'q* deyiladi.

$\vec{e}$  birlik vektor va  $O$  nuqta  $l$  o'qni bir qiymatli aniqlaydi.

$M$  nuqtadan  $l$  o'qqa tushirilgan  $AA_1$  perpendikulyarning  $A_1$  asosiga  $A$  nuqtaning  $l$  o'qdagi proeksiyasi deyiladi (7-shakl).

$\overline{AB}$  ( $\overline{AB} \neq 0$ ) ixtiyoriy vektor bo'lsin.  $A_1$  va  $B_1$  bilan mos ravishda  $\overline{AB}$  vektor boshi va oxirining  $l$  o'qdagi proeksiyalarini belgilaymiz.

$\overline{A_1B_1}$  vektorga  $\overline{AB}$  vektorning  $l$  o'qdagi tashkil etuvchisi deyiladi.

**4-ta'rif.**  $\overline{AB}$  vektorning  $l$  o'qdagi proeksiyasi deb  $\pm |\overline{A_1B_1}|$  songa aytiladi va  $\text{Pr}_l \overline{AB}$  bilan belgilanadi. Bu son

$\overline{A_1B_1}$  tashkil etuvchi va  $l$  o'q bir tomonga yo'nalgan bo'lsa musbat ishora bilan, aks holda manfiy ishora olinadi (7-shakl).

Demak,

$$\text{Pr}_l \overline{AB} = \pm |\overline{A_1B_1}|.$$

**5-ta'rif.**  $\vec{a}$  vektor bilan uning  $l$  o'qdagi tashkil etuvchisi  $\vec{a}_1$  vektor orasidagi  $\varphi$  burchakka  $\vec{a}$  vektor bilan  $l$  o'q orasidagi burchak (ikki  $\vec{a}$  va  $\vec{a}_1$  vektor orasidagi burchak) deyiladi. Ravshanki,  $0 \leq \varphi \leq \pi$  (8-shakl).

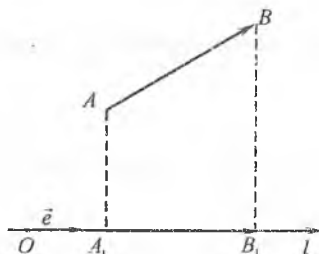
Vektorning o'qdagi proeksiyasining asosiy xossalari bilan tanishamiz.

**1-xossa.**  $\vec{a}$  vektorning  $l$  o'qdagi proeksiyasi  $\vec{a}$  vektor modulining bu vektor bilan o'q orasidagi  $\varphi$  burchak kosinusiga ko'paytmasiga teng, ya'ni

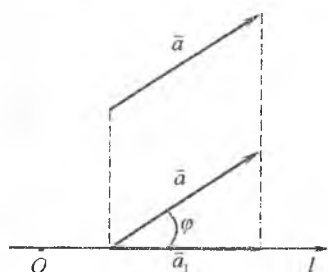
$$\text{Pr}_l \vec{a} = |\vec{a}| \cos \varphi.$$

**Isboti.** Agar  $\vec{a}$  vektor bilan  $l$  o'q orasidagi burchak o'tkir ( $0 \leq \varphi < \frac{\pi}{2}$ )

bo'lsa,  $\vec{a}_1$  tashkil etuvchi va  $l$  o'q bir tomonga yo'nalgan bo'ladi.



7-shakl.



8-shakl.

U holda

$$\text{Pr}_l \vec{a} = +|\vec{a}_1| = |\vec{a}| \cos \varphi.$$

Agar  $\vec{a}$  vektor bilan  $l$  o'q orasidagi burchak o'tmas ( $\frac{\pi}{2} < \varphi \leq \pi$ ) bo'lsa,  $\vec{a}_1$  tashkil etuvchi va  $l$  o'q qarama-qarshi tomonga yo'nalgan bo'ladi.

U holda

$$\text{Pr}_l \vec{a} = -|\vec{a}_1| = -|\vec{a}| \cos(\pi - \varphi) = |\vec{a}| \cos \varphi.$$

Agar  $\varphi = \frac{\pi}{2}$  bo'lsa, u holda

$$\text{Pr}_l \vec{a} = 0 = |\vec{a}| \cos \frac{\pi}{2} = |\vec{a}| \cos \varphi.$$

Bu xossadan quyidagi natijalar kelib chiqadi.

**1-natija.** Vektorning o'qdagi proeksiyasi:

1) vektor o'q bilan o'tkir burchak tashkil qilsa, musbat bo'ladi;

2) vektor o'q bilan o'tmas burchak tashkil qilsa, manfiy bo'ladi;

3) vektor o'q bilan to'g'ri burchak tashkil qilsa, nolga teng bo'ladi.

**2-natija.** Teng vektorlarning bitta o'qdagi proeksiyalari teng bo'ladi.

**2-xossa.** Bir nechta vektor yig'indisining berilgan o'qdagi proeksiyasi vektorlarning shu o'qdagi proeksiyalari yig'indisiga teng, masalan,

$$\text{Pr}_l(\vec{a} + \vec{b} + \vec{c}) = \text{Pr}_l \vec{a} + \text{Pr}_l \vec{b} + \text{Pr}_l \vec{c}.$$

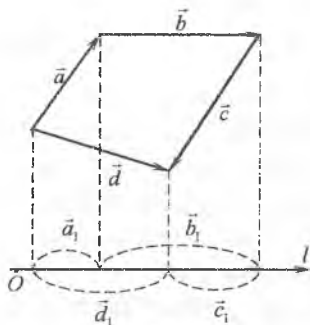
*Isboti.*  $\vec{d} = \vec{a} + \vec{b} + \vec{c}$  bo'lsin.

U holda (9-shakl)

$$\text{Pr}_l \vec{d} = +|\vec{d}_1| = +|\vec{a}_1| + |\vec{b}_1| - |\vec{c}_1|,$$

ya'ni

$$\text{Pr}_l(\vec{a} + \vec{b} + \vec{c}) = \text{Pr}_l \vec{a} + \text{Pr}_l \vec{b} + \text{Pr}_l \vec{c}.$$



9-shakl.

**3-xossa.** Vektor skalyar songa ko'paytirilsa, uning o'qdagi proektsiyasi ham shu songa ko'payadi, ya'ni

$$\text{Pr}_l(\lambda \vec{a}) = \lambda \cdot \text{Pr}_l \vec{a}.$$

*Isboti.* Vektorning o'qdagi proektsiyasining 1-xossasiga ko'ra,

$$\lambda > 0 \text{ da } \text{Pr}_l(\lambda \vec{a}) = |\lambda \vec{a}| \cos \varphi = \lambda |\vec{a}| \cos \varphi = \lambda \cdot \text{Pr}_l \vec{a}.$$

$$\lambda < 0 \text{ da } \text{Pr}_l(\lambda \vec{a}) = |\lambda \vec{a}| \cos(\pi - \varphi) = -\lambda |\vec{a}| (-\cos \varphi) = \lambda |\vec{a}| \cos \varphi = \lambda \cdot \text{Pr}_l \vec{a}.$$

$$\lambda = 0 \text{ da } \text{Pr}_l(0 \cdot \vec{a}) = 0 = 0 \cdot \text{Pr}_l \vec{a} = \lambda \cdot \text{Pr}_l \vec{a}.$$

**3-natija.** Vektorlar chiziqli kombinatsiyasining o'qdagi proektsiyasi bu vektorlar o'qdagi proektsiyalarining mos chiziqli kombinatsiyasiga teng, masalan,

$$\text{Pr}_l(k\vec{a} + n\vec{b}) = k \text{Pr}_l \vec{a} + n \text{Pr}_l \vec{b}.$$

**2-misol.**  $ABC$  to'g'ri burchakli uchburchakning katetlari  $BA = 5$ ,  $BC = 5\sqrt{3}$ .  $ABC$  uchburchak balandliklari bo'ylab yo'nalgan vektorlarning  $AC$  gipotenuza bo'ylab yo'nalgan  $l$  o'qdagi proektsiyalarini toping.

*Yechish.*

$$\angle BAC = \varphi, \angle BCA = \psi \text{ bo'lsin}$$

deylik.

U holda

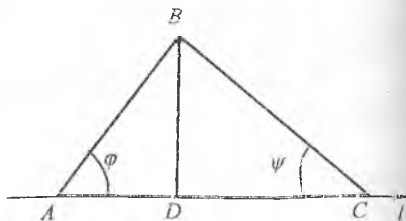
$$\cos \varphi = \frac{BA}{AC} = \frac{5}{\sqrt{5^2 + (5\sqrt{3})^2}} = \frac{1}{2}, \quad \cos \psi = \frac{BC}{AC} = \frac{5\sqrt{3}}{\sqrt{5^2 + (5\sqrt{3})^2}} = \frac{\sqrt{3}}{2}.$$

$ABC$  uchburchak balandliklari bo'ylab yo'nalgan vektorlar  $\vec{AB}$ ,  $\vec{CB}$ ,  $\vec{BD}$  bo'ladi (10-shakl). Bu vektorlarning  $l$  o'qdagi proektsiyalarini topamiz:

$$\text{Pr}_l \vec{AB} = |\vec{AB}| \cdot \cos \varphi = 5 \cdot \frac{1}{2} = \frac{5}{2},$$

$$\text{Pr}_l \vec{CB} = -|\vec{CB}| \cdot \cos \psi = -5\sqrt{3} \cdot \frac{\sqrt{3}}{2} = -\frac{15}{2},$$

$$\text{Pr}_l \vec{BD} = |\vec{BD}| \cdot \cos \frac{\pi}{2} = 0.$$



10-shakl.

#### 2.1.4. Vektorlarning chiziqli bog'liqligi. Bazis

Uch o'lchovli  $R^3$  fazoda  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar berilgan bo'lsin. Ixtiyoriy  $\alpha_1, \alpha_2, \alpha_3$  sonlarni olamiz va  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlardan

$$\vec{a} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 \quad (1.1)$$

vektorini tuzamiz. Bunda  $\vec{a}$  vektorga  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$  vektorlarning *chiziqli kombinatsiyasi*,  $\alpha_1, \alpha_2, \dots, \alpha_n$  sonlarga bu chiziqli kombinatsiyaning koeffitsiyentlari deyiladi.

**6-ta'rif.** Agar  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar uchun kamida bittasi nolga teng bo'lmagan shunday  $\alpha_1, \alpha_2, \alpha_3$  sonlar topilsa va bu sonlar uchun

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 = 0 \quad (1.2)$$

tenglik bajarilsa,  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar sistemasiga *chiziqli bog'liq vektorlar* deyiladi.

**7-ta'rif.** Agar (1.2) tenglik faqat  $\alpha_1 = \alpha_2 = \alpha_3 = 0$  bo'lganda o'rinli bo'lsa,  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar sistemasiga *chiziqli erkli vektorlar* deyiladi.

**1-teorema.**  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar chiziqli bog'liq bo'lishi uchun ulardan hech bo'lmaganida bittasi qolganlarining chiziqli kombinatsiyasidan iborat bo'lishi zarur va yetarli.

**Isbot. Zarurligi.** Berilgan vektorlar chiziqli bog'liq bo'lsin. U holda (1.2) tenglik bajariladi, bunda  $\alpha_k$  ( $k=1,2,3$ ) sonlardan kamida bittasi nolga teng bo'lmaydi.  $\alpha_3 \neq 0$  bo'lsin deylik.

U holda

$$\vec{a}_3 = -\frac{\alpha_1}{\alpha_3} \vec{a}_1 - \frac{\alpha_2}{\alpha_3} \vec{a}_2$$

bo'ladi, ya'ni  $\vec{a}_3$  vektor qolgan vektorlarning chiziqli kombinatsiyasidan iborat.

**Yetarfiligi.** Berilgan vektorlardan bittasi, masalan,  $\vec{a}_3$  qolganlarining chiziqli kombinatsiyasidan iborat bo'lsin:

$$\vec{a}_3 = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2.$$

Hundan

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + (-1) \vec{a}_3 = 0$$

kelib chiqadi, ya'ni (1.2) tenglik  $\alpha_3 = -1$  da bajariladi. Demak, berilgan vektorlar chiziqli bog'liq.

7-ta'rifga ko'ra bitta nol bo'lmagan  $\vec{a}$  vektordan iborat sistema chiziqli erkli bo'ladi, chunki  $\alpha \cdot \vec{a} = 0$  tenglik faqat va faqat  $\alpha = 0$  da bajariladi.

Demak, bitta  $\vec{a}$  vektordan iborat sistema faqat va faqat  $\vec{a} = 0$  bo'lganida chiziqli bog'liq bo'ladi.

Tekislikdagi vektorlarning chiziqli bog'liq yoki chiziqli erkli bo'lishi haqidagi masalani qaraymiz.

**2-teorema.** Ikkita  $\vec{a}_1$  va  $\vec{a}_2$  vektor chiziqli bog'liq bo'lishi uchun ular kollinear bo'lishi zarur va yetarli.

*Isboti. Zarurligi.*  $\vec{a}_1$  va  $\vec{a}_2$  vektorlar kollinear bo'lsin. U holda  $\vec{a}_2 = \lambda \vec{a}_1$  bo'ladi, bundan  $\lambda \vec{a}_1 + (-1)\vec{a}_2 = 0$  yoki  $\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 = 0$ . Demak, vektorlar chiziqli bog'liq.

*Yetarliligi.*  $\vec{a}_1$  va  $\vec{a}_2$  vektorlar chiziqli bog'liq bo'lsin. U holda  $\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 = 0$  tenglik  $\alpha_1$  va  $\alpha_2$  sonlardan kamida bittasi, masalan,  $\alpha_2$  noldan farqli bo'lganida bajariladi. Bundan  $\vec{a}_2 = -\frac{\alpha_1}{\alpha_2} \vec{a}_1$  yoki  $\vec{a}_2 = \lambda \vec{a}_1$  kelib chiqadi. Demak, vektorlar kollinear.

Bu teoremadan quyidagi natija kelib chiqadi.

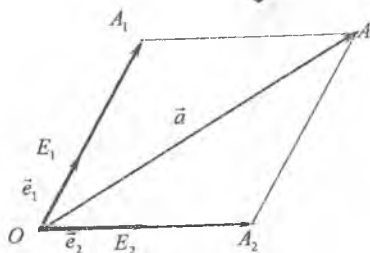
**4-natija.** Ikkita  $\vec{a}_1$  va  $\vec{a}_2$  vektor chiziqli erkli bo'lishi uchun ular kollinear bo'lmashligi zarur va yetarli.

**3-teorema.** Agar  $\vec{e}_1$  va  $\vec{e}_2$  biror tekislikning ikkita kollinear bo'lmagan vektorlari bo'lsa, u holda shu tekislikning istalgan uchinchi  $\vec{a}$  vektorini bu vektorlar bo'yicha yagona usulda

$$\vec{a} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 \quad (1.3)$$

ko'rinishda yoyish mumkin.

*Isboti.* Bitta  $O$  nuqtaga vektorlarni qo'yamiz:  $\overline{OE_1} = \vec{e}_1$ ,  $\overline{OE_2} = \vec{e}_2$ ,  $\overline{OA} = \vec{a}$ .  $A$  nuqtadan  $\vec{e}_1$  va  $\vec{e}_2$  vektorlarga parallel ikkita to'g'ri chiziq o'tkazamiz. Bu to'g'ri chiziq bilan  $\vec{e}_1$  va  $\vec{e}_2$  vektorlar yotgan to'g'ri chiziqning kesishish nuqtalarini mos ravishda  $A_1$  va  $A_2$  bilan belgilaymiz (11-shakl).



11-shakl.

Ikki vektor yig'indisi ta'rifiga ko'ra,  $\overline{OA} = \overline{OA_1} + \overline{A_1A}$ , bu yerda

$(A_1 = OA_1)$  ekani hisobga olinsa

$$\vec{a} = \overline{OA_1} + \overline{OA_2}. \quad (1.4)$$

$\vec{e}_1$  va  $\overline{OA_1}$  vektorlar kollinear bo'lgani uchun  $\overline{OA_1} = \alpha_1 \vec{e}_1$  va shu kabi  $\overline{OA_2} = \alpha_2 \vec{e}_2$  bo'ladi. Bu tengliklarga ko'ra, (1.4) tenglikdan (1.3) tenglik kelib chiqadi.

(1.3) yoyilmaning  $\alpha_1$  va  $\alpha_2$  koeffitsiyentlari bir qiymatli aniqlanishini ko'rsatamiz. Buning uchun

$$\vec{a} = \beta_1 \vec{e}_1 + \beta_2 \vec{e}_2 \quad (1.5)$$

yoyilma mavjud bo'lsin deb faraz qilamiz.

(1.5) tenglikni (1.3) tenglikdan hadma-had ayirib, topamiz:

$$(\alpha_1 - \beta_1) \vec{e}_1 + (\alpha_2 - \beta_2) \vec{e}_2 = 0.$$

Shartga ko'ra,  $\vec{e}_1$  va  $\vec{e}_2$  vektorlar kollinear emas. U holda 1-narijaga ko'ra, ular chiziqli erkli. Shu sababli oxirgi tenglik faqat  $\alpha_1 - \beta_1 = 0$  va  $\alpha_2 - \beta_2 = 0$  bo'lganida bajariladi. Bundan  $\alpha_1 = \beta_1$ ,  $\alpha_2 = \beta_2$ . Demak, (1.3) yoyilmaning  $\alpha_1$  va  $\alpha_2$  koeffitsiyentlari bir qiymatli aniqlanadi.

**4-teorema.** Tekislikdagi har qanday uchta vektor chiziqli bog'liq bo'ladi.

*Isboti.*  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlardan ikkitasi, masalan,  $\vec{a}_1$  va  $\vec{a}_2$  kollinear vektorlar bo'lsin. U holda  $\vec{a}_2 = \alpha \vec{a}_1$  yoki  $\vec{a}_2 = \alpha \vec{a}_1 + 0 \vec{a}_3$  bo'ladi. Demak,  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar chiziqli bog'liq.

$\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlardan istalgan ikkitasi kollinear bo'lmasin.

U holda 3-teoremaga ko'ra  $\vec{a}_3$  vektorni

$$\vec{a}_3 = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2,$$

ko'rtinishda yozish mumkin. Demak,  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  vektorlar chiziqli bog'liq.

2- va 4-teoremalardan quyidagi natijalar kelib chiqadi.

**5-natija.** Tekislikdagi chiziqli erkli vektorlar soni ko'pi bilan ikkiga teng.

**6-natija.** Tekislikdagi vektorlar uchun ko'rsatilgandagi kabi fazodagi vektorlar uchun quyidagi xulosalarni ko'rsatish mumkin:

1) Uchta vektor chiziqli bog'liq bo'lishi uchun ular komplanar bo'lishi zarur va yetarli;

2) Uchta vektor chiziqli erkli bo'lishi uchun ular komplanar bo'lmasligi zarur va yetarli;



3) Agar  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  vektorlar komplanar bo'lmasa, u holda istalgan  $\vec{a}$  vektor bu vektorlar bo'yicha yagona ko'rinishda yoyilishi mumkin, ya'ni

$$\vec{a} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \alpha_3 \vec{e}_3; \quad (1.6)$$

4) Uch o'lchovli fazodagi har qanday to'rtta vektor chiziqli bog'liq bo'ladi;

5) Fazodagi chiziqli erkli vektorlar soni ko'pi bilan uchga teng.

*Tekislikdagi bazis deb, istalgan ikkita chiziqli erkli vektorga aytiladi.*

5-natija va 3-teoremaga ko'ra, tekislikdagi istalgan  $\vec{a}$  vektorni  $\vec{e}_1, \vec{e}_2$  bazis bo'yicha yagona ko'rinishda yoyish mumkin, ya'ni

$$\vec{a} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2. \quad (1.7)$$

(1.7) tenglikka  $\vec{a}$  vektorning  $\vec{e}_1, \vec{e}_2$  bazis bo'yicha yoyilmasi,  $\alpha_1, \alpha_2$  sonlarga  $\vec{a}$  vektorning  $\vec{e}_1, \vec{e}_2$  bazisdagi *affin koordinatalari* deyiladi va  $\vec{a} = \{\alpha_1; \alpha_2\}$  deb yoziladi.

*Fazodagi bazis deb, istalgan uchta chiziqli erkli vektorga aytiladi.*

6-natijada keltirilgan xulosalarga ko'ra, uch o'lchovli fazodagi istalgan  $\vec{a}$  vektorni  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  bazis bo'yicha yagona ko'rinishda yoyish mumkin, ya'ni

$$\vec{a} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \alpha_3 \vec{e}_3. \quad (1.8)$$

Bunda  $\alpha_1, \alpha_2, \alpha_3$  sonlar  $\vec{a}$  vektorning  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  bazisdagi *affin koordinatalari* bo'ladi, ya'ni  $\vec{a} = \{\alpha_1; \alpha_2; \alpha_3\}$ .

*3-misol.* Uchburchakli muntazam piramidada  $AB, AC, AD - A$  uchning qirralari,  $DO - D$  uchdan tushirilgan balandlik (12-shakl). Agar  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  mos ravishda  $AB, AC, AD$  qirralar bo'ylab yo'nalgan vektorlar bo'lsin.  $\overline{DO}$  vektorning  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  bazis bo'yicha yoyilmasini toping.

*Yechish.* Vektorlarni songa ko'paytirish amalining xossasiga ko'ra:

$$\overline{AB} = \lambda_1 \vec{e}_1, \quad \overline{AC} = \lambda_2 \vec{e}_2, \quad \overline{AD} = \lambda_3 \vec{e}_3,$$

bu yerda  $\lambda_1, \lambda_2, \lambda_3$  - haqiqiy sonlar.

Piramidaning bir uchidan chiqqan qirralarida yotgan  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  vektorlar komplanar emas. Shu sababli  $\overline{DO}$  vektorini  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  bazis bo'yicha yoyish mumkin.

Piramida muntazam bo'lgani uchun uning balandligi asosining medianalari kesilish nuqtasiga tushadi, ya'ni  $O$  uchburchak medianalarining kesilish nuqtasi bo'ladi.

Vektorlarni qo'shish qoidasiga ko'ra

$$\overline{DO} = \overline{DA} + \overline{AO}.$$

Hunda

$$\overline{DA} = -\overline{AD} = -\lambda_3 \vec{e}_3,$$

$$\overline{AO} = \frac{2}{3} \overline{AM} = \frac{2}{3} \cdot \frac{\overline{AB} + \overline{AC}}{2} = \frac{1}{3} (\lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2).$$

Demak,

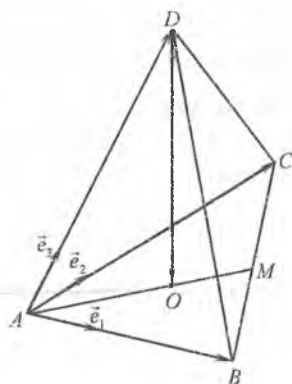
$$\overline{DO} = -\lambda_3 \vec{e}_3 + \frac{1}{3} (\lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2).$$

### 2.1.5. Dekart koordinatalar sistemasida vektorlar

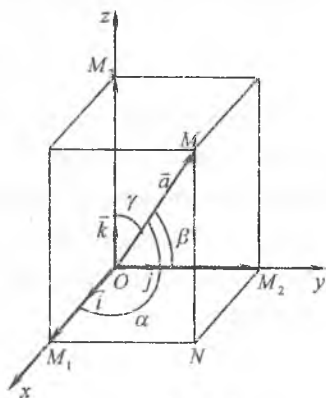
Fazodagi bazis sifatida o'zaro perpendikular bo'lgan  $\vec{i}, \vec{j}, \vec{k}$  birlik vektorlarni olaylik. Bunday bazis *ortonormallashgan bazis* deyiladi. Hunda  $\vec{i}, \vec{j}, \vec{k}$  vektorlar bazis ortlari deb ataladi.

Koordinatalar boshidan mos ravishda  $\vec{i}, \vec{j}, \vec{k}$  bazis ortlari yo'nalishida o'tkazilgan  $Ox, Oy, Oz$  o'qlarga koordinata o'qlari deyiladi.  $Ox, Oy, Oz$  o'qlardan tashkil topgan  $Oxyz$  koordinatalar sistemaga to'g'ri burchakli (yoki dekart) koordinatalar sistemasi deyiladi.

Fazoda ixtiyoriy  $\vec{a}$  vektorini olib,



12-shakl.



13-shakl.

uning boshini koordinatalar boshiga keltiramiz, ya'ni  $\vec{a} = \overline{OM}$  vektorni hosil qilamiz (13-shakl).

$\vec{a}$  vektorning koordinata o'qlaridagi proeksiyalarini topamiz. Buning uchun  $\overline{OM}$  vektorning oxiridan koordinata tekisliklariga parallel tekisliklar o'tkazamiz va ularning koordinata o'qlari bilan kesishish nuqtalarini mos ravishda  $M_1, M_2, M_3$  orqali belgilaymiz. Bu tekisliklar koordinata tekisliklari bilan birgalikda diagonallaridan biri  $\overline{OM}$  vektor bo'lgan to'g'ri burchakli parallelepipedni hosil qiladi.

Bunda

$$\text{Pr}_x \vec{a} = |\overline{OM}_1|, \text{Pr}_y \vec{a} = |\overline{OM}_2|, \text{Pr}_z \vec{a} = |\overline{OM}_3|.$$

Bir nechta vektorlarni qo'shish qoidasiga ko'ra,

$$\vec{a} = \overline{OM}_1 + \overline{M_1N} + \overline{NM}.$$

Yoki  $\overline{M_1N} = \overline{OM}_2$ ,  $\overline{NM} = \overline{OM}_3$  ni hisobga olsak,

$$\vec{a} = \overline{OM}_1 + \overline{OM}_2 + \overline{OM}_3. \quad (1.9)$$

Shuninddek,

$$\overline{OM}_1 = |\overline{OM}_1| \cdot \vec{i}, \overline{OM}_2 = |\overline{OM}_2| \cdot \vec{j}, \overline{OM}_3 = |\overline{OM}_3| \cdot \vec{k}. \quad (1.10)$$

$\vec{a} = \overline{OM}$  vektorning koordinata o'qlaridagi proeksiyalarini mos ravishda  $a_x, a_y$  va  $a_z$  orqali belgilaymiz, ya'ni  $|\overline{OM}_1| = a_x$ ,  $|\overline{OM}_2| = a_y$ ,  $|\overline{OM}_3| = a_z$ .

U holda (1.9) va (1.10) tengliklardan topamiz:

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}. \quad (1.11)$$

(1.11) tenglik  $\vec{a}$  vektorning  $\vec{i}, \vec{j}, \vec{k}$  bazis bo'yicha yoyilmasi deb yuritiladi.  $a_x, a_y, a_z$  sonlarga  $\vec{a}$  vektorning *dekart koordinatalari* (yoki oddiygina *koordinatalari*) deyiladi va  $\vec{a} = \{a_x; a_y; a_z\}$  kabi yoziladi.

$\vec{a} = \{a_x; a_y; a_z\}$  vektor berilgan bo'lsin.

To'g'ri burchakli parallelepipedning diagonali haqidagi teoremani qo'llaymiz:

$$|\overline{OM}|^2 = |\overline{OM}_1|^2 + |\overline{OM}_2|^2 + |\overline{OM}_3|^2,$$

Bundan

$$|\vec{a}|^2 = a_x^2 + a_y^2 + a_z^2$$

yoki

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \quad (1.12)$$

ya'ni vektorning uzunligi uning koordinatalari kvadratlarning yig'indisining kvadrat ildiziga teng.

$\vec{a}$  vektorning  $Ox, Oy, Oz$  o'qlar bilan tashkil qilgan burchaklari mos ravishda  $\alpha, \beta, \gamma$  bo'lsin (13-shakl).  $\cos \alpha, \cos \beta, \cos \gamma$  lar  $\vec{a}$  vektorning yo'naltiruvchi kosinuslari deb ataladi.

Vektorning o'qdagi proektsiyasi 1-xossasidan topamiz:

$$a_x = |\vec{a}| \cos \alpha, \quad a_y = |\vec{a}| \cos \beta, \quad a_z = |\vec{a}| \cos \gamma.$$

Bundan  $\vec{a} \neq 0$  bo'lganida

$$\cos \alpha = \frac{a_x}{|\vec{a}|}, \quad \cos \beta = \frac{a_y}{|\vec{a}|}, \quad \cos \gamma = \frac{a_z}{|\vec{a}|}. \quad (1.13)$$

(1.13) tenglikdan (1.12) formulani hisobga olib, topamiz:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \quad (1.14)$$

ya'ni nol bo'lmagan vektorning yo'naltiruvchi kosinuslari kvadratlarning yig'indisi birga teng.

Bundan  $\vec{a}^0$  birlik vektorning koordinatalari  $\cos \alpha, \cos \beta, \cos \gamma$  bo'lishi kelib chiqadi, ya'ni  $\vec{a}^0 = \{\cos \alpha; \cos \beta; \cos \gamma\}$ .

*4-misol.* Uzunligi  $|\vec{a}| = 2$  ga teng vektor  $Ox, Oy$  koordinata o'qlari bilan  $\alpha = 60^\circ, \beta = 120^\circ$  li burchaklar tashkil qiladi.  $\vec{a}$  vektorning koordinatalarini toping.

*Yechish.* Vektorning o'qdagi proyeksiyasining 1-xossasisiga ko'ra:

$$a_x = |\vec{a}| \cos \alpha = 2 \cos 60^\circ = 2 \cdot \frac{1}{2} = 1;$$

$$a_y = |\vec{a}| \cos \beta = 2 \cos 120^\circ = 2 \cdot \left(-\frac{1}{2}\right) = -1.$$

Vektorning uzunligini topamiz:

$$2 = \sqrt{1 + 1 + a_z^2}.$$

Bundan

$$a_z^2 = 2 \quad \text{yoki} \quad a_z = \sqrt{2}, \quad a_z = -\sqrt{2}.$$

Demak,

$$\bar{a} = \{1; -1; \sqrt{2}\} \text{ yoki } \bar{a} = \{1; -1; -\sqrt{2}\}.$$

$\bar{a}$  va  $\bar{b}$  vektorlar koordinatalari bilan berilgan bo'lsin:

$$\bar{a} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k}, \quad \bar{b} = b_x \bar{i} + b_y \bar{j} + b_z \bar{k}.$$

Vektorning o'qdagı proeksiyasining xossalariğa ko'ra:

$$\bar{a} \pm \bar{b} = (a_x \pm b_x) \bar{i} + (a_y \pm b_y) \bar{j} + (a_z \pm b_z) \bar{k}, \quad (1.15)$$

$$\lambda \bar{a} = \lambda a_x \bar{i} + \lambda a_y \bar{j} + \lambda a_z \bar{k}, \quad (1.16)$$

$$\bar{a} = \bar{b} \Leftrightarrow \begin{cases} a_x = b_x, \\ a_y = b_y, \\ a_z = b_z. \end{cases} \quad (1.17)$$

Shunday qilib,

1) vektorlar qo'shilganida (ayrilganida) ularning mos koordinatalari qo'shiladi (ayriladi);

2) vektor songa ko'paytirilganida uning barcha koordinatalari shu songa ko'payadi;

3) teng vektorlarning mos koordinatalari teng bo'ladi va aksincha.

*5-misol.*  $\bar{a} = -4\bar{i} - 2\bar{j} + 4\bar{k}$  vektor berilgan. Bu vektorga qarama-qarshi yo'nalgan, kollinear va uzunligi  $|\bar{b}| = 9$  bo'lgan vektorning koordinatalarini toping.

*Yechish.*  $\bar{b}$  vektorning koordinatalari  $b_x, b_y, b_z$ , ya'ni  $\bar{b} = \{b_x; b_y; b_z\}$  bo'lsin.

$\bar{a}$  va  $\bar{b}$  vektorlar kollinear bo'lsa  $\bar{a} = \lambda \bar{b}$  bo'ladi, bu yerda  $\lambda$  - biror son.

U holda ikki vektorning tengligi shartidan

$$b_x = \lambda a_x, \quad b_y = \lambda a_y, \quad b_z = \lambda a_z$$

yoki

$$b_x = -4\lambda, \quad b_y = -2\lambda, \quad b_z = 4\lambda.$$

Bu koordinatalarni va  $\bar{b}$  vektorning uzunligini hisobga olib, topamiz:

$$\sqrt{16\lambda^2 + 4\lambda^2 + 16\lambda^2} = 9, \quad \pm 6\lambda = 9 \quad \text{yoki} \quad \lambda = \pm \frac{3}{2}.$$

$\vec{a}$  va  $\vec{b}$  vektorlar qarama-qarshi yo'nalgani uchun  $\lambda < 0$ , ya'ni

$$\lambda = -\frac{1}{2}$$

Demak,  $\vec{b} = \{6; 3; -6\}$ .

Koordinatlar boshiga qo'yilgan va oxiri  $M$  nuqta bo'lgan  $\vec{r} = \overline{OM}$  vektorga  $M$  nuqtaning radius vektori deyiladi.  $M(x; y; z)$  nuqta radius vektorining koordinatalari  $r = \{x; y; z\}$  bo'ladi.

Boshi  $A(x_1; y_1; z_1)$  nuqtada va oxiri  $B(x_2; y_2; z_2)$  nuqtada bo'lgan  $\overline{AB}$  vektorni qaraymiz.  $A$  va  $B$  nuqtalarning radius vektorlarini mos ravishda  $r_1 = \{x_1; y_1; z_1\}$  va  $r_2 = \{x_2; y_2; z_2\}$  bo'ladi.

14-shaklga ko'ra,  $\overline{AB} = \vec{r}_2 - \vec{r}_1$ .  
 Bundan (1.15) tenglikka asosan

$$\overline{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

yoki

$$\overline{AB} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}. \quad (1.18)$$

Shunday qilib, vektorning koordinatalari uning oxiri va boshining mos koordinatalari ayimasiga teng.

(1.18) tenglikdan  $\overline{AB}$  vektorning modulini topamiz:

$$|\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \quad (1.19)$$

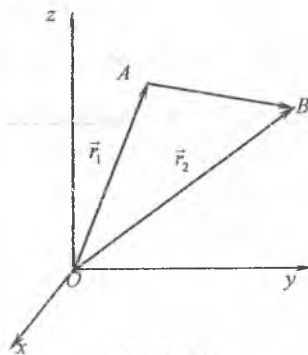
$\overline{AB}$  vektorning uzunligi  $A$  va  $B$  nuqtalar orasidagi masofani aniqlaydi. Shu sababli (1.19) tenglikka *ikki nuqta orasidagi masofani topish formulasi* deyiladi.

**6-misol.** Parallelogrammning uchta ketma-ket uchi berilgan:  $A(-1; 3; 1)$ ,  $B(-2; -5; 3)$ ,  $C(0; -1; 1)$ .  $BD$  diagonal uzunligini toping.

**Yechish.** Parallelogramm  $D$  uchning  $x; y; z$  koordinatalarini parallelogramm uchun  $\overline{AD} = \overline{BC}$  ekanidan aniqlaymiz.  $\overline{AD}$  va  $\overline{BC}$  vektorlarni (1.18) formula ko'rinishida yozamiz:

$$\overline{AD} = \{x + 1; y - 3; z - 1\},$$

$$\overline{BC} = \{0 - (-2); -1 - (-5); 1 - 3\} = \{2; 4; -2\}.$$



14-shakl.

U holda  $\overline{AD} = \overline{BC}$  tenglikdan  $x+1=2$ ,  $y-3=4$ ,  $z-1=-2$ , ya'ni  $D(1;7;-1)$  bo'lishi kelib chiqadi.  $\overline{BD}$  vektorning koordinatalarini topamiz:

$$\overline{BD} = \{1 - (-2); 7 - (-5); -1 - 3\} = \{3; 12; -4\}.$$

U holda

$$|\overline{BD}| = \sqrt{3^2 + 12^2 + (-4)^2} = \sqrt{9 + 144 + 16} = 13.$$

$A(x_1; y_1; z_1)$  va  $B(x_2; y_2; z_2)$  nuqtalarni tutashtiruvchi kesma berilgan bo'lsin.  $AB$  kesmani berilgan  $\lambda > 0$  nisbatda bo'luvchi, ya'ni  $\frac{AC}{CB} = \lambda$  tenglikni ta'minlaydigan  $C(x; y; z)$  nuqtani topish masalasini yechamiz.

Masalaning shartiga ko'ra,  $\overline{AC}$  va  $\overline{CB}$  vektorlar kollinear, ya'ni  $\overline{AC} = \lambda \overline{CB}$ .

U holda

$$\overline{AC} = \{x - x_1; y - y_1; z - z_1\}, \quad \overline{CB} = \{x_2 - x; y_2 - y; z_2 - z\}$$

inobatga olinsa,

$$x - x_1 = \lambda(x_2 - x), \quad y - y_1 = \lambda(y_2 - y), \quad z - z_1 = \lambda(z_2 - z).$$

Bu tengliklardan  $x, y, z$  larni topamiz:

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z = \frac{z_1 + \lambda z_2}{1 + \lambda}. \quad (1.20)$$

(1.20) formulaga kesmani berilgan  $\lambda$  nisbatda bo'lish formulasi deyiladi.

(1.20) tengliklardan  $\lambda = 1$  da kesma o'rtasining koordinatalarini topish formulalari kelib chiqadi:

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}. \quad (1.21)$$

**I-izoh.** Tekislikda yotuvchi  $AB$  kesma uchun (1.20) va (1.21) formulalar quyidagi ko'rinishlarni oladi:

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad (1.22)$$

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}. \quad (1.23)$$

*misol*  $A(2;5)$  va  $B(4;9)$  nuqtalarni tutashtiruvchi  $AB$  kesmani  $(\lambda + \mu = 1)$  nisbatda bo'luvchi  $C$  nuqtaning koordinatalarini toping.

*Yechish* Masalaning shartiga ko'ra,  $\lambda = \frac{1}{3}$ ,  $C(x; y)$  nuqtani

(1.12) formulalar bilan topamiz:

$$x = \frac{2 + \frac{1}{3} \cdot 4}{1 + \frac{1}{3}} = \frac{5}{2}, \quad y = \frac{5 + \frac{1}{3} \cdot 9}{1 + \frac{1}{3}} = 6, \quad \text{yoki } C\left(\frac{5}{2}; 6\right).$$

## 2.1.6. Mashqlar

1. Agar  $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$  bo'lsa,  $\vec{a}$  va  $\vec{b}$  vektorlar qanday shartni qanoqlantiradi?

2.  $|\vec{a}| = 13$ ,  $|\vec{b}| = 19$  va  $|\vec{a} + \vec{b}| = 24$  bo'lsa,  $|\vec{a} - \vec{b}|$  ni hisoblang.

3. Agar  $M$  —  $AB$  kesmaning o'rtasi bo'lsa, fazoning ixtiyoriy  $O$  nuqtasi uchun  $\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$  tenglikni isbotlang.

4.  $AD, BE, CF$  kesmalar  $ABC$  uchburchakning medianalari bo'lsa,  $\vec{AD} + \vec{BE} + \vec{CF} = \vec{0}$  tenglikni isbotlang.

5.  $AN$  —  $ABC$  uchburchakning bisektrisasi.  $\vec{AB} = \vec{a}$ ,  $\vec{AC} = \vec{b}$ ,  $|\vec{a}| = 2$ ,  $|\vec{b}| = 1$  bo'lsa,  $\vec{AN}$  vektorini toping.

6.  $ABCD$  teng yonli trapetsiyada  $\angle DAB = 60^\circ$ ,  $|AD| = |DC| = |CB| = 2$ ,  $M, N$  mos ravishda  $DC$  va  $BC$  tomonlarning o'rtalari.  $\vec{NM}$  vektorini mos o'qishda  $\vec{AB}$  va  $\vec{AD}$  tomonlar bo'ylab yo'nalgan  $\vec{m}$  va  $\vec{n}$  birlik vektorlar orqali ifodalang.

7.  $O$  nuqtada  $\vec{OA} = \vec{a}$  va  $\vec{OB} = \vec{b}$  vektorlar qo'yilgan.  $AOB$  burchak bisektrisasida yotuvchi biror  $\vec{OM}$  vektorini toping.

8.  $ABCD$  parallelogrammda  $\vec{AB} = \vec{a}$  va  $\vec{AD} = \vec{b}$ ,  $M$  — parallelogramm diagonallarining kesishish nuqtasi.  $\vec{MA}$ ,  $\vec{MB}$  vektorlarni  $\vec{a}$  va  $\vec{b}$  vektorlar orqali ifodalang.

9.  $m$  ning qanday qiymatida  $\vec{c} = \vec{a} - m\vec{b}$  va  $\vec{d} = -\sqrt{3}\vec{a} + 6\vec{b}$  vektorlar kollinear bo'ladi?

10. Biror bazisda  $\vec{a} = \{m; -1; 2\}$ ,  $\vec{b} = \{3; n; 6\}$  vektorlar berilgan.  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear bo'lsa  $m$  va  $n$  ni toping.



11.  $ABCD$  to'g'ri burchakli trapetsiya asoslari  $|AB|=4$  va  $|CD|=2$  va  $\angle ABC=45^\circ$ .  $\overline{AB}$ ,  $\overline{AD}$ ,  $\overline{DC}$ ,  $\overline{AC}$  vektorlarning  $\overline{CB}$  vektor bilan aniqlanuvchi o'qqa proyeksiyalarini toping.

12.  $ABC$  teng tomonli uchburchakning tomonlari  $2\sqrt{3}$  ga teng,  $AD, BF, CE$  - uchburchakning balandliklari.  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CA}$ ,  $\overline{AD}$ ,  $\overline{BF}$ ,  $\overline{CE}$  vektorlarning  $\angle BAC$  burchak bissiktrisasi bo'ylab yo'nalgan  $l$  o'qqa proyeksiyalarini toping.

13.  $ABCD$  parallelogrammda  $P$  va  $Q$  mos ravishda  $BC$  va  $CD$  tomonlarning o'rtalari. Bazis vektorlar  $\vec{e}_1 = \overline{AD}$  va  $\vec{e}_2 = \overline{AB}$  bo'lsa,  $\overline{PQ}$  vektorni bu bazis bo'yicha yoying.

14.  $OACB$  to'g'ri to'rtburchakda  $M$  va  $N$  mos ravishda  $BC$  va  $AC$  tomonlarning o'rtasi.  $\overline{OC}$  vektorning  $\overline{OM} = \vec{a}$  va  $\overline{ON} = \vec{b}$  vektorlar bo'yicha bazisga yoying.

15.  $ABCD$  piramidada  $P$  va  $Q$  mos ravishda  $AD$  va  $BC$  qirralarning o'rtasi. Bazis vektorlar  $\vec{e}_1 = \overline{AB}$ ,  $\vec{e}_2 = \overline{AC}$  va  $\vec{e}_3 = \overline{AD}$  bo'lsa,  $\overline{PQ}$  vektorni bu bazis bo'yicha yoying.

16.  $OABC$  tetraedr berilgan.  $\overline{OA}, \overline{OB}, \overline{OC}$  vektorlardan iborat bo'lgan bazisda  $\overline{OF}$  vektorni toping, bu yerda  $F$  - asos medianalarining kesishish nuqtasi.

17. Tekislikda uchta  $\vec{a} = \{3; -2\}$ ,  $\vec{b} = \{-2; 1\}$  va  $\vec{c} = \{7; -4\}$  vektorlar berilgan. Har bir vektorning qolgan ikki vektor bo'yicha yoyilmasini toping.

18.  $\vec{a} = \{2; 1; 0\}$ ,  $\vec{b} = \{1; -1; 2\}$ ,  $\vec{c} = \{2; 2; -1\}$  vektorlar bazis tashkil qilishini ko'rsating.  $\vec{d} = \{3; 7; -7\}$  vektorning shu bazis bo'yicha yoyilmasini toping.

19.  $\vec{a} = \{-1; 5; -2\}$  va  $\vec{b} = \{2; -1; 3\}$  vektor berilgan.  $3\vec{a} - 2\vec{b}$  va  $-\frac{1}{3}\vec{a} + \frac{2}{3}\vec{b}$  vektorlarning koordinata o'qlaridagi proeksiyalarini toping.

20.  $\vec{a} = \{1; -1; 2\}$ ,  $\vec{b} = \{-2; -3; 1\}$  va  $\vec{c} = \{0; -3; -2\}$  vektorlar berilgan.  $-2\vec{a} + 3\vec{b} - \vec{c}$  va  $3\vec{a} - 2\vec{b} + 2\vec{c}$  vektorlarning koordinata o'qlaridagi proeksiyalarini toping.

21. Tomonlari  $\vec{a} = \{-1; 0; 7\}$  va  $\vec{b} = \{5; -4; -5\}$  vektorlarga qurilgan parallelogramm diagonallarining uzunliklarini toping.

22.  $\overline{AB} = \{2; 6; 4\}$  va  $\overline{AC} = \{4; 2; -2\}$  bo'lsa,  $ABC$  uchburchakning  $CP$  medianasi uzunligini toping.

23. Fazoda  $M$  nuqtaning radius vektori koordinata o'qlari bilan bir xil burchak tashkil qiladi va uzunligi 3 ga teng.  $M$  nuqtaning koordinatalarini toping.

24.  $\vec{a}$  vektor  $OX$  va  $OZ$  o'qlari bilan mos ravishda  $60^\circ$  va  $120^\circ$  li burchak tashkil qiladi. Agar  $|\vec{a}| = 4$  bo'lsa, bu vektorning koordinatalarini toping.

15. Agar  $\vec{a} = \{2; -1; 1\}$  vektorning boshi  $A(3; -2; -4)$  bo'lsa, uning oxirining koordinatalarini toping.

16. Agar  $\vec{a} = \{2; 4; -1\}$  vektorning oxiri  $B(-1; 3; -4)$  bo'lsa, uning boshining koordinatalarini toping.

17.  $A$  va  $B$  nuqtalar berilgan.  $\overline{AB}$  vektorning ortini toping:

- 1)  $A(-4; -9; 6)$ ,  $B(8; 6; -10)$ ;                      2)  $A(6; -1; 9)$ ,  $B(2; -4; -3)$ .

18.  $A(2; -1; 0)$ ,  $B(1; -1; 2)$ ,  $C(0; 5; 3)$  nuqtalar berilgan.  $\vec{a} = \overline{AB} - \overline{CB}$  vektorning ortini toping.

19.  $\vec{a} = \{2; 3\}$ ,  $\vec{b} = \{1; -3\}$ ,  $\vec{c} = \{-1; 3\}$  vektorlar berilgan.  $\alpha$  ning qanday qiymatlarida  $\vec{m} = \vec{a} + \alpha\vec{b}$  va  $\vec{n} = \vec{a} + 3\vec{c}$  vektorlar kollinear bo'ladi.

20.  $\vec{a} = 16\vec{i} - 12\vec{j} + 15\vec{k}$  vektor bilan bir xil yo'nalgan va uzunligi  $|\vec{b}| = 15$  bo'lgan vektorning koordinatalarini toping.

21. Uchlari  $A(-1; -3)$ ,  $B(2; -3)$ ,  $C(2; 1)$  bo'lgan uchburchakning perimetrini toping.

22. Uchlari  $A(-3; -3)$ ,  $B(-1; 3)$ ,  $C(1; -1)$  bo'lgan uchburchakning to'g'ri burchakli ekanini ko'rsating.

23. Uchlari  $A(2; 1)$ ,  $B(-1; 1)$ ,  $C(-3; 2)$  nuqtalarda bo'lgan uchburchakka tashqi chizilgan aylananing markazini va radiusini toping.

24.  $M_1(-1; -2)$  va  $M_2(3; 4)$  nuqtalar berilgan.  $M_1M_2$  to'g'ri chiziqda yotuvchi va  $O_1$  nuqtaga nisbatan  $M_1$  nuqtaga 3 barobar yaqin bo'lgan  $M(x; y)$  nuqtani toping.

25. Uchlari  $A(1; 4)$ ,  $B(-5; 0)$ ,  $C(-2; -1)$  nuqtalarda bo'lgan uchburchak medianalarining kesishish nuqtasini toping.

26. Parallelogrammning uchta ketma-ket  $A(-6; -4)$ ,  $B(-4; 8)$ ,  $C(-1; 5)$  uchlari berilgan. Parallelogrammning to'rtinchi uchini toping.

27. Uchlari  $A(4; 1; -3)$ ,  $B(1; 4; -2)$ ,  $C(1; 10; -8)$  nuqtalarda bo'lgan  $ABC$  uchburchakning  $AD$  medianasi uzunligini toping.

## 2.2. VEKTORLARNING KO'PAYTMALARI

### 2.2.1. Ikki vektorning skalyar ko'paytmasi

#### *Skalyar ko'paytmaning ta'rifi*

**1-ta'rif.** Ikki  $\vec{a}$  va  $\vec{b}$  vektorning skalyar ko'paytmasi deb bu vektorlar uzunliklari bilan ular orasidagi burchak kosinusi ko'paytmasi

teng songa aytiladi va  $u \vec{a}\vec{b}$  (yoki  $\vec{a} \cdot \vec{b}$  yoki  $(\vec{a}, \vec{b})$ ) kabi belgilanadi, ya'ni

$$\vec{a}\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi, \quad (2.1)$$

bu yerda  $\varphi$  –  $\vec{a}$  va  $\vec{b}$  vektorlar orasidagi burchak (bunda vektorlarning boshi bir nuqtaga qo'yiladi).

(2.1) formulani boshqa ko'rinishda yozish mumkin.

Ma'lumki,

$$\text{Pr}_b \vec{a} = |\vec{a}| \cos\varphi \quad \text{va} \quad \text{Pr}_a \vec{b} = |\vec{b}| \cos\varphi.$$

Bundan

$$\vec{a}\vec{b} = |\vec{a}| \cdot \text{Pr}_b \vec{b} \quad (2.2)$$

yoki

$$\vec{a}\vec{b} = |\vec{b}| \cdot \text{Pr}_a \vec{a}, \quad (2.3)$$

ya'ni ikki vektorning skalyar ko'paytmasi ulardan birining moduli bilan ikkinchisining birinchi vektor yo'nalishidagi o'qqa proeksiyasining ko'paytmasiga teng.

### *Skalyar ko'paytmaning xossalari*

*1-xossa. Ko'paytuvchilarning o'rin almashtirish xossasi:*

$$\vec{a}\vec{b} = \vec{b}\vec{a}.$$

*Isboti.*  $\vec{a}\vec{b} = |\vec{a}| \cdot |\vec{b}| \cos(\vec{a}, \vec{b}) = |\vec{b}| \cdot |\vec{a}| \cos(\vec{b}, \vec{a}) = \vec{b}\vec{a}.$

*2-xossa. Skalyar ko'paytuvchiga nisbatan guruhlash xossasi:*

$$(\lambda\vec{a})\vec{b} = \lambda(\vec{a}\vec{b}).$$

*Isboti.* (2.2) formulaga ko'ra,  $(\lambda\vec{a})\vec{b} = |\vec{b}| \cdot \text{Pr}_b(\lambda\vec{a})$ . Vektorning o'qdagi proeksiyasining 3-xossasiga asosan  $\text{Pr}_b(\lambda\vec{a}) = \lambda \cdot \text{Pr}_b \vec{a}$ .

Bundan

$$(\lambda\vec{a})\vec{b} = |\vec{b}| \cdot \text{Pr}_b(\lambda\vec{a}) = \lambda \cdot |\vec{b}| \cdot \text{Pr}_b \vec{a} = \lambda \cdot (|\vec{b}| \text{Pr}_b \vec{a}) = \lambda(\vec{a}\vec{b}).$$

*3-xossa. Qo'shishga nisbatan taqsimot xossasi:*

$$\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}.$$

*Isboti.* Vektorning o'qdagi proeksiyasining 2-xossasiga ko'ra,

$$\text{Pr}_a(\vec{b} + \vec{c}) = \text{Pr}_a \vec{b} + \text{Pr}_a \vec{c}.$$

Demak,

$$\begin{aligned} \vec{a}(\vec{b} + \vec{c}) &= |\vec{a}| \cdot \text{Pr}_{\vec{a}}(\vec{b} + \vec{c}) = |\vec{a}| \cdot (\text{Pr}_{\vec{a}} \vec{b} + \text{Pr}_{\vec{a}} \vec{c}) = \\ &= |\vec{a}| \cdot \text{Pr}_{\vec{a}} \vec{b} + |\vec{a}| \cdot \text{Pr}_{\vec{a}} \vec{c} = \vec{a}\vec{b} + \vec{a}\vec{c}. \end{aligned}$$

**4-xossa.** Agar  $\vec{a}$  va  $\vec{b}$  vektorlar perpendikular bo'lsa, u holda ularning skalyar ko'paytmasi nolga teng bo'ladi. Shuningdek, teskari ta'rif ham o'rinli: agar  $\vec{a}\vec{b} = 0$  ( $|\vec{a}| \neq 0, |\vec{b}| \neq 0$ ) bo'lsa, u holda  $\vec{a} \perp \vec{b}$  bo'ladi.

$$\text{Nusxasi: } \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = \vec{j} \cdot \vec{i} = \vec{k} \cdot \vec{j} = \vec{i} \cdot \vec{k} = 0.$$

*Ishboti.*  $\vec{a} \perp \vec{b}$  bo'lganda  $\cos \varphi = 0$  bo'ladi. Bundan  $\vec{a}\vec{b} = 0$ .

$\vec{a}\vec{b} = 0$  ( $|\vec{a}| \neq 0, |\vec{b}| \neq 0$ ) bo'lsa,  $\cos \varphi = 0$  bo'ladi. Bundan  $\varphi = \frac{\pi}{2}$ , ya'ni  $\vec{a} \perp \vec{b}$ .

**5-xossa.** Vektorning skalyar kvadrati uning uzunligi kvadratiga teng, ya'ni  $\vec{a}^2 = |\vec{a}|^2$ .

$$\text{Nusxasi: } \vec{i}^2 = \vec{j}^2 = \vec{k}^2 = 1.$$

*Ishboti.*  $\vec{a}^2 = \vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \cos 0^\circ = |\vec{a}| \cdot |\vec{a}| = |\vec{a}|^2$ .

*Izoh.* Agar  $\vec{a}$  vektorni skalyar kvadratga oshirib, keyin kvadrat daraja chiqarilsa,  $\vec{a}$  vektorning o'zi emas, balki uning moduli hosil bo'ladi, ya'ni

$$\sqrt{\vec{a}^2} = |\vec{a}| (\sqrt{\vec{a}^2} \neq \vec{a}).$$

*1-misol.*  $|\vec{a}| = 4$ ,  $|\vec{b}| = 6$ ,  $\varphi = (\vec{a}, \vec{b}) = \frac{\pi}{3}$  bo'lsin.  $(3\vec{a} - \vec{b}) \cdot (2\vec{a} + 4\vec{b})$

ko'paytmani toping.

*Yechish.* Avval 3-xossa yordamida qavslarni ochamiz va keyin skalyar ko'paytmaning ta'rifini va xossalardan foydalanib, topamiz:

$$\begin{aligned} (\vec{a} - \vec{b}) \cdot (2\vec{a} + 4\vec{b}) &= 3\vec{a} \cdot 2\vec{a} - \vec{b} \cdot 2\vec{a} + 3\vec{a} \cdot 4\vec{b} - \vec{b} \cdot 4\vec{b} = 6\vec{a}^2 + 10\vec{a}\vec{b} - 4\vec{b}^2 = \\ &= 6|\vec{a}|^2 + 10|\vec{a}| \cdot |\vec{b}| \cos \frac{\pi}{3} - 4|\vec{b}|^2 = \\ &= 6 \cdot 4^2 + 10 \cdot 4 \cdot 6 \cdot \frac{1}{2} - 4 \cdot 6^2 = 96 + 120 - 144 = 72. \end{aligned}$$

*2-misol.*  $|\vec{a}| = 4$ ,  $|\vec{b}| = 3$ ,  $\varphi = (\vec{a}, \vec{b}) = \frac{2\pi}{3}$  bo'lsin. Bu vektorlarga qurilgan parallelogramm diagonallarining uzunliklarini toping.

*Yechish.*  $\vec{a}$  va  $\vec{b}$  vektorlarga qurilgan parallelogramm diagonallarini  $\vec{a} + \vec{b}$  va  $\vec{a} - \vec{b}$  vektorlar orqali ifodalash mumkin.

Skalyar ko'paytmaning xossalaridan foydalanib, topamiz:

$$\begin{aligned}
 |\vec{a} + \vec{b}| &= \sqrt{(\vec{a} + \vec{b})^2} = \sqrt{\vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2} = \sqrt{|\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\varphi + |\vec{b}|^2} = \\
 &= \sqrt{16 + 2 \cdot 4 \cdot 3 \cdot \left(-\frac{1}{2}\right) + 9} = \sqrt{13}, \\
 |\vec{a} - \vec{b}| &= \sqrt{(\vec{a} - \vec{b})^2} = \sqrt{\vec{a}^2 - 2\vec{a}\vec{b} + \vec{b}^2} = \sqrt{|\vec{a}|^2 - 2|\vec{a}||\vec{b}|\cos\varphi + |\vec{b}|^2} = \\
 &= \sqrt{16 + 2 \cdot 4 \cdot 3 \cdot \frac{1}{2} + 9} = \sqrt{37}.
 \end{aligned}$$

### **Koordinatalari bilan berilgan vektorlarning skalyar ko'paytmasi**

Ikkita  $\vec{a} = \{a_x; a_y; a_z\}$  va  $\vec{b} = \{b_x; b_y; b_z\}$  vektor berilgan bo'lsin.

U holda bu vektorlarni  $\vec{i}, \vec{j}, \vec{k}$  birlik vektorlar orqali ifodalab skalyar ko'paytmaning xossalarini va  $\vec{i}, \vec{j}, \vec{k}$  vektorlarning skalyar ko'paytmalarini hisobga olib, topamiz:

$$\begin{aligned}
 \vec{a}\vec{b} &= (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \cdot (b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) = a_x b_x \vec{i}\vec{i} + a_x b_y \vec{i}\vec{j} + a_x b_z \vec{i}\vec{k} + \\
 &+ a_y b_x \vec{j}\vec{i} + a_y b_y \vec{j}\vec{j} + a_y b_z \vec{j}\vec{k} + a_z b_x \vec{k}\vec{i} + a_z b_y \vec{k}\vec{j} + a_z b_z \vec{k}\vec{k} = \\
 &= a_x b_x + a_y b_y + a_z b_z.
 \end{aligned}$$

Demak,

$$\vec{a}\vec{b} = a_x b_x + a_y b_y + a_z b_z, \quad (2.4)$$

ya'ni koordinatalari bilan berilgan ikki vektorning skalyar ko'paytmasi ularning mos koordinatalari ko'paytmalarining yig'indisiga teng.

*3-misol.*  $\vec{a} = \{4; -2; 3\}$ ,  $\vec{b} = \{1; -2; 0\}$ ,  $\vec{c} = \{2; 1; -3\}$  bo'lsin.

$(\vec{a} + 3\vec{b}) \cdot (\vec{a} - \vec{b} + \vec{c})$  ko'paytmani toping.

*Yechish.* Avval  $\vec{m} = \vec{a} + 3\vec{b}$  va  $\vec{n} = \vec{a} - \vec{b} + \vec{c}$  vektorlarning koordinatalarini topamiz:

$$\vec{m} = \{4 + 3 \cdot 1; -2 + 3 \cdot (-2); 3 + 3 \cdot 0\} = \{7; -8; 3\},$$

$$\vec{n} = \{4 - 1 + 2; -2 + 2 + 1; 3 - 0 - 3\} = \{5; 1; 0\}.$$

Bundan (2.4) formulaga ko'ra

$$\vec{m} \cdot \vec{n} = 7 \cdot 5 + (-8) \cdot 1 + 3 \cdot 0 = 27.$$

## Skalyar ko'paytmaning ayrim tatbiqlari

### 1. Ikki vektor orasidagi burchak

$\vec{a} = \{a_x, a_y, a_z\}$  va  $\vec{b} = \{b_x, b_y, b_z\}$  vektorlar orasidagi  $\varphi$  burchak kosinusini (2.1) va (2.4) tengliklardan topamiz:

$$\cos \varphi = \frac{\vec{a}\vec{b}}{|\vec{a}| \cdot |\vec{b}|} \quad (2.5)$$

yoki

$$\cos \varphi = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}. \quad (2.6)$$

Shu kabi, fazodagi ikki yo'nalish orasidagi burchak kosinusi bu yo'nalishlarning mos (bir nomdagi) yo'naltiruvchi kosinuslari ko'paytmalarining yig'indisiga teng:

$$\cos \varphi = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2. \quad (2.7)$$

### 2. Ikki vektorning perpendikulyarlik sharti

$\vec{a} \perp \vec{b}$  bo'lsin. U holda  $\cos \varphi = 0$  bo'lgani uchun (2.6) tenglikdan

$$a_x b_x + a_y b_y + a_z b_z = 0 \quad (2.8)$$

kelib chiqadi.

Fazodagi ikki yo'nalishlarning perpendikulyarlik shartini (2.7) tenglikdan topamiz:

$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0. \quad (2.9)$$

### 3. Vektorning berilgan yo'nalishdagi proeksiyasi

(2.3) tenglikdan topamiz:

$$\text{Pr}_{\vec{b}} \vec{a} = \frac{\vec{a}\vec{b}}{|\vec{b}|} \left( \text{Pr}_{\vec{a}} \vec{b} = \frac{\vec{a}\vec{b}}{|\vec{a}|} \right) \quad (2.10)$$

yoki

$$\text{Pr}_{\vec{b}} \vec{a} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{b_x^2 + b_y^2 + b_z^2}} \left( \text{Pr}_{\vec{a}} \vec{b} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \right). \quad (2.11)$$

Shu kabi  $\vec{a}(x;y;z)$  vektorining yo'nalishdagi (o'qdag) proeksiyasi:  $\cos\alpha, \cos\beta, \cos\gamma$  bo'lgan  $l$  yo'nalishdagi (o'qdag) proeksiyasi:

$$\text{Pr}_l \vec{a} = x \cos\alpha + y \cos\beta + z \cos\gamma. \quad (2.12)$$

#### 4. Kuchning bajargan ishi

$\vec{MN}$  vektor bilan  $\varphi$  burchak tashkil etuvchi  $\vec{F}$  kuch ta'sirida moddiy nuqta  $M$  nuqtadan  $N$  nuqtaga to'g'ri chiziq bo'ylab ko'chayotgan bo'lsin (15-shakl).

Fizika kursidan ma'lumki,  $\vec{F}$  kuchning  $\vec{MN} = \vec{S}$  ko'chishdagi bajargan ishi

$$A = |\vec{F}| \cdot |\vec{S}| \cdot \cos\varphi \quad \text{yoki} \quad A = \vec{F}\vec{S} \quad (2.13)$$

formula bilan aniqlanadi.

Demak, moddiy nuqtaning to'g'ri chizikli harakatida o'zgarmas kuchning bajargan ishi kuch vektori va ko'chish vektorining skalyar ko'paytmasiga teng. Bu jumla skalyar ko'paytmaning mexanik ma'nosini anglatadi.

4-misol. Moddiy nuqta  $A(1;-2;2)$  nuqtadan  $B(5;-5;-3)$  nuqtaga  $\vec{F} = \{2;-1;-3\}$  kuch ta'sirida to'g'ri chiziq bo'ylab ko'chgan.

Quyidagilarni toping: 1)  $\vec{F}$  kuchning bajargan ishini; 2)  $\vec{F}$  kuchning ko'chish yo'nalishidagi proeksiyasini; 3)  $\vec{F}$  kuchning ko'chish yo'nalishi bilan tashkil qilgan burchagini.

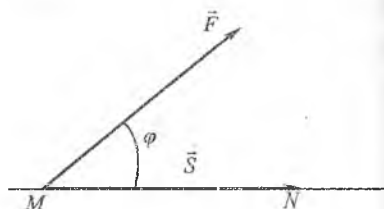
*Yechish.* Avval moddiy nuqta ko'chish vektorini, uning va  $\vec{F}$  kuchning uzunligini topamiz:

$$\vec{S} = \vec{AB} = \{4;-3;-5\}, \quad |\vec{S}| = \sqrt{16+9+25} = 5\sqrt{2}, \quad |\vec{F}| = \sqrt{4+1+9} = \sqrt{14}.$$

U holda:

$$1) A = \vec{F}\vec{S} = 2 \cdot 4 + (-1) \cdot (-3) + (-3) \cdot (-5) = 26 \quad (\text{ish b.});$$

$$2) \text{Pr}_S \vec{F} = \frac{\vec{F}\vec{S}}{|\vec{S}|} = \frac{26}{5\sqrt{2}} = \frac{13\sqrt{2}}{5};$$



15-shakl.

$$\cos \varphi = \frac{FS}{|F||S|} = \frac{26}{5\sqrt{2} \cdot \sqrt{14}} = \frac{13\sqrt{7}}{35}, \quad \varphi = \arccos \frac{13\sqrt{7}}{35}.$$

1) misol  $\vec{m} = \vec{a} + 2\vec{b}$  va  $\vec{n} = 5\vec{a} - 4\vec{b}$  o'zaro perpendikular vektorlar bo'lsa,  $\vec{a}$  va  $\vec{b}$  birlik vektorlar orasidagi burchakni toping.

**Yechish:**  $\vec{m} \perp \vec{n}$  bo'lgani uchun  $(\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 4\vec{b}) = 0$  bo'ladi.

Himlan

$$5\vec{a} \cdot \vec{a} + 6\vec{a} \cdot \vec{b} - 8\vec{b} \cdot \vec{b} = 0 \quad \text{yoki} \quad 5|\vec{a}|^2 + 6|\vec{a}| \cdot |\vec{b}| \cos \varphi - 8|\vec{b}|^2 = 0.$$

$\vec{a}$  va  $\vec{b}$  birlik vektorlar bo'lgani sababli:  $5 + 6\cos \varphi - 8 = 0$ .

U holda

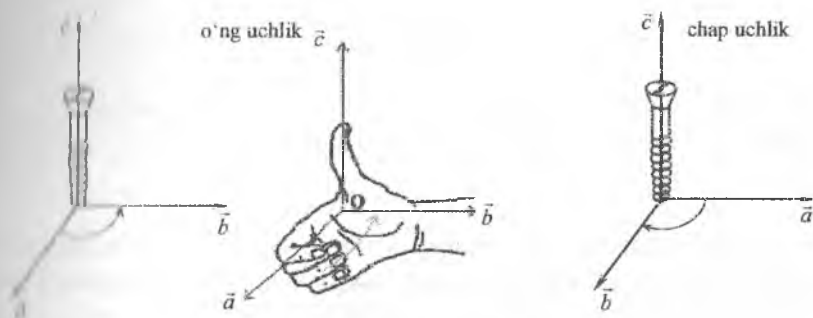
$$\cos \varphi = \frac{1}{2} \quad \text{yoki} \quad \varphi = \frac{\pi}{3}.$$

## 2.2.2. Ikki vektorning vektor ko'paytmasi

### Vektor kopaytmaning ta'rifi

Agar uchta vektordan qaysi biri birinchi, qaysi biri ikkinchi va qaysi biri uchinchi ekani ko'rsatilgan bo'lsa, bu vektorlarga tartiblangan uchlik deyiladi.

Tartiblangan uchlikda vektorlar joylashish tartibida yoziladi.



16-shakl.

Agar komplanar bo'lmagan vektorlar tartiblangan uchligining uchinchi vektori uchidan qaralganda birinchi vektordan ikkinchi vektorga qisqa burilish soat strelkasi yo'nalishiga teskari bo'lsa, bunday



uchlikka o'ng uchlik, agar soat strelkasi yo'nalishida bo'lsa chap uchlik deyiladi (16-shakl).

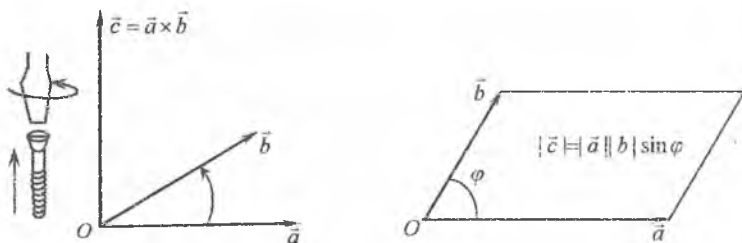
**2-ta'rif.**  $\vec{a}$  vektorning  $\vec{b}$  vektorga vektor ko'paytmasi deb quyidagi shartlar bilan aniqlanadigan  $\vec{c}$  vektorga aytiladi (17-shakl):

- 1)  $\vec{c}$  vektor  $\vec{a}$  va  $\vec{b}$  vektorlarga perpendikular, ya'ni  $\vec{c} \perp \vec{a}$  va  $\vec{c} \perp \vec{b}$ ;
- 2)  $\vec{c}$  vektorning uzunligi son jihatidan tomonlari  $\vec{a}$  va  $\vec{b}$  vektorlardan iborat bo'lgan parallelogrammning yuziga teng, ya'ni

$$|\vec{c}| = |\vec{a}| \cdot |\vec{b}| \sin \varphi, \text{ bu yerda } \varphi = (\vec{a}, \vec{b});$$

- 3)  $\vec{a}, \vec{b}, \vec{c}$  vektorlar o'ng uchlik tashkil qiladi.

$\vec{a}$  va  $\vec{b}$  vektorlarning vektor ko'paytmasi  $\vec{a} \times \vec{b}$  yoki  $[\vec{a}, \vec{b}]$  kabi belgilanadi.



17-shakl.

### Vektor ko'paytmaning xossalari

**1-xossa.** Ko'paytuvchilarning o'rinlari almashtirilsa vektor ko'paytma ishorasini qarama-qarshisiga o'zgartiradi, ya'ni

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

**Isboti.** Vektor ko'paytmaning ta'rifiga ko'ra,  $\vec{a} \times \vec{b}$  va  $\vec{b} \times \vec{a}$  vektorlar bir xil uzunlikka ega (parallelogrammning yuzi o'zgarmaydi), kollinear, ammo qarama-qarshi yo'nalgan, chunki  $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$  vektorlar ham,  $\vec{b}, \vec{a}, \vec{b} \times \vec{a}$  vektorlar ham o'ng uchlik tashkil qiladi.

Demak,

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

**2-xossa.** Skalyar ko'paytuvchiga nisbatan guruhlash xossasi:

$$(\lambda \vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}).$$

*Isboti.*  $\lambda > 0$  bo'lsin. U holda  $(\lambda\vec{a}) \times \vec{b}$  va  $\lambda(\vec{a} \times \vec{b})$  vektorlar  $\vec{a}$  va  $\vec{b}$  vektorlarga perpendikular bo'ladi, chunki  $\vec{b}$ ,  $\lambda\vec{a}$  va  $\vec{a}$  vektorlar bir tekislikda yotadi. Shu sababli  $(\lambda\vec{a}) \times \vec{b}$  va  $\lambda(\vec{a} \times \vec{b})$  vektorlar kollinear. Shuningdek, bu vektorlar yo'nalishdosh ( $\lambda\vec{a}$  va  $\vec{a}$  vektorlar yo'nalishdosh) hamda ular bir xil uzunlikka ega:

$$|(\lambda\vec{a}) \times \vec{b}| = |\lambda\vec{a}| \cdot |\vec{b}| \sin(\widehat{(\lambda\vec{a}, \vec{b})}) = \lambda |\vec{a}| \cdot |\vec{b}| \sin(\widehat{\vec{a}, \vec{b}}),$$

$$|\lambda(\vec{a} \times \vec{b})| = \lambda |\vec{a} \times \vec{b}| = \lambda |\vec{a}| \cdot |\vec{b}| \sin(\widehat{\vec{a}, \vec{b}}).$$

Damak,

$$(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}).$$

Xotira  $\lambda < 0$  da ham shu kabi isbotlanadi.

*4-xavssa.*  $(\vec{a})$  'shishga nisbatan taqsimot xossasi:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

Bu xossaning isbotini keltirmaymiz.

*4-xavssa.* Agar  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear bo'lsa, u holda ularning vektor ko'paytmasi nolga teng bo'ladi. Shuningdek, teskari tasdiq o'rinli: agar  $\vec{a} \times \vec{b} = 0$  ( $|\vec{a}| \neq 0, |\vec{b}| \neq 0$ ) bo'lsa, u holda  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear bo'ladi.

*Isboti.*  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear bo'lsa, ular orasidagi burchak  $\varphi = 0^\circ$  yoki  $\varphi = 180^\circ$  ga teng va  $\sin \varphi = 0$  bo'ladi. U holda  $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin \varphi = 0$ . Bundan

$$\vec{a} \times \vec{b} = 0.$$

$\vec{a} \times \vec{b} = 0$  bo'lsa,  $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin \varphi = 0$  bo'ladi. U holda  $|\vec{a}| \cdot |\vec{b}| \neq 0$  bo'lgani uchun  $\sin \varphi = 0$ . Bundan  $\varphi = 0^\circ$  yoki  $\varphi = 180^\circ$ , ya'ni  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear.

*6-misol.*  $\vec{i}, \vec{j}, \vec{k}$  vektorlarning vektor ko'paytmalarini toping.

*Yechish.* Bunda vektor ko'paytmaning ta'rifidan quyidagi tengliklar bevosita kelib chiqadi:

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}.$$

Haqiqatan ham, masalan,  $\vec{i} \times \vec{j} = \vec{k}$  tenglik o'rinli, chunki:

$$1) \vec{k} \perp \vec{i}, \quad \vec{k} \perp \vec{j};$$

$$2) |\vec{k}| = |\vec{i}| \cdot |\vec{j}| \sin 90^\circ = 1; \quad 3) \vec{i}, \vec{j}, \vec{k} \text{ vektorlar o'ng uchlik tashkil qiladi.}$$

Shuningdek, 1- xossaga ko'ra,

$$\vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j}.$$

Vektor ko'paytmaning 4-xossasidan topamiz:

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0.$$

7-misol.  $|\vec{a}|=3, |\vec{b}|=2, \varphi = (\vec{a}, \vec{b}) = \frac{\pi}{6}$  bo'lsin.  $|(\vec{a} + 2\vec{b}) \times (\vec{a} - 3\vec{b})|$  ni

hisoblang.

*Yechish.* Vektor ko'paytmaning ta'rifi va xossalariidan foydalanib, topamiz:

$$(\vec{a} + 2\vec{b}) \times (\vec{a} - 3\vec{b}) = \vec{a} \times \vec{a} + 2\vec{b} \times \vec{a} - 3\vec{a} \times \vec{b} - 6\vec{b} \times \vec{b} = -5\vec{a} \times \vec{b}.$$

Bundan

$$|(\vec{a} + 2\vec{b}) \times (\vec{a} - 3\vec{b})| = |-5\vec{a} \times \vec{b}| = 5|\vec{a}| \cdot |\vec{b}| \sin \varphi = 5 \cdot 3 \cdot 2 \cdot \sin \frac{\pi}{6} = 15.$$

### **Koordinatalari bilan berilgan vektorlarning vektor ko'paytmasi**

Ikkita  $\vec{a} = \{a_x; a_y; a_z\}$  va  $\vec{b} = \{b_x; b_y; b_z\}$  vektor berilgan bo'lsin.

$\vec{i}, \vec{j}, \vec{k}$  vektorlarning vektor ko'paytmalari formulalaridan foydalanib, topamiz:

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \times (b_x \vec{i} + b_y \vec{j} + b_z \vec{k}) = a_x b_x (\vec{i} \times \vec{i}) + a_x b_y (\vec{i} \times \vec{j}) + a_x b_z (\vec{i} \times \vec{k}) + \\ &+ a_y b_x (\vec{j} \times \vec{i}) + a_y b_y (\vec{j} \times \vec{j}) + a_y b_z (\vec{j} \times \vec{k}) + a_z b_x (\vec{k} \times \vec{i}) + a_z b_y (\vec{k} \times \vec{j}) + a_z b_z (\vec{k} \times \vec{k}) = \\ &= a_x b_y \vec{k} - a_x b_z \vec{j} - a_y b_x \vec{k} + a_y b_z \vec{i} + a_z b_x \vec{j} - a_z b_y \vec{i} = (a_y b_z - a_z b_y) \vec{i} - (a_x b_z - a_z b_x) \vec{j} + \\ &+ (a_x b_y - a_y b_x) \vec{k} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k}, \end{aligned}$$

ya'ni

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k}. \quad (2.14)$$

Oxirgi tenglikni quyidagicha yozish mumkin:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \quad (2.15)$$

*Misol*  $\vec{a} = \{3; -1; -2\}, \vec{b} = \{0; -2; 4\}$  bo'lsin.  $(\vec{a} + 2\vec{b}) \times (2\vec{a} - 3\vec{b})$  ko'paytmani toping.

*Yechish* Avval  $\vec{m} = \vec{a} + 2\vec{b}$  va  $\vec{n} = 2\vec{a} - 3\vec{b}$  vektorlarning koordinatalarini topamiz:

$$\vec{m} = \{1 \cdot 3 + 2 \cdot 0; 1 \cdot (-1) + 2 \cdot (-2); 1 \cdot (-2) + 2 \cdot 4\} = \{3; -5; 6\},$$

$$\vec{n} = \{2 \cdot 3 - 3 \cdot 0; 2 \cdot (-1) - 3 \cdot (-2); 2 \cdot (-2) - 3 \cdot 4\} = \{6; -8; -16\}.$$

Bundan (2.14) formulaga ko'ra

$$\vec{m} \times \vec{n} = \begin{vmatrix} -5 & 6 \\ -8 & -16 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 6 \\ 6 & -16 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -5 \\ 6 & -8 \end{vmatrix} \vec{k} = 128\vec{i} + 84\vec{j} + 6\vec{k}.$$

### Vektor ko'paytmaning ayrim tatbiqlari

#### 1. Ikki vektorning kollinearlik sharti

Vektor ko'paytmaning 4-xossasiga ko'ra,  $\vec{a}$  va  $\vec{b}$  vektorlar kollinear bo'lsa

$$\vec{a} \times \vec{b} = 0$$

yoki

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\vec{i} - (a_x b_z - a_z b_x)\vec{j} + (a_x b_y - a_y b_x)\vec{k} = 0$$

bo'ladi.

Bundan

$$a_y b_z - a_z b_y = 0, \quad a_x b_z - a_z b_x = 0, \quad a_x b_y - a_y b_x = 0$$

yoki

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}, \quad (2.16)$$

ya'ni kollinear vektorlarning koordinatalari proporsional bo'ladi va aksincha proporsional koordinatalarga ega vektorlar kollinear bo'ladi.

#### 2. Parallelogramm va uchburchakning yuzlari

Vektor ko'paytmaning ta'rifiga ko'ra  $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin \varphi$ , ya'ni

$$S_{par} = |\vec{a} \times \vec{b}|.$$

Bundan

$$S_{par} = |\vec{a} \times \vec{b}|, \quad S_{uchb} = \frac{1}{2} |\vec{a} \times \vec{b}|. \quad (2.17)$$

### 3. Nuqtaga nisbatan kuch momenti

$O$  nuqtasi mahkamlangan qattiq jism  $A$  nuqtasiga qo'yilgan  $\vec{F}$  kuch ta'sirida  $O$  nuqta atrofida aylanma harakat qilayotgan bo'lsin, masalan, bolt kalit yordamida buralayotgan bo'lsin (18-shakl).

Fizika kursidan ma'lumki,  $\vec{F}$  kuchning  $O$  nuqtaga nisbatan momenti deb  $O$  nuqtadan o'tuvchi va quyidagi shartlarni qanoatlantiruvchi  $\vec{M}$  vektorga aytiladi:

1)  $\vec{M} \perp \vec{r}$  va  $\vec{M} \perp \vec{F}$ , bu yerda  $\vec{r} = \vec{OA}$  -  $A$  nuqtaning radius vektori;

2)  $|\vec{M}| = |\vec{r}| \cdot |\vec{F}| \sin \varphi$ , bu yerda

$\varphi = (\vec{r}, \vec{F})$ ;

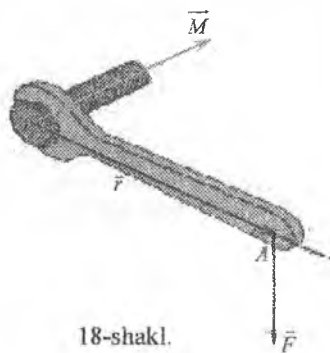
3)  $\vec{r}, \vec{F}, \vec{M}$  vektorlar o'ng uchlik tashkil qiladi.

Shunday qilib,

$$\vec{M} = \vec{r} \times \vec{F},$$

ya'ni qo'zg'almas nuqtaga nisbatan kuch momenti kuch qo'yilgan nuqta radius vektorining kuch vektoriga vektor ko'paytmasiga teng.

Bu jumla vektor ko'paytmaning mexanik ma'nosini anglatadi.



18-shakl.

### 4. Aylanma harakatning chiziqli tezligi

Yuqorida keltirilgandagi kabi qo'zg'almas  $O$  nuqta atrofida  $\vec{\omega}$  burchak tezlik bilan aylanma harakat qilayotgan qattiq jism  $M$  nuqtasining chiziqli tezligi Eylar formulasi bilan topiladi:

$$\vec{v} = \vec{\omega} \times \vec{r},$$

bu yerda  $\vec{r} = \vec{OM}$  -  $M$  nuqtaning radius vektori.

9-misol.  $m, n$  ning qanday qiymatlarida  $\vec{a} = \{-2; 3; n\}$  va  $\vec{b} = \{m; -6; 2\}$  vektorlar kollinear bo'ladi?

*Yechish.* Ikki vektorning kollinearlik shartiga ko'ra,  $\frac{-2}{m} = \frac{3}{-6} = \frac{n}{2}$ .

Bundan  $m = 4, n = -1$ .

10-misol.  $\vec{a} = 2\vec{j} - 3\vec{k}$  va  $\vec{b} = 4\vec{i} + 3\vec{j}$  vektorlarga qurilgan parallelogramning yuzini hisoblang.

*Yechish* Yuzani (2.17) formula bilan hisoblaymiz:

$$S_{\text{yuzi}} = \sqrt{\begin{vmatrix} 2 & -3 \\ 3 & 0 \end{vmatrix}^2 + \begin{vmatrix} 0 & -3 \\ 4 & 0 \end{vmatrix}^2 + \begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix}^2} = \sqrt{9^2 + 12^2 + (-8)^2} = 17(\text{y.b.}).$$

### 2.2.3. Uchta vektorning aralash ko'paytmasi

#### *Aralash ko'paytmaning ta'rif va geometrik ma'nosi*

*Ta'rif.* Uchta  $\vec{a}, \vec{b}$  va  $\vec{c}$  vektorning aralash ko'paytmasi deb  $\vec{a} \times \vec{b}$  vektorning  $\vec{c}$  vektorga skalyar ko'paytmasiga teng songa aytiladi va  $[\vec{a}, \vec{b}, \vec{c}]$  kabi belgilanadi.

Uchta komplanar bo'lmagan  $\vec{a}, \vec{b}, \vec{c}$  vektorlar berilgan bo'lsin. Bu vektorlarga parallelepiped quramiz va  $\vec{d} = \vec{a} \times \vec{b}$  vektorni yasaymiz (19-shakl).

Vektor ko'paytmaning ta'rifiga ko'ra,

$$\vec{d} \perp \vec{a}, \quad \vec{d} \perp \vec{b}, \quad |\vec{d}| = S_{\text{par}},$$

bu yerda  $S_{\text{par}}$  – parallelepiped asosining yuzi.

Ta'rifga ko'ra  $\vec{d} \cdot \vec{c} = |\vec{d}| \cdot |\vec{c}| \cos \varphi$ ,

bu yerda  $\varphi$  –  $\vec{c}$  va  $\vec{d}$  vektorlar orasidagi burchak.

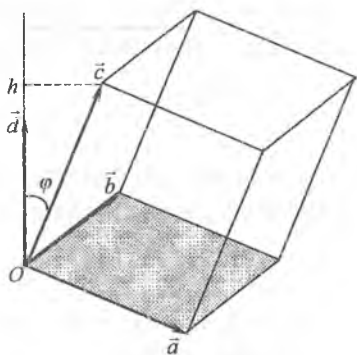
19-shaklda  $\vec{a}, \vec{b}, \vec{c}$  vektorlar o'ng uchlik tashkil qiladi va  $\varphi < \frac{\pi}{2}$ , ya'ni  $\cos \varphi > 0$ . U holda  $|\vec{c}| \cos \varphi = h$  va  $\vec{d} \cdot \vec{c} = S_{\text{par}} \cdot h = V$ . Ikkinchi holatdan  $\vec{d} \cdot \vec{c} = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \vec{b} \vec{c}$ . Demak,  $V = \vec{a} \vec{b} \vec{c}$ .

Agar  $\vec{a}, \vec{b}, \vec{c}$  vektorlar chap uchlik tashkil qilsa,  $\varphi > \frac{\pi}{2}$  va  $\cos \varphi < 0$  bo'ladi. U holda  $|\vec{c}| \cos \varphi = -h$ ,  $V = -\vec{a} \vec{b} \vec{c}$ .

Shunday qilib, komplanar bo'lmagan uchta vektor aralash ko'paytmasining moduli qirralari bu vektorlarning uzunliklaridan iborat bo'lgan parallelepiped hajmiga teng:

$$V = |\vec{a} \vec{b} \vec{c}|. \quad (2.18)$$

Bu jumla aralash ko'paytmaning geometrik ma'nosini anglatadi.



19-shakl.

### Aralash ko'paytmaning xossalari

**1-xossa.** Amallarining o'rinlari almashtirilsa aralash ko'paytma o'zgarmaydi, ya'ni

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}).$$

*Isboti.* Skalyar ko'paytmaning o'rin almashtirish xossasiga ko'ra,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{c} \cdot (\vec{a} \times \vec{b}).$$

(2.18) formulaga ko'ra,

$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|, \quad V = |(\vec{b} \times \vec{c}) \cdot \vec{a}|.$$

Bunda  $\vec{a}, \vec{b}, \vec{c}$  va  $\vec{b}, \vec{c}, \vec{a}$  uchliklarning har ikkalasi bir vaqtda yoki o'ng uchlik yoki chap uchlik tashkil qiladi. Shu sababli  $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a}$ .

Bundan

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}).$$

**2-xossa.** Ko'paytuvchilarning o'rinlari doiraviy almashtirilsa, aralash ko'paytma o'zgarmaydi, ya'ni

$$\vec{a}\vec{b}\vec{c} = \vec{b}\vec{c}\vec{a} = \vec{c}\vec{a}\vec{b}.$$

*Isboti.* 1-xossa va skalyar ko'paytmaning o'rin almashtirish xossasidan topamiz:

$$\vec{a}\vec{b}\vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{b}\vec{c}\vec{a},$$

$$\vec{b}\vec{c}\vec{a} = \vec{b} \cdot (\vec{c} \times \vec{a}) = (\vec{c} \times \vec{a}) \cdot \vec{b} = \vec{c}\vec{a}\vec{b}.$$

**3-xossa.** Ikkita qo'shni ko'paytuvchining o'rinlari almashtirilsa, aralash ko'paytma ishorasi qarama-qarshisiga almashadi. Masalan,  $\vec{a}\vec{b}\vec{c} = -\vec{b}\vec{a}\vec{c}$ .

$$\textit{Isboti.} \quad \vec{a}\vec{b}\vec{c} = (\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{b} \times \vec{a}) \cdot \vec{c} = -\vec{b}\vec{a}\vec{c}.$$

**4-xossa.** Agar nolga teng bo'lmagan  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar bo'lsa, u holda ularning aralash ko'paytmasi nolga teng bo'ladi.

Shunindek, teskari tasdiq o'rinli: agar  $\vec{a}\vec{b}\vec{c} = 0$  ( $|\vec{a}| \neq 0, |\vec{b}| \neq 0, |\vec{c}| \neq 0$ ) bo'lsa, u holda  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar bo'ladi.

*Isboti.*  $\vec{a}\vec{b}\vec{c} = 0$  ( $|\vec{a}| \neq 0, |\vec{b}| \neq 0, |\vec{c}| \neq 0$ ) bo'lsin.  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar emas deb faraz qilamiz.

U holda bu vektorlarga hajmi  $V \neq 0$  bo'lgan parallelopiped qurish mumkin.  $V = \pm \vec{a}\vec{b}\vec{c}$  dan  $\vec{a}\vec{b}\vec{c} \neq 0$  kelib chiqadi. Bu  $\vec{a}\vec{b}\vec{c} = 0$  shartga ega. Demak, qilingan faraz noto'g'ri va  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar.

U holda  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar bo'lsin.

U holda  $\vec{d} = \vec{a} \times \vec{b}$  vektor  $\vec{a}, \vec{b}, \vec{c}$  vektorlar yotgan tekislikka perpendikulyar bo'ladi.

Hundan  $\vec{d} \perp \vec{c}$ . Shu sababli  $\vec{d} \cdot \vec{c} = 0$  yoki  $\vec{a}\vec{b}\vec{c} = 0$ .

### **Koordinatalari bilan berilgan vektorlarning aralash ko'paytmasi**

U holda  $\vec{a} = \{a_x; a_y; a_z\}$ ,  $\vec{b} = \{b_x; b_y; b_z\}$  va  $\vec{c} = \{c_x; c_y; c_z\}$  vektor berilgan bo'lsin.

U holda

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k},$$

$$\vec{a}\vec{b}\vec{c} = \left( \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k} \right) \cdot (c_x \vec{i} + c_y \vec{j} + c_z \vec{k}) =$$

$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z$$

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \quad (2.19)$$

*11 misol.*  $\vec{a} = \{-2; 2; 1\}$ ,  $\vec{b} = \{3; -2; 5\}$ ,  $\vec{c} = \{1; -1; 3\}$  vektorlar berilgan.  $\vec{a}\vec{b}\vec{c}$  ko'paytmani hisoblang.

*Yechish.*  $\vec{a}\vec{b}\vec{c}$  ni (2.19) formula bilan topamiz:

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} -2 & 2 & 1 \\ 3 & -2 & 5 \\ 1 & -1 & 3 \end{vmatrix} = 12 + 10 - 3 + 2 - 10 - 18 = -7.$$



## Aralash ko'paytmaning ayrim tathbiqlari

### 1. Fazodagi vektorlarning o'zaro joylashishi

$\vec{a}, \vec{b}, \vec{c}$  vektorlarning fazoda o'zaro joylashishini aniqlash  $V = \pm \vec{a}\vec{b}\vec{c}$  bo'lishiga asoslanadi. Bunda agar  $\vec{a}\vec{b}\vec{c} > 0$  bo'lsa, u holda vektorlar o'ng uchlik tashkil qiladi, agar  $\vec{a}\vec{b}\vec{c} < 0$  bo'lsa, u holda vektorlar chap uchlik tashkil qiladi.

### 2. Uchta vektorning komplanarlik sharti

Aralash ko'paytmaning 4-xossasiga ko'ra nolga teng bo'lmagan  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar bo'lsa, u holda  $\vec{a}\vec{b}\vec{c} = 0$  yoki

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0. \quad (2.20)$$

### 3. Parallelepiped va piramidaning hajmlari

Aralash ko'paytmaning geometrik ma'nosiga ko'ra,  $\vec{a}, \vec{b}, \vec{c}$  vektorlarga qurilgan parallelepiped hajmini  $V_{par} = |\vec{a}\vec{b}\vec{c}|$  bilan va piramida hajmini  $V_{pir} = \frac{1}{6} |\vec{a}\vec{b}\vec{c}|$  bilan topish mumkin.

Shunday qilib,

$$V_{par} = \text{mod} \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}, \quad V_{pir} = \frac{1}{6} \text{mod} \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \quad (2.21)$$

**12-misol.** Uchlari  $A(2;3;1)$ ,  $B(4;1;-2)$ ,  $C(6;3;7)$ ,  $D(-5;-4;8)$  nuqtalarda bo'lgan piramidaning  $D$  uchidan tushirilgan  $h$  balandligi uzunligini toping.

**Yechish.** Avval piramida qirralarini ifodalovchi vektorlarni topamiz:

$$\vec{AB} = \{2; -2; -3\}, \quad \vec{AC} = \{4; 0; 6\}, \quad \vec{AD} = \{-7; -7; 7\}.$$

Uchburchakda hajmini hisoblaymiz:

$$V = \frac{1}{6} \operatorname{mod} \begin{vmatrix} 2 & -2 & -3 \\ 4 & 0 & 6 \\ -7 & -7 & 7 \end{vmatrix} = \frac{1}{6} |84 + 84 + 84 + 56| = \frac{154}{3}.$$

Uchburchak yuzini hisoblaymiz:

$$S = \frac{1}{2} |\overline{AB} \times \overline{AC}| = \frac{1}{2} \sqrt{\begin{vmatrix} -2 & -3 \\ 0 & 6 \end{vmatrix}^2 + \begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix}^2 + \begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix}^2} = \\ = \frac{1}{2} \sqrt{(-12)^2 + 24^2 + 8^2} = 14.$$

Piramida uchun  $V = \frac{1}{3} hS$ .

Demakdan

$$h = \frac{3V}{S} = \frac{3 \cdot \frac{154}{3}}{14} = \frac{154}{14} = 11 \text{ (u.b.)}$$

*1.3-misol.* Fazoda  $A, B, C, D$  nuqtalar koordinatalari bilan berilgan.  $A(3;3;2)$ ,  $B(5;7;7)$ ,  $C(4;6;10)$ ,  $D(2;3;7)$ . Quyidagilarni toping:

- 1)  $\overline{AB}$  vektor proeksiyalari va yo'nalishini;
- 2)  $\overline{AB} \cdot \overline{AC}$ ,  $\overline{AB} \times \overline{AC}$  ko'paytmalarni;
- 3)  $ABC$  uchburchak yuzasini;
- 4)  $ABCD$  piramida hajmini.

*Yechish.* 1)  $\overline{AB}$  vektor proeksiyasini (1.18) formula bilan topamiz:

$$\overline{AB} = \{a_x; a_y; a_z\} = \{5-3; 7-3; 7-2\} = \{-2; 5; 5\}.$$

Demakdan (1.12) formulaga ko'ra,

$$|\overline{AB}| = |\vec{a}| = \sqrt{(-2)^2 + 5^2 + 5^2} = 3\sqrt{6}.$$

$\overline{AB}$  vektor yo'nalishini (1.13) formulalar bilan topamiz:

$$\cos \alpha = \frac{a_x}{|\vec{a}|} = -\frac{\sqrt{6}}{9}, \quad \cos \beta = \frac{a_y}{|\vec{a}|} = \frac{5\sqrt{6}}{18}, \quad \cos \gamma = \frac{a_z}{|\vec{a}|} = \frac{5\sqrt{6}}{18}.$$

Topilgan yechimlarni *Maple* paketida bajaramiz:

> with(geom3d):

> point(A,7,2,2),point(B,5,7,7), point(C,4,6,10), point(D,2,3,7):

> with(LinearAlgebra):

> v := <a,b,c>;

$$v := (a)e_x + (b)e_y + (c)e_z$$

> VectorNorm(v,2,conjugate=false);

$$\sqrt{a^2 + b^2 + c^2}$$

> v1 := <5-7,7-2,7-2>;

$$v1 := -2e_x + 5e_y + 5e_z$$

> VectorNorm(v1,2,conjugate=false);

$$3\sqrt{6}$$

> Normalize(<a,b,c>,Euclidean,conjugate=false);

$$\begin{bmatrix} a \\ \sqrt{a^2 + b^2 + c^2} \\ b \\ \sqrt{a^2 + b^2 + c^2} \\ c \\ \sqrt{a^2 + b^2 + c^2} \end{bmatrix}$$

> Normalize(v1,Euclidean,conjugate=false);

$$\begin{bmatrix} -\frac{1}{9}\sqrt{6} \\ \frac{5}{18}\sqrt{6} \\ \frac{5}{18}\sqrt{6} \end{bmatrix}$$

2) (2.2.8) formuladan  $\overline{AB} = \{-2;5;5\}$ ,  $\overline{AC} = \{-3,4,8\}$  larni hisobga olib, topamiz:

$$\overline{AB} \cdot \overline{AC} = (-2) \cdot (-3) + 5 \cdot 4 + 5 \cdot 8 = 66$$

> vAC := <4-7,6-2,10-2>;

$$vAC := -3e_x + 4e_y + 8e_z$$

> AB.AC:=Dotproduct(<5-7,7-2,7-2>, <4-7,6-2,10-2>);

$$\text{VectorCalculus}:-.(AB, AC) := 66$$

10)  $\vec{n}$  vektor ko'paytmani (2.15) formula bilan topamiz:

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 5 & 5 \\ -3 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 5 \\ 4 & 8 \end{vmatrix} \vec{i} - \begin{vmatrix} -2 & 5 \\ -3 & 8 \end{vmatrix} \vec{j} - \begin{vmatrix} -2 & 5 \\ -3 & 4 \end{vmatrix} \vec{k} = 20\vec{i} + \vec{j} + 7\vec{k}$$

11)  $A := \langle\langle i, 0 \rangle\rangle \langle\langle j, m \rangle\rangle \langle\langle k, n, q \rangle\rangle$ ;

$$A := \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ l & m & n \\ o & p & q \end{vmatrix}$$

12)  $\text{Determinant}(A)$ ;  $imq - inp + lkp - ljq + ojn - okm$

13)  $axb := \langle\langle i, 2, -3 \rangle\rangle \langle\langle j, 5, 4 \rangle\rangle \langle\langle k, 5, 8 \rangle\rangle$ ;

$$axb = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 5 & 5 \\ -3 & 4 & 8 \end{vmatrix}$$

14)  $\text{Determinant}(axb)$ ;  $20i + 7k + j$

15)  $ABC$  uchburchak yuzasini (2.2.17) formula bilan hisoblaymiz:

$$S = \frac{1}{2} |\overline{AB} \times \overline{AC}| = \frac{1}{2} \sqrt{20^2 + 1^2 + 7^2} = \frac{15\sqrt{2}}{2}$$

16)  $\text{modN} := \text{VectorNorm}(N, 2, \text{conjugate} = \text{false})$ ;  $\text{modN} := 15\sqrt{2}$

17)  $s := \text{modN}/2$ ;  $s := \frac{15}{2}\sqrt{2}$

18)  $ABCD$  piramida hajmini topamiz:

$$V = \frac{1}{6} \text{mod} \begin{vmatrix} -2 & 5 & 5 \\ -3 & 4 & 8 \\ -5 & 1 & 5 \end{vmatrix} = \frac{1}{6} \cdot 64 = \frac{32}{3}$$

19)  $\text{with}(\text{geom3d})$ ;

20)  $abc := \langle\langle -5, -2, -3 \rangle\rangle \langle\langle 1, 5, 4 \rangle\rangle \langle\langle 5, 5, 8 \rangle\rangle$ ;

$$abc := \begin{vmatrix} -2 & 5 & 5 \\ -3 & 4 & 8 \\ -5 & 1 & 5 \end{vmatrix}$$

> Determinant(abc);

-64

> VABCD:={ABC(Determinant(abc))/6};

$$VABCD = \frac{32}{3}.$$

## 2.2.4. Mashqlar

1. Tomonlari birga teng bo'lgan teng tomonli  $ABC$  uchburchak berilgan.  $\overline{AB} \cdot \overline{BC} + \overline{BC} \cdot \overline{CA} + \overline{CA} \cdot \overline{AB}$  ifodaning qiymatini toping.

2. Tomonlari  $BC = 5$ ,  $CA = 6$ ,  $AB = 7$  ga teng bo'lgan  $ABC$  uchburchak berilgan.  $\overline{AB} \cdot \overline{BC}$  skalyar ko'paytmani toping.

3. Agar  $|\vec{a}| = 6$ ,  $|\vec{b}| = 4$ ,  $\varphi = (\vec{a}, \vec{b}) = \frac{2\pi}{3}$  bo'lsin. Toping:

1)  $(2\vec{a} + \vec{b})^2$ ;

2)  $(2\vec{a} - 3\vec{b}) \cdot (\vec{a} - 2\vec{b})$ .

4.  $\vec{a} = \{1; -2; 2\}$  va  $\vec{b} = \{2; 4; -5\}$  vektorlar berilgan. Toping:

1)  $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + \vec{b})$ ;

2)  $(\vec{a} - \vec{b})^2$ .

5. Berilgan vektorlar  $m$  ning qanday qiymatlarida perpendikular bo'ladi?

1)  $\vec{a} = \{1; -2m; 0\}$ ,  $\vec{b} = \{4; 2; 3m\}$ ;

2)  $\vec{a} = \{m; -5; 2\}$ ,  $\vec{b} = \{m - 2; m; m + 3\}$ .

6.  $\vec{e}_1$ ,  $\vec{e}_2$ ,  $\vec{e}_3$  birlik vektorlar uchun  $\vec{e}_1 + \vec{e}_2 + \vec{e}_3 = 0$  bo'lsa,  $\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_3 + \vec{e}_3 \vec{e}_1$  ni toping.

7. Tomonlari  $\vec{a} = 2\vec{i} + \vec{j}$  va  $\vec{b} = -\vec{j} + 2\vec{k}$  vektorlardan iborat bo'lgan parallelogramning diagonallari orasidagi burchakni toping.

8. Uchlari  $A(-1; -2; 4)$ ,  $B(-4; -2; 0)$ ,  $C(3; -2; 1)$  bo'lgan  $ABC$  uchburchak berilgan.  $\angle B$  ni toping.

9.  $xOz$  va  $yOz$  burchaklarning bissektoralari qanday burchak tashkil qiladi?

10. Koordinata o'qlari bilan tashkil qilgan burchaklari berilgan fazodagi ikki yo'nalish orasidagi burchakni toping:

1)  $l_1: \left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\right)$  va  $l_2: \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ ;

2)  $l_1: \left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}\right)$  va  $l_2: \left(\frac{5\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{2}\right)$ .

2.11.  $\vec{a} = \{3; -6; -1\}$ ,  $\vec{b} = \{1; 4; -5\}$ ,  $\vec{c} = \{3; -4; 12\}$  vektorlar berilgan. Quyidagilarni toping:

1)  $\text{Pr}_{\vec{c}} \vec{a}$ ;

2)  $\text{Pr}_{\vec{c}} (2\vec{a} - 3\vec{b})$ .

13.  $a = (5; 6; -1)$  nuqtani  $B(5; 6; -1)$  nuqtaga to'g'ri chiziq bo'ylab ko'chirishda

14.  $a = (1; 2; -3)$  kuchning bajargan ishini toping.

14.  $a = (1; 2; -3)$  va  $b = \{1; 2; -3\}$  vektorlar berilgan. Agar  $\bar{x} \cdot \bar{a} = 9$ ,

15.  $a$  va  $b$  vektorlar  $c$  oqiga perpendikular bo'lsa,  $\bar{x}$  vektorning koordinatalarini toping.

14.  $a = (1; 2; -1)$ ,  $b = \{1; -2; 3\}$  va  $c = \{1; 2; -7\}$  vektorlar berilgan. Agar

15.  $a = (1; 2; -1)$ ,  $b = \{1; -2; 3\}$  va  $c = \{1; 2; -7\}$  bo'lsa,  $\bar{x}$  vektorni toping.

14. Agar  $|\bar{a}|=4$ ,  $|\bar{b}|=6$ ,  $\varphi = (\bar{a}, \bar{b}) = \frac{5\pi}{6}$  bo'lsa, quyidagilarni toping:

1)  $|\bar{a} + \bar{b}|$ ; 2)  $|(3\bar{a} - 3\bar{b}) \times (\bar{a} + 4\bar{b})|$ .

16. Tomonlari  $\bar{a}$  va  $\bar{b}$  vektorlar uzunliklaridan iborat bo'lgan uchburchakning yuzini toping:

1)  $\bar{a} = 3\bar{m} + 2\bar{n}$ ,  $\bar{b} = 2\bar{m} + \bar{n}$ , bu yerda  $|\bar{m}|=1$ ,  $|\bar{n}|=1$ ,  $\varphi = (\bar{m}, \bar{n}) = \frac{\pi}{6}$ ;

2)  $\bar{a} = 3\bar{m} + 2\bar{n}$ ,  $\bar{b} = 5\bar{m} + 4\bar{n}$ , bu yerda  $|\bar{m}|=2$ ,  $|\bar{n}|=3$ ,  $\varphi = (\bar{m}, \bar{n}) = \frac{\pi}{3}$ .

17. Agar  $|\bar{a}|=5$ ,  $|\bar{b}|=10$ ,  $\bar{a}\bar{b}=25$  bo'lsa,  $|\bar{a} \times \bar{b}|$  ni toping.

18. Agar  $|\bar{a}|=3$ ,  $|\bar{b}|=13$ ,  $|\bar{a} \times \bar{b}|=36$  bo'lsa,  $\bar{a}\bar{b}$  ni toping.

19.  $\bar{a} = \{-1; 2; 3\}$  va  $\bar{b} = \{2; -1; 3\}$  vektorlar berilgan. Vektor ko'paytmalarini toping.

1)  $\bar{a} + \bar{b}$ ; 2)  $(2\bar{a} + \bar{b}) \times (3\bar{b} - \bar{a})$ .

20. Tomonlari  $\bar{a}$  va  $\bar{b}$  vektorlar uzunliklaridan iborat bo'lgan uchburchakning yuzini toping:

1)  $\bar{a} = \{2; -2; 1\}$ ,  $\bar{b} = \{8; 4; 1\}$ ; 2)  $\bar{a} = \{3; 5; -8\}$ ,  $\bar{b} = \{6; 3; -2\}$ .

21. Uchburchak uchlari  $A(1; 2; 0)$ ,  $B(3; 0; -3)$ ,  $C(5; 2; 6)$  berilgan. Uning  $B$  uchidan  $AC$  tomonga tushirilgan balandlik uzunligini toping.

22.  $A$  nuqtaga  $\bar{F}$  kuch qo'yilgan. Bu kuchning  $B$  nuqtaga nisbatan momentini toping.

1)  $\bar{F} = \{2; -4; 5\}$ ,  $A(0; 2; 1)$ ,  $B(-1; 2; 3)$ ; 2)  $\bar{F} = \{1; 2; -1\}$ ,  $A(-1; 4; -2)$ ,  $B(2; 3; -1)$ .

23.  $\bar{F} = \{2; 2; 4\}$  kuch  $A(4; 2; -3)$  qo'yilgan.  $B(0; 2; 4)$  nuqtaga nisbatan kuch momentining qiymatini va yo'naltiruvchi kosinuslarini toping.

24. Kollinear bo'lmagan  $\bar{m}$  va  $\bar{n}$  vektorlar berilgan.  $\bar{a} = \alpha \cdot \bar{m} + 6\bar{n}$  va  $\bar{b} = 3\bar{m} - 2\bar{n}$  vektorlar  $\alpha$  ning qanday qiymatida kollinear bo'ladi?

25.  $\bar{a} = \{-1; 3; \alpha\}$  va  $\bar{b} = \{\beta; -6; -3\}$  vektorlar  $\alpha$  va  $\beta$  ning qanday qiymatlarida kollinear bo'ladi?

26. Ikkita  $\vec{a} = \{2; -3\}$ ,  $\vec{b} = \{-1; 5\}$  vektorlar berilgan. Quyidagi shartlarni qanoatlantiruvchi  $\vec{x}$  vektorning koordinatalarini toping:

1)  $\vec{x} \perp \vec{a}$  va  $\vec{b} \cdot \vec{x} = 7$ ;

2)  $\vec{x} \parallel \vec{a}$  va  $\vec{b} \cdot \vec{x} = 17$ .

27.  $\vec{a} = \{4; -2; -3\}$  va  $\vec{b} = \{0; 1; 3\}$  vektorlarga perpendikular, uzunligi 26 ga teng va  $Oy$  o'q bilan o'tmas burchak tashkil qiluvchi  $\vec{x}$  vektorning koordinatalarini toping.

28. Berilgan vektorlarning komplanar yoki komplanar emasligini aniqlang.

1)  $\vec{a} = \{3; -2; 1\}$ ,  $\vec{b} = \{2; 1; 2\}$ ,  $\vec{c} = \{3; -1; -2\}$ ;

2)  $\vec{a} = \{2; -1; 2\}$ ,  $\vec{b} = \{3; -4; 7\}$ ,  $\vec{c} = \{1; 2; -3\}$ .

29.  $\alpha$  ning qanday qiymatlarida  $\vec{a}, \vec{b}, \vec{c}$  vektorlar komplanar bo'ladi?

1)  $\vec{a} = \{1; 1; \alpha\}$ ,  $\vec{b} = \{0; 1; 0\}$ ,  $\vec{c} = \{3; 0; 1\}$ ;

2)  $\vec{a} = \{\alpha; 3; 1\}$ ,  $\vec{b} = \{5; -1; 2\}$ ,  $\vec{c} = \{-1; 5; 4\}$ .

30. Piramida uchlarining koordinatalari berilgan. Piramidaning hajmini va  $D$  uchidan tushirilgan balandligini toping:

1)  $A(1; -2; 2)$ ,  $B(-1; 1; 2)$ ,  $C(-1; -2; 8)$ ,  $D(1; 1; 10)$ ;

2)  $A(1; 1; 1)$ ,  $B(2; 0; 2)$ ,  $C(2; 2; 2)$ ,  $D(3; 4; -3)$ .

31.  $\vec{a}, \vec{b}, \vec{c}$  vektorlar berilgan. Bu vektorlar qanday uchlik tashkil etishini aniqlang va qirralari bu vektorlardan iborat bo'lgan parallelepiped hajmini toping:

1)  $\vec{a} = \{1; -2; 1\}$ ,  $\vec{b} = \{3; 2; 1\}$ ,  $\vec{c} = \{-1; 0; 1\}$ ;

2)  $\vec{a} = \{1; 3; 3\}$ ,  $\vec{b} = \{-1; 2; 0\}$ ,  $\vec{c} = \{1; 2; -3\}$ .

# ANALITIK GEOMETRIYA

3

• Tekislikdagi to'g'ri  
chiziq

• Ikki to'liq tartibli  
chiziqqa

• Tekislik

• Uch to'liqdagi to'g'ri  
chiziq

• Ikki to'liq tartibli  
sirtlar



*René Dekart  
(1596-1650) –*

*frantsuz faylasufi,  
matematigi, mexanigi,  
filolgi va fiziologi.*

*René Dekart zamonaviy analitik geometriya va algebraik raqamlarga asos solgan.*

*Dekart ismonidan asallik geometriyaning sarallilishi egri chiziqlar tenglamalarini biror koordinatalar sistemasida tahlil etish imkonini berdi va funksiya tushunchasi tomon hal qiluvchi qadam bo'ldi.*

Analitik geometriya – matematikaning bo'limlaridan biri bo'lib, u geometriya bilan algebra birlashtiradi, ya'ni ayrim geometrik tushunchalarni algebraik tahlil qilish va ayrim algebraik bog'lanishlarni geometrik izohlash imkonini beradi. Bunda asosiy e'tibor ikkita masalaga, xossalari ko'ra geometrik shaklning tenglamasini keltirib chiqarishga va tenglamasiga ko'ra geometrik shaklning ko'rinishi va xossalari o'rganishga, qaratiladi.

Fransuz matematigi *René Dekart analitik geometriyaning asoschisi* hisoblanadi. Dekart tomonidan 1637-yilda kiritilgan koordinatalar usuli nuqtaning o'rini biror koordinatalar sistemasiga nisbatan aniqlashga asoslanadi.

## 3.1. TEKISLIKDAGI TO'G'RI CHIZIQ

### 3.1.1. Tekislikdagi chiziq

Umumiy boshlang'ich  $O$  nuqtaga ega bo'lgan o'zaro perpendikulyar  $Ox$  va  $Oy$  koordinata o'qlari tekislikda to'g'ri burchakli  $Oxy$  koordinatalar sistemasini hosil qiladi.



$Oxy$  koordinatalar sistemasida ikkita  $x$  va  $y$  sonlari tekislikdagi har qanday  $M$  nuqtaning o'rnini to'liq aniqlaydi. Bunda nuqta  $M(x; y)$  kabi belgilanadi:  $x$  ga  $M$  nuqtaning *absissasi*,  $y$  ga  $M$  nuqtaning *ordinatasi* deyiladi.

$Oxy$  tekislikdagi chiziq tenglamasi deb aynan shu chiziq nuqtalarining  $x$  va  $y$  koordinatalari orasidagi bog'lanishni aniqlovchi ikki noma'lumli

$$F(x, y) = 0$$

ko'rinishdagi tenglamaga aytiladi.

Shu kabi, koordinatalari ikki noma'lumli  $F(x, y) = 0$  tenglamani qanoatlantiruvchi  $Oxy$  tekislikning barcha  $M(x; y)$  nuqtalari to'plamiga *tekislikda* shu tenglama bilan aniqlanuvchi *chiziq* deyiladi.

Ayrim hollarda tekislikdagi chiziq  $y = f(x)$  tenglama bilan beriladi. Bunda chiziq  $y = f(x)$  funksiyaning grafigi deb ataladi.

Tekislikdagi chiziq ikkita  $x = x(t)$ ,  $y = y(t)$ ,  $t \in T$  tenglamalar bilan ham berilishi mumkin. Bunda barcha  $M(x(t); y(t))$ ,  $t \in T$  nuqtalar to'plami tekislikdagi chiziqni ifodalaydi.  $x = x(t)$ ,  $y = y(t)$  funksiyalarga bu chiziqning parametrik tenglamalari,  $t$  o'zgaruvchiga parametr deyiladi.

Tekislikdagi chiziqning ikkita  $x = x(t)$ ,  $y = y(t)$  parametrik (skalyar) tenglamalarini bitta  $\vec{r} = \vec{r}(t)$  vektor tenglama bilan berish mumkin. Bunda  $t$  parametr (vaqt) o'zgarishi bilan  $\vec{r} = \vec{r}(t)$  vektorning oxiri biror chiziqni chizadi. Bu chiziqqa nuqtaning traektoriyasi,  $\vec{r} = \vec{r}(t)$  tenglamaga harakat tenglamasi deyiladi. Bu jumla chiziqning vektor va parametrik tenglamalarining *mexanik ma'nosini* bildiradi.

Shunday qilib, tekislikdagi har qanday chiziqqa ikki o'zgaruvchining biror  $F(x, y) = 0$  tenglamasi mos keladi va aksincha, ikki o'zgaruvchining har qanday  $F(x, y) = 0$  tenglamasiga, umuman olganda, tekislikdagi biror chiziq mos keladi. Bunda «umuman olganda» iborasi aytilganlarda mustasnoga yo'l qo'yilishi mumkinligini bildiradi. Masalan,  $(x-1)^2 + (y-4)^2 = 0$  tenglamaga chiziq emas, balki  $M(1; 4)$  nuqta mos keladi;  $x^2 + y^2 + 3 = 0$  tenglamaga tekislik nuqtalarining hech bir geometrik o'rnini mos kelmaydi.

### 3.1.2. Tekislikdagi to'g'ri chiziq tenglamalari

To'g'ri chiziqning tekislikdagi o'rni turli parametrlar bilan bir qiyamli aniqlanishi mumkin. Masalan, to'g'ri chiziqda yotuvchi nuqta va to'g'ri chiziqqa perpendikular vektor bilan, to'g'ri chiziqning koordinata o'qlarida ajratgan kesmalari bilan va hokazo. Bunday parametrlar to'g'ri chiziqning tenglamalarini keltirib chiqarish uchun asos bo'ladi. Quyida berilgan parametrlariga ko'ra to'g'ri chiziq tenglamalarini keltirib chiqarish bilan tanishamiz.

1- To'g'ri chiziqda yotuvchi  $M_0(x_0; y_0)$  nuqta va to'g'ri chiziqqa perpendikular bo'lgan  $\vec{n} = \{A; B\}$  vektor berilgan.

1- to'g'ri chiziqda yotuvchi ixtiyoriy  $M(x; y)$  nuqtani olamiz va  $\vec{M} = \{x - x_0; y - y_0\}$  vektorni yasaymiz (1-shakl).

Bunda  $\vec{n} \perp \vec{M}$  bo'ladi. Ikki vektorning perpendikulyarlik shartiga asosan to'g'ri chiziq tenglamasini topamiz:

$$A(x - x_0) + B(y - y_0) = 0. \quad (1.1)$$

(1.1) tenglamaga berilgan nuqtadan o'tuvchi va berilgan vektorga perpendikular to'g'ri chiziq tenglamasi deyiladi.

To'g'ri chiziqqa perpendikular bo'lgan har qanday vektorga to'g'ri chiziqning normal vektori deyiladi.

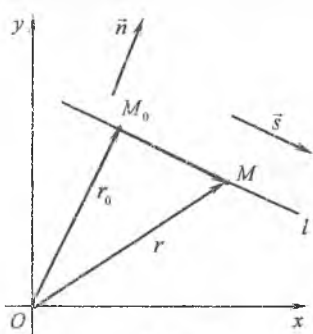
Demak,  $\vec{n} = \{A; B\}$  vektor (1.1) tenglama bilan aniqlanuvchi to'g'ri chiziqning normal vektori bo'ladi.

1-misol.  $M_1(2; 3)$  va  $M_2(-1; 0)$  nuqtalar berilgan.  $M_2$  nuqtadan o'tuvchi va  $\vec{M_1M_2}$  vektorga perpendikular to'g'ri chiziq tenglamasini tuzing.

*Yechish.* Avval  $\vec{M_1M_2}$  vektorini topamiz:

$$\vec{M_1M_2} = \{-1 - 2; 0 - 3\} = \{-3; -3\}.$$

Bundan  $A = -3, B = -3.$



1-shakl.

Izlanayotgan to'g'ri chiziq tenglamasini (1.1) formula bilan tuzamiz:

$$-3(x - (-1)) - 3(y - 0) = 0$$

yoki

$$x + y + 1 = 0.$$

II. To'g'ri chiziqda yotuvchi  $M_0(x_0; y_0)$  nuqta va to'g'ri chiziqqa parallel bo'lgan  $\vec{s} = \{p; q\}$  vektor berilgan.

I to'g'ri chiziqda yotuvchi  $M_0(x_0; y_0)$  va  $M(x; y)$  nuqtalardan  $\overline{M_0M} = \{x - x_0; y - y_0\}$  vektorni yasaymiz (1-shakl).

Bunda  $\vec{s}$  va  $\overline{M_0M}$  vektorlar kollinear bo'ladi. Ikki vektorning kollinearlik shartidan quyidagini topamiz:

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}. \quad (1.2)$$

(1.2) tenglamaga berilgan nuqtadan o'tuvchi va berilgan vektorga parallel to'g'ri chiziq tenglamasi deyiladi.

Shunindek, bu tenglama to'g'ri chiziqning kanonik tenglamasi deb ataladi.

To'g'ri chiziqqa parallel bo'lgan (yoki to'g'ri chiziqda yotuvchi) nolga teng bo'lmagan har qanday vektorga to'g'ri chiziqning yo'naltiruvchi vektori deyiladi.

Demak,  $\vec{s} = \{p; q\}$  vektor (1.2) tenglama bilan aniqlanuvchi to'g'ri chiziqning yo'naltiruvchi vektori bo'ladi.

**I-izoh.** (1.2) tenglamadan to'g'ri chiziqning keltirilgan II shartni qanoatlaniruvchi boshqa tenglamalarini hosil qilish mumkin.

1. (1.2) tenglamada

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = t, \quad t \in (-\infty; +\infty)$$

belgilash kiritamiz.

Bundan

$$x = x_0 + tp, \quad y = y_0 + tq \quad (1.3)$$

tenglamalar kelib chiqadi, bu yerda  $t$  - parametr.

(1.3) tenglamalarga to'g'ri chiziqning parametrik tenglamalari deyiladi.

2. Ma'lumki, tekislikdagi chiziqning ikkita parametrik (skalyar) tenglamalarini bitta vektor tenglama bilan berish mumkin, ya'ni (1.1) tenglamalarini

$$\vec{r} = \vec{r}_0 + t\vec{s} \quad (1.4)$$

ko'rinishda yozish mumkin, bu yerda  $\vec{r} = \{x; y\}$ ,  $\vec{r}_0 = \{x_0; y_0\}$  - mos ravishda  $M(x; y)$ ,  $M_0(x_0; y_0)$  nuqtalarning radius vektorlari;  $\vec{s} = \{s_x; s_y\}$  to'g'ri chiziqning yo'naltiruvchi vektori (1-shakl).

(1.4) tenglamaga to'g'ri chiziqning vektor tenglamasi deyiladi.

*2-misol.*  $M(-2; 4)$  nuqtadan o'tuvchi va  $\vec{s} = \{1; -3\}$  vektorga parallel to'g'ri chiziqning kanonik, parametrik va vektor tenglamalarini tuzing.

*Yechish.* To'g'ri chiziqning kanonik, parametrik va vektor tenglamalarini (1.2), (1.3) va (1.4) formulalar bilan topamiz:

$$\frac{x+2}{1} = \frac{y-4}{-3};$$

$$x = -2 + t, \quad y = 4 - 3t, \quad t \in T;$$

$$r = \vec{r}_0 + t\vec{s}, \quad r_0 = \{-2; 4\}.$$

III To'g'ri chiziqda yotuvchi ikkita  $M_1(x_1; y_1)$  va  $M_2(x_2; y_2)$  nuqta berilgan.

$l$  to'g'ri chiziqda yotuvchi ixtiyoriy  $M(x; y)$  nuqtani olib,  $\overline{M_1M} = \{x - x_1; y - y_1\}$

va  $\overline{M_1M_2} = \{x_2 - x_1; y_2 - y_1\}$  vektorlarni qaralaymiz (2-shakl). Bunda  $\overline{M_1M}$  va

$\overline{M_1M_2}$  vektorlar kollinear bo'ladi.

Ikki vektorning kollinearlik shartidan topamiz:

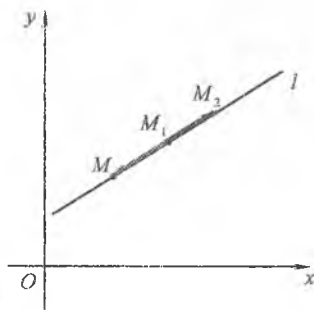
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad (1.5)$$

bo'ladi.

(1.5) tenglamaga berilgan ikki nuqtadan o'tuvchi to'g'ri chiziqning tenglamasi deyiladi.

IV To'g'ri chiziqning  $Ox$  va  $Oy$  o'qlaridan ajratgan kesmalari  $a$  va  $b$  berilgan.

$l$  to'g'ri chiziqda yotuvchi ixtiyoriy  $M(x; y)$  nuqtani topamiz (3-shakl).



2-shakl.

$\triangle CBM$  va  $\triangle OBA$  o'xshash. U holda uchburchaklarning o'xshashlik alomatiga ko'ra,

$$\frac{CB}{OB} = \frac{CM}{OA} \Rightarrow \frac{OB - OC}{OB} = \frac{OD}{OA} \Rightarrow \frac{OC}{OB} + \frac{OD}{OA} = 1.$$

Bundan  $OC = x$ ,  $OB = a$ ,  $OD = y$ ,  $OA = b$  o'rniga qo'yish bajarib, topamiz:

$$\frac{x}{a} + \frac{y}{b} = 1. \quad (1.6)$$

(1.6) tenglamaga to'g'ri chiziqning kesmalarga nisbatan tenglamasi deyiladi.

3- misol.  $4x + 3y - 12 = 0$  tenglama bilan berilgan to'g'ri chiziqni chizmada tasvirlang.

*Yechish.* Tekislikdagi to'g'ri chiziqni chizish uchun uning ikkita nuqtasini bilish yetarli bo'ladi.

To'g'ri chiziq tenglamasida, masalan  $x=0$  deb,  $y=4$  ni, ya'ni  $A(0;4)$  nuqtani va shu kabi  $B\left(2; \frac{4}{3}\right)$  nuqtani topamiz. Bu nuqtalarni tutashtirib, berilgan tenglamaga mos to'g'ri chiziqni chizamiz (4-shakl).

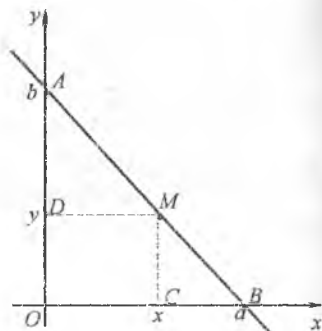
Bu masalani boshqacha, ya'ni to'g'ri chiziq tenglamasini kesmalarga nisbatan tenglamaga keltirib yechish mumkin. Buning uchun tenglamaning ozod hadi  $(-12)$ ni o'ng tomonga o'tkazamiz va hosil bo'lgan tenglikning har ikkala tomonini 12 ga bo'lamiz:

$$4x + 3y = 12, \quad \frac{4x}{12} + \frac{3y}{12} = 1$$

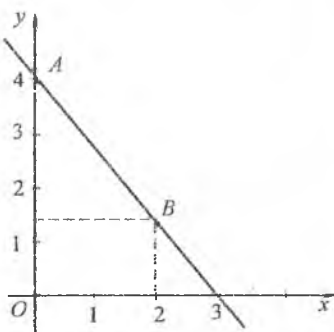
yoki

$$\frac{x}{3} + \frac{y}{4} = 1.$$

Bu tenglama bilan aniqlanuvchi to'g'ri chiziq  $Ox$  o'qidan koordinatalar boshiga nisbatan o'ng



3-shakl.



4-shakl.

Ushbu to'g'ri chiziqning  $Oy$  o'qidan esa koordinatalar boshiga tekislik yuzasiga  $d$  ga teng kesma ajratadi (4-shakl).

U to'g'ri chiziqning og'ish burchagi  $\varphi$  va  $Oy$  o'qidan ajratgan kesma  $b$  berilgan.

Ushbu o'g'ning musbat yo'nalishidan berilgan to'g'ri chiziqqa soat so'rtishga teskari yo'nalishda hisoblangan  $\varphi$  burchakka to'g'ri chiziqning og'ish burchagi deyiladi.

U og'ish burchagining tangensi, ya'ni  $k = tg\varphi$  son to'g'ri chiziqning burchak koeffitsiyenti deb ataladi.

U to'g'ri chiziqda yotuvchi to'g'ri chiziq  $M(x; y)$  nuqtani olamiz va burchak tangensi ta'rifidan topilganamiz (5-shakl):

$$\frac{y-b}{x} = tg\varphi$$

yoki

$$y = tg\varphi x + b.$$

Bundan

$$y = kx + b. \quad (1.7)$$

Bu tenglamaga to'g'ri chiziqning burchak koeffitsiyentli tenglamasi deyiladi.

**2.4.oh.** (1.7) tenglamadan to'g'ri chiziqning  $k$  burchak koeffitsiyentga ega bo'lgan yana bir tenglamasini keltirib chiqaramiz. Bu to'g'ri chiziq  $M_1(x_1; y_1)$  nuqtadan o'tsin. U holda bu nuqtaning koordinatalari (1.7) tenglamani qanoatlantiradi:  $y_1 = kx_1 + b$ .

Bundan  $b = y_1 - kx_1$ .

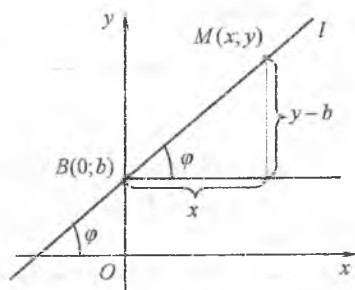
U holda (1.7) tenglamadan topamiz:

$$y = kx - kx_1 + y_1$$

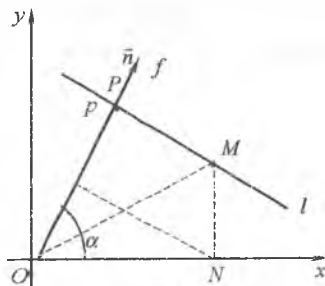
yoki

$$y - y_1 = k(x - x_1). \quad (1.8)$$

(1.8) tenglamaga berilgan nuqtadan berilgan yo'nalish bo'yicha o'tuvchi to'g'ri chiziq tenglamasi deyiladi.



5-shakl.



6-shakl.

Suningdek, bu tenglama *to'g'ri chiziqlar dastasi tenglamasi* deb ataladi.

VI. *To'g'ri chiziq*  $\bar{n} = \overline{OP}$  *normalining yo'nalishi*  $\alpha$  *va uzunligi*  $p$  *berilgan.*

$l$  *to'g'ri chiziqda yotuvchi ixtiyoriy*  $M(x; y)$  *nuqtani olamiz.*  
6-shaklga asosan:

$$\Pr_f \overline{OP} = \Pr_f \overline{ON} + \Pr_f \overline{NM} + \Pr_f \overline{MP},$$

bu yerda  $\Pr_f \overline{OP} = p$ ,  $\Pr_f \overline{ON} = x \cos \alpha$ ,  $\Pr_f \overline{NM} = y \sin \alpha$ ,  $\Pr_f \overline{MP} = 0$ .

Bundan,

$$p = x \cos \alpha + y \sin \alpha$$

yoki

$$x \cos \alpha + y \sin \alpha - p = 0. \quad (1.9)$$

(1.9) tenglamaga *to'g'ri chiziqning normal tenglamasi* deyiladi.

Keltirib chiqarilgan (1.1)-(1.9) tenglamalar asosida ushbu xulosa kelib chaqadi:

$x, y$  o'zgaruvchilarning har qanday birinchi darajali tenglamasi tekislikdagi biror to'g'ri chiziqni ifodalaydi va aksincha, tekislikdagi har qanday to'g'ri chiziq  $x, y$  o'zgaruvchilarning biror birinchi darajali tenglamasi bilan aniqlanadi.

Demak, tekislikdagi har bir  $l$  to'g'ri chiziq tenglamasini

$$Ax + By + C = 0 \quad (1.10)$$

ko'rinishda yozish mumkin, bu yerda  $C$  - ozod had;  $A^2 + B^2 \neq 0$ .

(1.10) tenglamada  $A$  va  $B$  sonlar to'g'ri chiziq normal vektorining koordinatalari bo'lishini (1.1) tenglama yordamida ko'rsatish mumkin:

$$A(x - x_0) + B(y - y_0) = 0,$$

$$Ax + By - (Ax_0 + By_0) = 0,$$

$$Ax + By + C = 0, \quad C = -(Ax_0 + By_0).$$

Demak,  $\bar{n} = \{A; B\}$ .

(1.10) tenglamaga *to'g'ri chiziqning umumiy tenglamasi* deyiladi.

(1.10) tenglamada:

1)  $C=0$  bo'lsa, tenglama  $By+C=0$  ko'rinishga keladi. Bunda berilgan chiziqning normal vektori  $Ox$  o'qqa perpendikular bo'ladi. Shu sababli to'g'ri chiziq  $Ox$  o'qqa parallel,  $Oy$  o'qqa perpendikular bo'ladi. Shu kabi  $B=0$  da kelib chiqadigan  $Ax+C=0$  to'g'ri chiziq  $Oy$  o'qqa parallel,  $Ox$  o'qqa perpendikular bo'ladi;

2)  $C=0$  bo'lsa, tenglama  $Ax+By=0$  ko'rinishni oladi. Bu tenglamani  $O(0;0)$  nuqtaning koordinatalari qanoatlantiradi. Demak, to'g'ri chiziq koordinatalar boshidan o'tadi;

3)  $A=0$  va  $C=0$  bo'lsa, tenglamadan  $y=0$  kelib chiqadi. Bu to'g'ri chiziq  $Ox$  o'qda yotadi. Shu kabi  $B=0$  va  $C=0$  da hosil bo'ladigan tenglama to'g'ri chiziq  $Oy$  o'qda yotadi.

*Yechilish.* Misolning qanday qiymatlarida  $(a^2+4a)x+(a-5)y-2a+4=0$  berilgan to'g'ri chiziq: 1)  $Ox$  o'qqa parallel bo'ladi; 2)  $Ox$  o'qqa perpendikular bo'ladi; 3) koordinatalar boshidan o'tadi.

*Yechilish.* Misolning shartiga ko'ra:  $A=a^2+4a$ ,  $B=a-5$ ,  $C=-2a+4$ .

U holda:

1)  $a^2+4a=0$  yoki  $a=-4$ ,  $a=0$  da  $A=0$  bo'ladi. Shu sababli berilgan to'g'ri chiziq  $Ox$  o'qqa parallel bo'ladi.

2)  $a-5=0$  yoki  $a=5$  da  $B=0$  va berilgan to'g'ri chiziq  $Ox$  o'qqa perpendikular bo'ladi.

3)  $-2a+4=0$  yoki  $a=2$  da  $C=0$  bo'ladi. Demak,  $a=2$  da to'g'ri chiziq koordinatalar boshidan o'tadi.

To'g'ri chiziqning (1.1)-(1.10) tenglamalaridan har birini tenglamalaridan keltirib chiqarish mumkin.

Misol tariqasida (1.10) tenglamadan (1.9) tenglamani keltirib chiqaramiz. Buning uchun (1.10) tenglikning chap va o'ng tomonini

normallovchi ko'paytuvchi deb ataluvchi  $M = \pm \frac{1}{\sqrt{A^2+B^2}}$  songa ko'paytiramiz.

Hosil bo'lgan  $\frac{Ax+By+C}{\sqrt{A^2+B^2}}=0$  tenglamada

$$\cos \alpha = \pm \frac{A}{\sqrt{A^2+B^2}}, \quad \sin \alpha = \pm \frac{B}{\sqrt{A^2+B^2}}, \quad p = \pm \frac{C}{\sqrt{A^2+B^2}}$$

Qisqartirishlar kiritsak, (1.9) tenglama kelib chiqadi.



Bunda  $M$  ko'paytuvchining ishorasi  $C$  koeffitsiyentning ishorasiga qarama-qarshi qilib tanlanadi.

5-misol. To'g'ri chiziqning  $5x - 12y + 8 = 0$  tenglamasini normal ko'rinishga keltiring.

*Yechish.*

$$\text{Tenglamaning chap va o'ng tomonini } M = -\frac{1}{\sqrt{5^2 + (-12)^2}} = -\frac{1}{13}$$

(chunki  $C > 0$ ) soniga ko'paytiramiz.

Bundan

$$-\frac{5x}{13} + \frac{12y}{13} - \frac{8}{13} = 0$$

yoki

$$x \cos \alpha + y \sin \alpha - p = 0,$$

$$\text{bu yerda } \cos \alpha = -\frac{5}{13}, \quad \sin \alpha = \frac{12}{13}, \quad p = \frac{8}{13}.$$

### 3.1.3. Tekislikda ikki to'g'ri chiziqning o'zaro joylashishi

#### *Ikki to'g'ri chiziq orasidagi burchak*

Tekislikdagi ikki  $l_1$  va  $l_2$  to'g'ri chiziqlar orasidagi burchak  $\varphi$  bo'lsin.

Bu burchak to'g'ri chiziq tenglamalarining berilishiga ko'ra, turli formulalar bilan aniqlanishi mumkin.

I. To'g'ri chiziqlar umumiy tenglamalari

$$A_1x + B_1y + C_1 = 0 \quad \text{va} \quad A_2x + B_2y + C_2 = 0$$

bilan berilgan bo'lsin.

Bunda to'g'ri chiziqlarning  $\vec{n}_1 = \{A_1; B_1\}$ ,  $\vec{n}_2 = \{A_2; B_2\}$  normal vektorlari orasidagi burchak to'g'ri chiziqlar orasidagi burchakka teng, ya'ni  $\varphi = (\hat{l}_1, \hat{l}_2) = (\vec{n}_1, \vec{n}_2)$  bo'ladi (7-shakl).

Ikki vektor orasidagi burchak kosinusi formulasidan topamiz:

$$\cos \varphi = \frac{\vec{n}_1 \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1 A_2 + B_1 B_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} \quad (1.11)$$

## II. To'g'ri chiziqlar kanonik tenglamalari

$$\frac{x-x_0}{p_1} = \frac{y-y_0}{q_1} \quad \text{va} \quad \frac{x-x_0}{p_2} = \frac{y-y_0}{q_2}$$

bilan berilgan bo'lsin.

Bunda  $\vec{s}_1 = \{p_1; q_1\}$ ,  $\vec{s}_2 = \{p_2; q_2\}$  bo'ladi.

U holda  $\varphi = (\hat{l}_1, \hat{l}_2) = (\vec{s}_1, \vec{s}_2)$  (7-shakl) ekanini hisobga olib, topamiz:

$$\cos \varphi = \frac{|\vec{s}_1 \vec{s}_2|}{|\vec{s}_1| \cdot |\vec{s}_2|} = \frac{p_1 p_2 + q_1 q_2}{\sqrt{p_1^2 + q_1^2} \sqrt{p_2^2 + q_2^2}}. \quad (1.12)$$

## III. To'g'ri chiziqlar burchak koeffitsiyentli

$$y = k_1 x + b_1 \quad \text{va} \quad y = k_2 x + b_2$$

tenglamalari bilan berilgan bo'lsin.

7-shaklga ko'ra,  $\varphi = \varphi_2 - \varphi_1$ .

Hundan

$$\operatorname{tg} \varphi = \operatorname{tg}(\varphi_2 - \varphi_1), \quad \operatorname{tg} \varphi = \frac{\operatorname{tg} \varphi_2 - \operatorname{tg} \varphi_1}{1 + \operatorname{tg} \varphi_2 \operatorname{tg} \varphi_1}$$

ya'ni

$$\operatorname{tg} \varphi = \frac{k_2 - k_1}{1 + k_1 k_2} \quad (1.13)$$

kelib chiqadi.

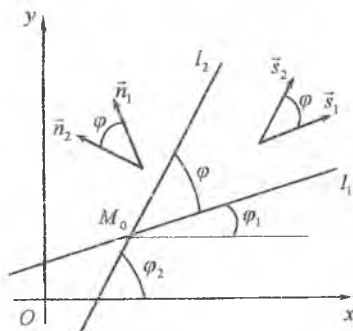
Agar bunda to'g'ri chiziqlardan qaysi biri birinchi va qaysi biri ikkinchi ekani ko'rsatilmasdan ular orasidagi o'tkir burchakni topish talab qilinsa, u holda (1.13) formulaning o'ng tomoni modulga olinadi, ya'ni

$$\operatorname{tg} \varphi = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right| \quad (1.14)$$

Shunday qilib, to'g'ri chiziqlar tenglamalarining ko'rinishiga qarab, ular orasidagi burchak (1.11) - (1.14) formulalardan biri bilan topiladi.

6-misol.  $y = -4x + 1$  va  $5x - 3y - 7 = 0$  to'g'ri chiziqlar orasidagi burchakni toping.

*Yechish.* Birinchi tenglamaga ko'ra,



7-shakl.

$k_1 = -4$ . Ikkinchi tenglamadan topamiz:

$$5x - 3y - 7 = 0, \quad y = \frac{5}{3}x - \frac{7}{3}, \quad \text{bunda } k_2 = \frac{5}{3}.$$

U holda

$$\operatorname{tg}\varphi = \frac{\frac{5}{3} - (-4)}{1 + (-4) \cdot \frac{5}{3}} = -1.$$

$$\text{Demak, } \varphi = \frac{3\pi}{4}.$$

### ***Ikki to'g'ri chiziqning perpendikularlik sharti***

Tekislikdagi ikki to'g'ri chiziqning perpendikularlik shartlarini ikki to'g'ri chiziq orasidagi burchakni topish formulalaridan keltirib chiqaramiz.

$l_1 \perp l_2$  bo'lsin. U holda  $\cos\varphi = 0$  va (1.11) tenglikdan topamiz:

$$A_1A_2 + B_1B_2 = 0. \quad (1.15)$$

Shu kabi (1.12) tenglikdan

$$p_1p_2 + q_1q_2 = 0 \quad (1.16)$$

kelib chiqadi.

(1.13) tenglikdan

$$\operatorname{ctg}\varphi = \frac{1 + k_1k_2}{k_1 - k_2}.$$

U holda  $l_1 \perp l_2$  da  $\operatorname{ctg}\varphi = 0$  yoki

$$1 + k_1k_2 = 0 \quad (1.17)$$

bo'ladi.

Demak, to'g'ri chiziqlar tenglamalarining ko'rinishiga qarab, ularning

perpendikular bo'lishi (1.15)-(1.17) shartlardan biri bilan aniqlanadi.

### ***Ikki to'g'ri chiziqning parallellik sharti***

I.  $l_1$  va  $l_2$  to'g'ri chiziqlar parallel bo'lsin. U holda ularning normal vektorlari  $\vec{n}_1 = \{A_1; B_1\}$  va  $\vec{n}_2 = \{A_2; B_2\}$  kollinear bo'ladi. Ikki

chiziqning kollinearlik shartidan ikki to'g'ri chiziqning parallellik tenglamasini topamiz:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \quad (1.18)$$

II Agar  $l_1$  va  $l_2$  to'g'ri chiziqlar parallel bo'lsa, u holda ularning o'rtak o'tuvchi vektorlari  $\vec{s}_1 = \{p_1; q_1\}$  va  $\vec{s}_2 = \{p_2; q_2\}$  kollinear bo'ladi.

Hundan

$$\frac{p_1}{p_2} = \frac{q_1}{q_2} \quad (1.19)$$

III  $l_1 \perp l_2$  bo'lganida ular orasidagi burchak uchun  $\operatorname{tg} \varphi = 0$  bo'ladi.

IV holda (1.14) tenglikdan topamiz:

$$k_1 = k_2 \quad (1.20)$$

Shunday qilib, (1.18)-(1.20) shartlardan biri to'g'ri chiziqlar tenglamalarining berilishiga ko'ra, ularning parallel bo'lishini aniqlaydi.

*Misol.*  $M_0(2;1)$  nuqtadan o'tuvchi va  $2x + 3y + 4 = 0$  to'g'ri chiziqqa perpendikular to'g'ri chiziq tenglamasini tuzing.

*Yechish.* To'g'ri chiziq tenglamasini  $Ax + By + C = 0$  ko'rinishda olaymiz.

To'g'ri chiziq  $M_0(2;1)$  nuqtadan o'tgani sababli  $2A + B + C = 0$  va  $2x + 3y + 4 = 0$  to'g'ri chiziqqa perpendikular bo'lgani uchun  $3A - 2B = 0$  bo'ladi.

Bu tenglamalarni birgalikda yechib topamiz:  $A = -\frac{3}{4}C$ ,  $B = \frac{1}{2}C$ .

$A$  va  $B$  koeffitsiyentlarni izlanayotgan tenglamaga qo'yamiz:

$$-\frac{3}{4}Cx + \frac{1}{2}Cy + C = 0.$$

Hundan

$$(-3x + 2y + 4)C = 0 \quad \text{yoki} \quad 3x - 2y - 4 = 0.$$

### ***Ikki to'g'ri chiziqning kesishishi***

To'g'ri chiziqlar umumiy tenglamalari

$$A_1x + B_1y + C_1 = 0 \quad \text{va} \quad A_2x + B_2y + C_2 = 0$$

bilan berilgan bo'lsin va  $M_0(x_0; y_0)$  nuqtada kesishsin (7-shakl).

U holda  $M_0(x_0; y_0)$  nuqtaning koordinatalari har ikkala tenglamani qanoatlantiradi. Shu sababli ikki to'g'ri chiziqning kesishish nuqtasi koordinatalari

$$\begin{cases} A_1x_0 + B_1y_0 + C_1 = 0, \\ A_2x_0 + B_2y_0 + C_2 = 0 \end{cases} \quad (1.21)$$

sistemadan topiladi.

Bunda  $M_0(x_0; y_0)$  kesishish nuqtasi orqali o'tuvchi to'g'ri chiziqlar dastasi

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0 \quad (1.22)$$

tenglama bilan aniqlanadi, bu yerda  $\lambda$  – sonli ko'paytuvchi.

*8-misol.*  $2x - y - 2 = 0$  a to'g'ri chiziq bo'ylab yo'naltirilgan yorug'lik nuri  $x - 2y + 2 = 0$  to'g'ri chiziqda sinadi va qaytadi. Qaytuvchi nur yo'nalgan to'g'ri chiziq tenglamasini tuzing.

*Yechish.* Yorug'lik nurining qaytish nuqtasi  $2x - y - 2 = 0$  va  $x - 2y + 2 = 0$  to'g'ri chiziqlarning kesishish nuqtasi bo'ladi.

Bu nuqta  $M(x; y)$  bo'lsin. Uni quyidagi sistemadan topamiz:

$$\begin{cases} 2x - y - 2 = 0, \\ x - 2y + 2 = 0. \end{cases}$$

Bundan  $M(2; 2)$ .

Yorug'lik nuri sinadigan va yo'naltirilgan to'g'ri chiziqlar orasidagi burchak tangensini topamiz.

Berilgan to'g'ri chiziqlarning burchak koeffitsiyentlari  $k_1 = \frac{1}{2}$ ,  $k_2 = 2$  bo'ladi.

Bundan

$$\operatorname{tg}\varphi = \frac{\frac{1}{2} - 2}{1 + \frac{1}{2} \cdot 2} = -\frac{3}{4}.$$

Bu son yorug'lik nuri qaytuvchi va sinuvchi to'g'ri chiziqlar orasidagi burchak tangensiga teng bo'ladi.

U holda

$$-\frac{3}{4} = \frac{k - \frac{1}{2}}{1 + \frac{1}{2} \cdot k},$$

bu yerda  $k$  — nur qaytuvchi to'g'ri chiziqning burchak koeffitsiyenti.

$$\text{Demak, } k = -\frac{2}{11}.$$

Demak, izlanayotgan to'g'ri chiziq uchun:  $M(2;2)$ ,  $k = -\frac{2}{11}$ .

Bu parametrlar bilan aniqlanuvchi to'g'ri chiziq tenglamasini tuzaniz:

$$y - 2 = -\frac{2}{11}(x - 2)$$

yoki

$$2x + 11y - 18 = 0.$$

### ***Ikki to'g'ri chiziqning ustma-ust tushishi***

$l_1$  va  $l_2$  to'g'ri chiziqlar umumiy tenglamalari

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0$$

talab berilgan bo'lsin va ustma-ust tushsin.

Hunda:

— birinchidan  $l_1 \parallel l_2$  bo'ladi va  $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \lambda$  tengliklardan  $A_1 - \lambda A_2 = 0$ ,

$B_1 - \lambda B_2 = 0$  kelib chiqadi;

— ikkinchidan  $l_1$  to'g'ri chiziqning har bir nuqtasi, jumladan,  $M_0(x_0, y_0)$  nuqtasi,  $l_2$  to'g'ri chiziqda ham yotadi, ya'ni

$$A_1x_0 + B_1y_0 + C_1 = 0, \quad A_2x_0 + B_2y_0 + C_2 = 0$$

bo'ladi.

Bu tengliklarning ikkinchisini  $\lambda$  ga ko'paytiramiz va birinchidan ayiramiz:

$$(A_1 - \lambda A_2)x_0 + (B_1 - \lambda B_2)y_0 + (C_1 - \lambda C_2) = 0.$$

Hundan  $C_1 = \lambda C_2$  kelib chiqadi.

Demak, to'g'ri chiziqlarning ustma-ust tushish sharti

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \quad (1.23)$$

tengliklar bilan ifodalanadi.

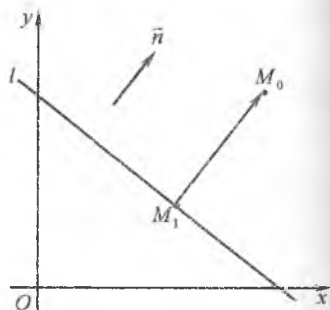
### 3.1.4. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa

Nuqtadan to'g'ri chiziqqa tushirilgan perpendikularning uzunligiga *nuqtadan to'g'ri chiziqqacha bo'lgan masofa* deyiladi.

$M_0(x_0; y_0)$  nuqta va  $Ax + By + C = 0$  tenglama bilan  $l$  to'g'ri chiziq berilgan bo'lsin.  $M_0$  nuqtadan  $l$  to'g'ri chiziqqa tushirilgan perpendikularning asosini  $M_1(x_1; y_1)$  bilan belgilaymiz (8-shakl).

U holda  $\overline{M_1M_0} = \{x_0 - x_1; y_0 - y_1\}$  va  $M_1(x_1; y_1)$  nuqta  $l$  to'g'ri chiziqda yotgani sababli  $Ax_1 + By_1 + C = 0$ , ya'ni  $C = -Ax_1 - By_1$  bo'ladi.

$\vec{n} = \{A; B\}$  vektorning  $l$  to'g'ri chiziqqa perpendikular bo'lishi ma'lum. Shu sababli  $M_0$  nuqtadan  $l$  to'g'ri chiziqqacha bo'lgan masofani vektorning o'qdagi proeksiyasining xossaligidan foydalanib topamiz:



8-shakl.

$$d = \left| \text{Pr}_{\vec{n}} \overline{M_1M_0} \right| = \frac{|\overline{M_1M_0} \cdot \vec{n}|}{|\vec{n}|} = \frac{|(x_0 - x_1)A + (y_0 - y_1)B|}{\sqrt{A^2 + B^2}} = \frac{|Ax_0 + By_0 - Ax_1 - By_1|}{\sqrt{A^2 + B^2}} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Shunday qilib, *nuqtadan to'g'ri chiziqqacha bo'lgan masofa*

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \quad (1.24)$$

formula bilan topiladi.

*9-misol.*  $3x + 4y - 4 = 0$  va  $6x + 8y + 5 = 0$  parallel to'g'ri chiziqlar orasidagi masofani toping.

*Yechish.*  $3x + 4y - 4 = 0$  to'g'ri chiziqda ixtiyoriy, masalan  $M(0;1)$  nuqtani olamiz. U holda berilgan parallel to'g'ri chiziqlar orasidagi  $d$  masofa  $M(0;1)$  nuqtadan  $6x + 8y + 5 = 0$  to'g'ri chiziqqacha bo'lgan masofaga teng bo'ladi. Uni (1.24) formula bilan hisoblaymiz:

$$d = \frac{|6 \cdot 0 + 8 \cdot 1 + 5|}{\sqrt{6^2 + 8^2}} = \frac{13}{10} (u.b).$$

### 3.1.5. Mashqlar

1. To'g'ri chiziqlarning burchak koeffitsiyentini va koordinata o'qlarida bo'lgan kesimlarini toping:

$$1) x-1, \quad 2) x=3y-2; \quad 3) \frac{y+1}{2} = \frac{x-3}{4}; \quad 4) \frac{x}{5} + \frac{y}{3} = \frac{1}{2}$$

2. To'g'ri chiziqning tenglamasini tuzing: 1)  $M_1(2;-3)$  nuqtadan o'tuvchi va  $l: x+y=0$  normal vektorga ega bo'lgan; 2)  $M_2(-2;-3)$  nuqtadan o'tuvchi va  $l: x+y=0$  yo'naltiruvchi vektorlarga ega bo'lgan; 3)  $M_3(-2;3)$  nuqtadan o'tuvchi va  $l: x+y=0$ ga perpendikular bo'lgan; 4)  $M_4(3;2)$  nuqtadan o'tuvchi  $Oy$  o'qda  $b=5$  ga teng kesima ajratuvchi.

3. Tenglamalardan qaysilari to'g'ri chiziqning normal tenglamasini ifodalaydi?

$$1) x+y=0; \quad 2) x-2,5=0; \quad 3) \frac{3}{5}x - \frac{4}{5}y - 3=0; \quad 4) \frac{12}{13}x + \frac{5}{13}y + 2=0.$$

4. To'g'ri chiziqlarning kesishish nuqtalarini va ular orasidagi burchakni toping:

$$1) x+y-3=0, \quad 2) 2x-3y+4=0; \quad 2) y = \frac{3}{4}x - \frac{5}{2}, \quad 4) 4x+3y-5=0;$$

$$1) \frac{x+1}{y} = \frac{y-1}{1}, \quad x-3y+9=0; \quad 4) \frac{x-1}{1} = \frac{y+3}{5}, \quad \frac{x-2}{-2} = \frac{y-2}{3}.$$

5.  $m$  va  $n$  ning qanday qiymatlarida  $mx+9y+n=0$  va  $4x+my-2=0$  to'g'ri chiziqlar: 1) parallel bo'ladi; 2) ustma-ust tushadi; 3) perpendikular bo'ladi?

6.  $m$  ning qanday qiymatlarida to'g'ri chiziqlar: 1) parallel bo'ladi; 2) perpendikular bo'ladi?

$$1) x-my+5=0, \quad 2x+3y+3=0; \quad 2) 2x-3y+4=0, \quad mx-6y+7=0.$$

7.  $x+y-7=0$  to'g'ri chiziqda koordinatalari  $2x-y+4=0$  tenglik bilan topilgan nuqtani toping.

8.  $A(4;2)$  nuqtadan o'tuvchi va koordinata o'qlari bilan yuzi 2 kvadrat bo'lgan teng uchburchak ajratuvchi to'g'ri chiziq tenglamasini tuzing.

9.  $A(-3;2), B(5;-2), C(0;4)$  bo'lsa,  $ABC$  uchburchakda  $BD$  balandlik tenglamasini tuzing.

10.  $A(-2;0), B(5;3), C(1;-1)$  bo'lsa,  $ABC$  uchburchakda  $AD$  mediana tenglamasini tuzing.

11.  $2x-y+3=0$  va  $x+y-2=0$  to'g'ri chiziqlarning kesishish nuqtasidan o'tuvchi va  $3x-4y-7=0$  to'g'ri chiziqqa perpendikular to'g'ri chiziq tenglamasini tuzing.



12. To'g'ri burchakli teng yonli uchburchak gipotenuzasi orqali o'tgan to'g'ri chiziq tenglamasi  $3x+2y-6=0$  dan va uchlaridan biri  $A(-1;-2)$  nuqtadan iborat. Uchburchakning katetlari tenglamalarini tuzing.

13. Parallelogrammning ikki uchi  $A(1;1)$  va  $B(2;-2)$  nuqtalarda yotadi va diagonallari  $(-1;0)$  nuqtada kesishadi. Parallelogrammning tomonlari tenglamalarini tuzing.

14. Agar  $A(5;3)$ ,  $B(1;1)$ ,  $C(3;5)$ ,  $D(6;6)$  to'rtburchakning uchlari bo'lsa, uning diagonallari kesishish nuqtasini va diagonallari orasidagi burchakni toping.

15. Uchburchakning uchlari berilgan:  $A(8;3)$ ,  $B(2;5)$ ,  $C(5;-1)$ . Uchburchak medianalarining kesishish nuqtasidan o'tuvchi va  $x+y-2=0$  to'g'ri chiziqqa perpendikular to'g'ri chiziq tenglamasini tuzing.

16. Burchak tomonlaridan birining tenglamasi  $4x-3y+9=0$  dan va bissektrisasining tenglamasi  $x-7y+21=0$  dan iborat. Burchak ikkinchi tomonining tenglamasini tuzing.

17. Uchburchakning ikki uchi  $A(5;1)$ ,  $B(1;3)$  va medianalari kesishish nuqtasi  $M(3;4)$  berilgan. Uchburchak tomonlarining tenglamalarini tuzing.

18. Uchburchakning ikki uchi  $A(2;-2)$ ,  $B(-6;2)$  va balandliklari kesishish nuqtasi  $M(1;2)$  berilgan. Uchburchakning  $C$  uchidan tushirilgan balandlik tenglamasini tuzing.

19. Uchburchak tomonlarining o'rtalari berilgan:  $M_1(1;-3)$ ,  $M_2(2;-2)$ ,  $M_3(-3;4)$ . Uchburchak tomonlarining tenglamalarini tuzing.

20. Parallelogrammning ikki tomoni  $2x+y-2=0$ ,  $x-y+17=0$  to'g'ri chiziqlarda yotadi va diagonallari  $M(-3;5;3;5)$  nuqtada kesishadi. Parallelogramm qolgan tomonlari yotgan to'g'ri chiziqlarning tenglamalarini tuzing.

21.  $x-2y+5=0$  to'g'ri chiziq bo'ylab yo'nalgan yorug'lik nuri  $3x-2y+7=0$  to'g'ri chiziqda sinadi va qaytadi. Qaytuvchi nur yo'nalgan to'g'ri chiziq tenglamasini tuzing.

22. Uchlari  $A(2;3)$ ,  $B(-1;4)$ ,  $C(5;5)$  nuqtalarda bo'lgan uchburchak og'irlik markazidan o'tuvchi va  $AC$  tomonga parallel to'g'ri chiziq tenglamasini tuzing.

23. Bir uchi  $A(3;4)$  nuqtada bo'lgan va bir tomoni  $2x+5y+3=0$  to'g'ri tenglama bilan berilgan to'g'ri chiziqda yotgan kvadratning yuzini toping.

24.  $4x-3y+8=0$  va  $8x-6y-7=0$  to'g'ri chiziqlar orasidagi masofani toping.

25. Kvadratning ikki tomoni  $5x+12y-61=0$  va  $5x+12y+17=0$  tenglamalar bilan berilgan to'g'ri chiziqlarda yotadi. Kvadrat diagonalining uzunligini toping.

16.  $M(2,4)$  nuqtadan o'tuvchi va koordinatalar boshidan 2 birlik masofada yotuvchi to'g'ri chiziq tenglamasini tuzing.

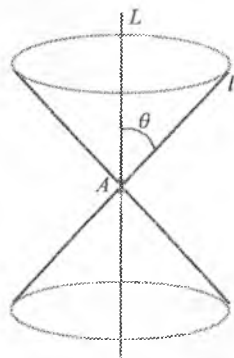
17.  $B(6,3)$  nuqtadan o'tuvchi va  $B(5,-1), C(3;7)$  nuqtalardan teng masofada yotuvchi to'g'ri chiziq tenglamasini tuzing.

18.  $M(-8,12)$  nuqtaning  $A(-5;1)$  va  $B(2;-3)$  nuqtalardan o'tuvchi to'g'ri chiziqning tekislikdagi proyeksiyasini toping.

## 3.2. IKKINCHI TARTIBLI CHIZIQLAR

3D koordinatalar sistemasida  $x, y$  o'zgaruvchilarning ikkinchi darajali tenglamasi bilan aniqlanuvchi chiziq tekislikdagi *ikkinchi tartibli chiziq* deyiladi.

Har qanday ikkinchi tartibli chiziqni darajaviy konusning tekislik bilan kesishish chiziqi sifatida hosil qilish mumkin. Shu sababli ikkinchi tartibli chiziqlar *konus kesimlar* deb ham ataladi. Berilgan  $l$  to'g'ri chiziqni uni o'tkazuvchi nuqtaga kesuvchi boshqa bir fiksirlangan  $l$  to'g'ri chiziq atrofida o'zgarimas  $\theta$  burchak ostida aylantirish natijasida hosil qilingan sirt *darajaviy konus* deyiladi (9-shakl).



9-shakl.

Danda  $l$  to'g'ri chiziqqa *konusning yasovchisi*,  $L$  to'g'ri chiziqqa *konusning o'qi*,  $A$  nuqtaga *konusning uchi*, konusning  $A$  nuqta bilan ajratilgan qismlariga *konusning pallalari* deyiladi.

Agar konus tekislik bilan kesilganida (10-shakl):

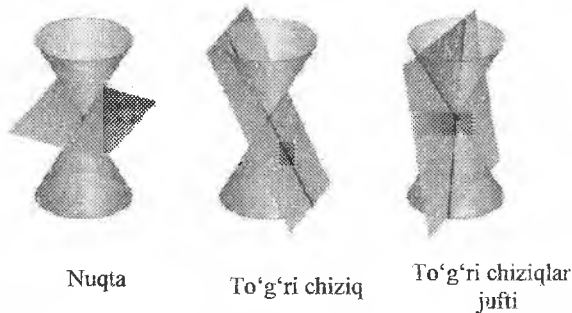
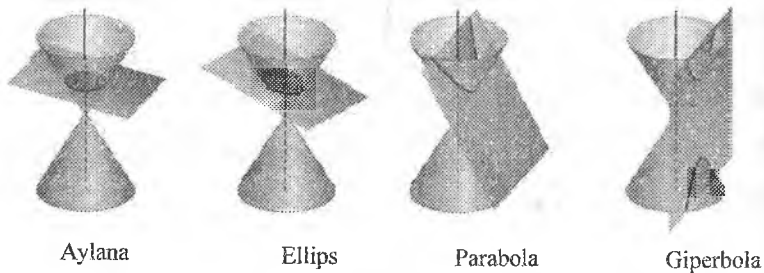
– tekislik konusning  $A$  uchidan o'tmasa va konus o'qiga perpendikular bo'lsa, kesimda *aylana* hosil bo'ladi;

– tekislik konus o'qiga perpendikular bo'lmay, konusning faqat bitta pallasini kessa va uning yasovchilaridan birortasiga parallel bo'lmasa, kesimda *ellips* hosil bo'ladi;

– tekislik konus yasovchilaridan biriga parallel ravishda uning pallalaridan birini kessa, kesimda *parabola* hosil bo'ladi;

– tekislik konusning ikkala pallasini kessa, kesimda *giperbola* hosil bo'ladi;

– tekislik konusning  $A$  uchidan o'tsa, kesimda *nuqta, to'g'ri chiziq, to'g'ri chiziqlar jufti* hosil bo'ladi.



10-shakl.

Ikkinchi tartibli chiziqlar fan va texnikaning ko'p sohalarida keng qo'llaniladi.

Bunga misollar keltiramiz.

1. Ma'lumki, avtomobil g'ildiraklari aylana shaklida yasaladi.
2. Quyosh sistemasining planetalari quyosh joylashgan umumiy fokusga ega ellipslar bo'yicha harakat qiladi.
3. Agar parabola fokusiga yorug'lik manbai joylashtirilsa, nurlar uning o'qiga parallel ravishda qaytadi. Projektorning tuzilishi bu xossaga asoslangan.

4. Mexanikada isbot qilinganidek, yer yuzidan gorizontga qarab burchak ostida  $v_0 = 11,2 \text{ km/c}$  (ikkinchi kosmik tezlik) boshlang'ich tezlik bilan chiqarilgan raketa parabola bo'ylab yer yuzidan cheksiz uzoqlashsa,  $v_0 > 11,2 \text{ km/c}$  boshlang'ich tezlik bilan chiqarilgan raketa giperbola bo'ylab yer yuzidan cheksiz uzoqlashadi,  $v_0 < 11,2 \text{ km/c}$  boshlang'ich tezlik bilan chiqarilgan raketa esa yerga qaytib tushadi yoki yerning sun'iy yo'ldoshi bo'lib qoladi.

### 3.2.1. Aylana

1-to'rif. Tekislikda markaz deb ataluvchi berilgan nuqtadan teng masofada yotuvchi nuqtalarning geometrik o'rniga *aylana* deyiladi.

Tekislikda  $M_0(x_0; y_0)$  nuqtadan  $R$  masofada yotuvchi nuqtalarni qaraymiz. Bu nuqtalardan biri  $M(x; y)$  nuqta bo'lsin (11-shakl).

Aylana ta'rifiga ko'ra,  $|M_0M| = R$ .

Bu tenglikka ikki nuqta orasidagi masofa formulasini qo'llaymiz:

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = R.$$

Shundan

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \quad (2.1)$$

(2.1) tenglamaga *aylananing kanonik tenglamasi* deyiladi. Bunda  $M_0(x_0; y_0)$  nuqta *aylana markazi*,  $R$  masofa *aylana radiusi* deb ataladi.

$x_0 = 0$ ,  $y_0 = 0$  da (2.1) tenglamadan topamiz:

$$x^2 + y^2 = R^2. \quad (2.2)$$

(2.2) tenglama markazi koordintalar boshidan o'tuvchi va radiusi  $R$  ga teng aylananani aniqlaydi.

*Umisol.* Koordinatalari  $x = R \cos t$ ,  $y = R \sin t$  tenglamalar bilan aniqlanuvchi  $M(x; y)$  nuqta aylana nuqtasi bo'lishini ko'rsating.

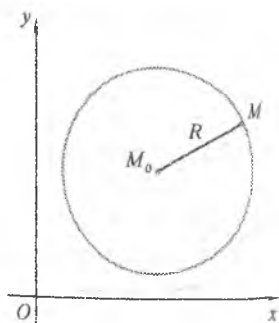
*Yechish.*  $M(x; y)$  nuqta koordinatalarining har ikkala tomonini kvadratga ko'taramiz va hadlab qo'shamiz:

$$x^2 + y^2 = R^2 \cos^2 t + R^2 \sin^2 t = R^2 (\sin^2 t + \cos^2 t) = R^2$$

yoki

$$x^2 + y^2 = R^2.$$

Demak, koordinatalari  $x = R \cos t$ ,  $y = R \sin t$ ,  $t \in R$  tenglamalar bilan aniqlanuvchi  $M(x; y)$  nuqta markazi koordintalar boshida yotuvchi va radiusi  $R$  ga teng aylanada yotadi.



11-shakl.

Aylanani aniqlovchi ushbu

$$\begin{cases} x = R \cos t, \\ y = R \sin t, \quad t \in [0; 2\pi] \end{cases} \quad (2.3)$$

tenglamalar sistemasiga aylananing parametrik tenglamalari deyiladi.

### 3.2.2. Ellips

**2-ta'rif.** Tekislikda fokuslar deb ataluvchi berilgan ikki nuqttagacha bo'lgan masofalarning yig'indisi o'zgarmas kattalikka teng bo'lgan nuqtalarning geometrik o'rniga *ellips* deyiladi.

$F_1$  va  $F_2$  ellipsning fokuslari,  $M$  ellipsning ixtiyoriy nuqtasi bo'lsin.  $F_1F_2 = 2c$ ,  $F_1M = r_1$ ,  $F_2M = r_2$  belgilashlar kiritamiz.

Ellipsning ta'rifiga ko'ra,  $F_1M + F_2M = 2a$ , ya'ni

$$r_1 + r_2 = 2a, \quad (2.4)$$

bu yerda  $a$  - o'zgarmas son bo'lib,  $a > c$ .

$Oxy$  koordinatalar sistemasini  $Ox$  o'q fokuslardan,  $Oy$  o'q  $F_1F_2$  kesmaning o'rtasidan o'tadigan qilib tanlaymiz (12-shakl).

U holda  $F_2(-c;0)$  va  $F_1(c;0)$  bo'ladi.

$M$  nuqtaning koordinatalari  $x$  va  $y$  bo'lsin deylik, ya'ni  $M(x;y)$ .

Ikki nuqta orasidagi masofa formulasi ko'ra

$$r_1 = \sqrt{(x-c)^2 + y^2}, \quad r_2 = \sqrt{(x+c)^2 + y^2}.$$

$r_1$  va  $r_2$  ning bu ifodalarini (2.4) tenglikka qo'yib, almashtirishlar bajaramiz:

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a,$$

$$(x-c)^2 + y^2 = (2a - \sqrt{(x+c)^2 + y^2})^2,$$

$$x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2,$$

$$a\sqrt{(x+c)^2 + y^2} = a^2 + xc,$$

$$a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2,$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

$b^2 = a^2 - c^2$  (chunki  $a > c$ ) belgilash kiritib, topamiz:

$$b^2x^2 + a^2y^2 = a^2b^2,$$

ya'ni

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (2.5)$$

(2.5) tenglamaga *ellipsning kanonik tenglamasi* deyiladi.

$x = acost$ ,  $y = bsint$  tenglamalar bilan aniqlanuvchi  $M(x; y)$  nuqta ellips nuqtasi bo'lishini 1-misoldagi kabi ko'rsatish mumkin.

Ellipsni aniqlovchi ushbu

$$\begin{cases} x = acost, \\ y = bsint, \quad t \in [0; 2\pi] \end{cases} \quad (2.6)$$

tenglamalar sistemasiga *ellipsning parametrik tenglamalari* deyiladi.

Ellipsning shaklini uning kanonik tenglamasidan foydalanib aniqlaymiz.

(2.5) tenglikda  $x$  va  $y$  ning faqat juft darajalari qatnashgani uchun ellips  $Ox, Oy$  o'qlarga va  $O(0; 0)$  nuqtaga nisbatan simmetrik bo'ladi. Shu sababli (2.5) tenglamani  $x \geq 0$ ,  $y \geq 0$  da (I-chorakda) tekshirish yetarli bo'ladi.

I-chorakda (2.5) tenglamadan  $y = \frac{b}{a} \sqrt{a^2 - x^2}$  kelib chiqadi.

Bunda  $x$  koordinata 0 dan  $a$  gacha o'sganida  $y$  koordinata  $b$  dan 0 gacha kamayadi. Ellipsning qolgan choraklardagi shaklini koordinata o'qlariga nisbatan simmetrik qilib chizamiz (12-shakl).

Ellipsda  $O(0; 0)$  nuqtaga markaz,  $A_1(a; 0)$ ,  $A_2(-a; 0)$ ,  $B_1(0; b)$ ,  $B_2(0; -b)$  nuqtalarga uchlar,  $A_1A_2$ ,  $B_1B_2$  kesmalarining  $2a$ ,  $2b$  uzunliklariga mos ravishda katta va kichik o'qlar,  $a, b$  sonlarga mos ravishda katta va kichik yarim o'qlar,  $F_1M$ ,  $F_2M$  kesmalarining  $r_1, r_2$  uzunliklariga fokus radiuslar deyiladi.

Ellipsning shakli  $\frac{b}{a}$  nisbatga bog'liq bo'ladi, ammo ellipsning

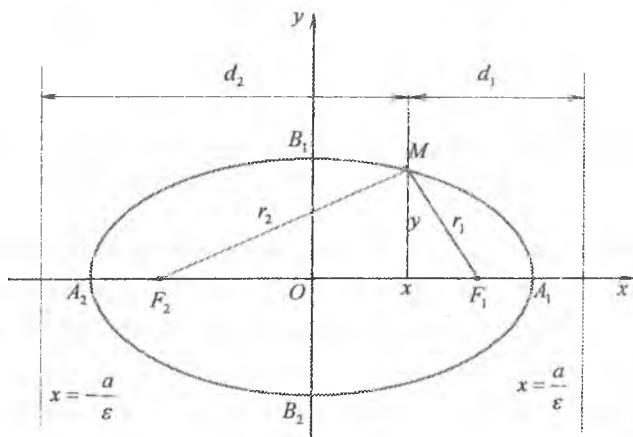
shaklini  $\frac{e}{a}$  nisbat yordamida tekshirish qulaylikka ega.

$e = \frac{c}{a}$  kattalikka *ellipsning eksstsentrtsiteti* deyiladi. Bunda  $0 < e < 1$ ,

chunki  $0 < c < a$ .  $b^2 = a^2 - c^2$  dan  $\frac{b}{a} = \sqrt{1 - \left(\frac{c}{a}\right)^2}$ , ya'ni  $\frac{b}{a} = \sqrt{1 - \varepsilon^2}$ .

Demak,  $\varepsilon \rightarrow 1$  da  $\frac{b}{a} \rightarrow 0$ , ya'ni  $b$  kichiklashib, ellips  $Oy$  o'qiga parallel ravishda  $Ox$  o'qqa tomon siqilib boradi, aksincha  $\varepsilon \rightarrow 0$  da  $\frac{b}{a} \rightarrow 1$ , ya'ni ellips aylanaga yaqinlashib boradi.

$x = \pm \frac{a}{\varepsilon}$  to'g'ri chiziqlar *ellipsning direktrisalari* deb ataladi.



12-shakl.

Ellipsning  $M$  nuqtasidan direktrisalargacha bo'lgan  $d_1$  va  $d_2$  masofalar uchun ushbu

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon$$

tengliklar bajariladi (12-shakl).

Bu tengliklardan ellipsning fokal radiuslari uchun

$$r_1 = a - \varepsilon x, \quad r_2 = a + \varepsilon x$$

formulalar hosil qilinadi.

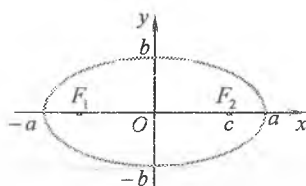
Fokuslari  $Oy$  o'qida va markazi koordinatalar boshda yotuvchi ellipsning kanonik tenglamalari shu kabi aniqlanadi.

Har ikkala hol uchun ellipsning tenglamalarini va asosiy xossalari keltiramiz.

**Markazi  $O(0;0)$  nuqtada bo'lgan ellipsning kanonik tenglamalari va asosiy xossalari**

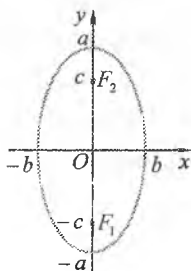
$$1. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b > 0$$

- katta o'q  $Ox$  da yotadi
- va  $2a$  ga teng;
- kichik o'q  $Oy$  da yotadi
- va  $2b$  ga teng;
- fokal:  $F_1(-c;0), F_2(c;0)$
- $c^2 = a^2 - b^2$ .



$$2. \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a > b > 0$$

- katta o'q  $Oy$  da yotadi
- va  $2a$  ga teng;
- kichik o'q  $Ox$  da yotadi
- va  $2b$  ga teng;
- fokal:  $F_1(0;-c), F_2(0;c)$
- $c^2 = a^2 - b^2$



Agar  $a = b$  bo'lsa, u holda (2.5) tenglamadan  $x^2 + y^2 = a^2$  tenglama, ya'ni markazi koordinata boshida yotuvchi va radiusi  $a$  ga teng aylana tenglamasi kelib chiqadi. Demak, aylana ellipsning xususiy holi hisoblanadi.

**2-misol.**  $4x^2 + 9y^2 = 144$  ellipsning o'qlari uzunliklarini, fokuslarining koordinatalarini va eksentrisitetini toping.

**Yechish.** Ellipsning tenglamasini kanonik ko'rinishga keltiramiz:

$$\frac{x^2}{36} + \frac{y^2}{16} = 1.$$

Bundan  $a^2 = 36, b^2 = 16$ .

Demak,  $a = 6, b = 4, 2a = 12, 2b = 8$ .

Shunday qilib, ellips o'qlarining uzunliklari mos ravishda 12 va 8 ga teng.



$a$  va  $b$  ni bilgan holda  $c$  ni aniqlaymiz:

$$c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = 2\sqrt{5}.$$

Bundan fokuslarning koordinatalarini va eksentrisitetni topamiz:

$$F_1(2\sqrt{5}; 0), F_2(-2\sqrt{5}; 0);$$

$$e = \frac{c}{a} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}.$$

### 3.2.3. Giperbola

**3-ta'rif.** Tekislikda fokuslar deb ataluvchi berilgan ikki nuqttagacha bo'lgan masofalar ayirmasining moduli o'zgarmas kattalikka teng bo'lgan nuqtalarning geometrik o'rniga *giperbola* deyiladi.

$Oxy$  koordinatalar sistemasini  $Ox$  o'q  $F_1$  va  $F_2$  fokuslardan,  $Oy$  o'q  $F_1F_2$  kesmaning o'rtasidan o'tadigan qilib tanlaymiz (13-shakl).

$M(x, y)$  giperbolaning ixtiyoriy nuqtasi bo'lsin.  $F_1F_2 = 2c$ ,  $F_1M = r_1$ ,  $F_2M = r_2$  belgilashlar kiritamiz. Giperbolaning ta'rifiga ko'ra,

$$|r_1 - r_2| = 2a, \quad (2.7)$$

bu yerda  $a$  — o'zgarmas son bo'lib,  $a < c$ .

(2.7) ifodada (2.4) ifodada bajarilgan almashtirishlar kabi almashtirishlar bajarib, quyidagi tenglamani keltirib chiqaramiz:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad (2.8)$$

bu yerda  $b^2 = c^2 - a^2$ .

(2.8) tenglamaga *giperbolaning kanonik tenglamasi* deyiladi.

Giperbolaning shaklini uning kanonik tenglamasidan foydalanib aniqlaymiz.

(2.8) tenglikda  $x$  va  $y$  ning faqat juft darajalari qatnashgani uchun giperbola ellips kabi  $Ox, Oy$  o'qlarga va  $O(0; 0)$  nuqtaga nisbatan simmetrik bo'ladi. Shu sababli (2.8) tenglamani  $x \geq 0$ ,  $y \geq 0$  da (I-chorakda) tekshiramiz.

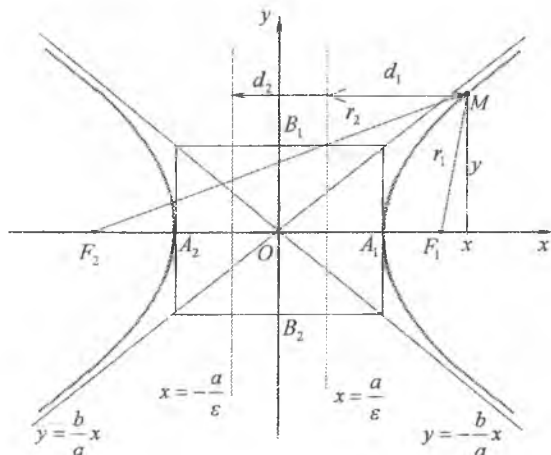
I-chorakda (2.8) tenglamadan  $y = \frac{b}{a}\sqrt{x^2 - a^2}$  kelib chiqadi. Bunda  $x \geq a$  va  $x$  koordinata  $a$  dan boshlab o'sishi bilan  $y$  koordinata ham

o'tib boradi, ya'ni  $M(x;y)$  nuqta cheksizlikka intiladi. Bu intilish qanday yuz berishini ko'rsatish uchun koordinatalar boshidan o'tuvchi va burchak koeffitsiyenti  $k = \frac{b}{a}$  ga teng bo'lgan  $y = \frac{b}{a}x$  to'g'ri chiziqni qaraymiz. Bu chiziq ushbu xossaga ega:  $M$  nuqta giperbola bo'ylab harakat qilib koordinata boshidan cheksiz uzoqlashgani sari bu to'g'ri chiziqqa juda yaqinlashib boradi, lekin uni kesib o'tmaydi, ya'ni asimptotik yaqinlashadi.

Shunday qilib, giperbola I-chorakda  $A_1(a;0)$  nuqtadan o'tib,  $y = \frac{b}{a}x$  to'g'ri chiziqqa asimptotik yaqinlashgani holda o'ngga va yuqoriga qarab o'tib boradi.

Giperbolaning qolgan choraklardagi shaklini koordinata o'qlariga nisbatan simmetrik qilib chizamiz (13-shakl).

Shunday qilib, giperbola ikki qismdan iborat bo'ladi. Bu qismlarga giperbolaning tarmoqlari deyiladi.



13-shakl.

$y = \pm \frac{b}{a}x$  tenglama bilan aniqlanuvchi to'g'ri chiziq'larga giperbolaning asimptotalari deyiladi.

Giperbolada  $A_1(a;0)$ ,  $A_2(-a;0)$ ,  $B_1(0;b)$ ,  $B_2(0;-b)$  nuqtalarga uchlar,  $A_1A_2$  kesmaning  $2a$  uzunligiga haqiqiy o'q,  $B_1B_2$  kesmaning  $2b$  uzunligiga mavhum o'q,  $a$ ,  $b$  sonlarga mos ravishda haqiqiy va

mavhum yarim o'qlar,  $F_1M$ ,  $F_2M$  kesmalarning  $r_1$ ,  $r_2$  uzunliklariga fokal radiuslar deyiladi.

$\varepsilon = \frac{c}{a}$  kattalikka giperbolaning eksentrisiteti deyiladi.

Bunda  $\varepsilon > 1$ , chunki  $c > a$ .

$$b^2 = c^2 - a^2 \text{ dan } \frac{b}{a} = \sqrt{\left(\frac{c}{a}\right)^2 - 1}, \text{ ya'ni } \frac{b}{a} = \sqrt{\varepsilon^2 - 1}.$$

Demak, eksentrisitet birga qanchalik yaqin bo'lsa,  $\frac{b}{a}$  shunchalik kichik bo'ladi, ya'ni  $\varepsilon \rightarrow 1$  da  $\frac{b}{a} \rightarrow 0$  va giperbola haqiqiy o'qi tomon siqilib boradi, aksincha  $\varepsilon$  kattalashgan sayin  $\frac{b}{a}$  ham kattalashadi va giperbolaning tarmoqlari kengayib boradi.

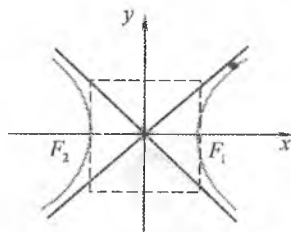
$x = \pm \frac{a}{\varepsilon}$  to'g'ri chiziqlar giperbolaning direktrisalari deb ataladi.

Fokuslari  $Oy$  o'qida va markazi koordinatalar boshda yotuvchi giperbolaning kanonik tenglamasi shu kabi aniqlanadi.

**Markazi  $O(0;0)$  nuqtada bo'lgan giperbolaning kanonik tenglamalari va asosiy xossalari**

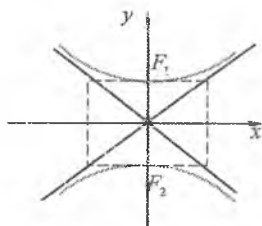
1.  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- haqiqiy o'q  $Ox$  da yotadi va  $2a$  ga teng;
- mavhum o'q  $Oy$  da yotadi va  $2b$  ga teng;
- fokuslar:  $F_1(-c;0)$ ,  $F_2(c;0)$ ;
- $c^2 = a^2 + b^2$ ;
- asimptotalari:  $y = \pm \frac{b}{a}x$



2.  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

- haqiqiy o'q  $Oy$  da yotadi va  $2a$  ga teng;
- mavhum o'q  $Ox$  da yotadi va  $2b$  ga teng;
- fokuslar:  $F_1(0;-c)$ ,  $F_2(0;c)$ ;
- $c^2 = a^2 + b^2$ ;
- asimptotalari:  $y = \pm \frac{a}{b}x$ .



Giperbolaning  $M$  nuqtasidan direktrisalargacha bo'lgan  $d_1$  va  $d_2$  masofalar uchun ushbu

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon$$

tengliklar bajariladi (13-shakl).

Bu tengliklardan giperbolaning fokal radiuslari uchun ushbu

$$x > 0 \text{ bo'lganda } r_1 = \varepsilon x - a, r_2 = \varepsilon x + a;$$

$$x < 0 \text{ bo'lganda } r_1 = -a - \varepsilon x, r_2 = a - \varepsilon x$$

formulalar hosil qilinadi.

Yarim o'qlari teng bo'lgan giperbolaga *teng tomonli giperbola* deyiladi.

Teng tomonli giperbola

$$x^2 - y^2 = a^2 \quad (2.9)$$

tenglama bilan aniqlanadi.

*Umisol.* Ekssentrisiteti  $\sqrt{2}$  ga teng va  $M(\sqrt{3}; \sqrt{2})$  nuqtadan o'tuvchi giperbolaning kanonik tenglamasini tuzing. Uning yarim o'qlari uzunligini, fokuslari koordinatalarini toping va asimptotalarining, direktrisalarining tenglamalarini tuzing.

*Yechish.* Ma'lumki,  $\varepsilon = \frac{c}{a} = \sqrt{2}$  yoki  $c^2 = 2a^2$ . Ikkinchi tomondan  $c^2 = a^2 + b^2$ . Bundan  $a^2 = b^2$ . Demak, izlanayotgan giperbola teng tomonli,  $M(\sqrt{3}; \sqrt{2})$  nuqta giperbolada yotgani uchun  $\frac{(\sqrt{3})^2}{a^2} - \frac{(\sqrt{2})^2}{a^2} = 1$ , y'ni  $a^2 = 1$ .

Demak, izlanayotgan giperbolaning kanonik tenglamasi

$$x^2 - y^2 = 1$$

ko'rinishini oladi.

Bu tenglama bilan aniqlanuvchi giperbolaning yarim o'qlari  $a = b = 1$  uzunlikka, fokuslari  $F_1(\sqrt{2}; 0)$ ,  $F_2(-\sqrt{2}; 0)$  koordinatalarga ega bo'ladi, asimptotalari  $y = \pm x$  tenglamalar bilan, direktrisalari  $x = \pm \frac{1}{\sqrt{2}}$  tenglamalar bilan topiladi.

### 3.2.4. Parabola

**4-ta'rif.** Tekislikda fokus deb ataluvchi berilgan nuqtadan va direktrisa deb ataluvchi berilgan to'g'ri chiziqdan teng uzoqlikda yotuvchi nuqtalarning geometrik o'rniga *parabola* deyiladi.

Parabolaning fokusidan direktrisasigacha bo'lgan masofani  $p$  ( $p > 0$ ) bilan belgilaymiz.

$p$  kattalikka *parabolaning parametri* deyiladi.

$Oxy$  koordinatalar sistemasini  $Ox$  o'q direktrisaqa perpendikular va fokusdan o'tadigan,  $O(0;0)$  nuqta fokus va direktrisaning o'rtasida yotadigan qilib tanlaymiz. Tanlangan koordinatalar sistemasida

$F\left(\frac{p}{2}; 0\right)$  nuqta fokus,  $x = -\frac{p}{2}$  to'g'ri chiziq direktrisa bo'ladi (17-shakl).

$M(x; y)$  parabolaning ixtiyoriy nuqtasi bo'lsin.

$M$  nuqtaning direktrisadagi proyeksiyasini  $N$  bilan belgilaymiz.

Parabolaning ta'rifiga ko'ra  $NM = MF$ .

Bundan

$$\left|x + \frac{p}{2}\right| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2},$$

$$x^2 + px + \frac{p^2}{4} = x^2 - px + \frac{p^2}{4} + y^2$$

yoki

$$y^2 = 2px. \quad (2.10)$$

(2.10) tenglamaga *parabolaning kanonik tenglamasi* deyiladi.

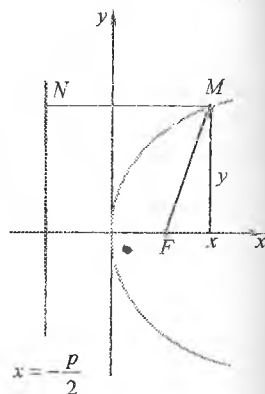
Parabolaning shaklini uning kanonik tenglamasidan foydalanib aniqlaymiz.

(2.10) tenglikda  $y$  ning juft darajasi

qatnashgani uchun parabola  $Ox$  o'qqa nisbatan simmetrik bo'ladi.

Shu sababli (2.10) tenglamani  $x \geq 0, y \geq 0$  da tekshiramiz.

I chorakda (2.10) tenglamadan  $y = \sqrt{2px}$  kelib chiqadi. Bunda  $x \geq 0$  va  $x$  koordinata 0 dan boshlab o'sishi bilan  $y$  koordinata ham o'sib boradi. Shunday qilib,  $y \geq 0$  bo'lganda  $M(x; y)$  nuqta  $O(0;0)$  nuqtadan chiqadi va  $x$  o'sishi bilan o'ngga va yuqoriga qarab bu nuqtadan cheksiz uzoqlashadi. Parabolaning  $y \leq 0$  dagi shaklini  $Ox$  o'qqa nisbatan



14-shakl.

simmetrik qilib chizamiz (14-shakl). Bunda  $O(0;0)$  nuqta parabolaning o'qi,  $O'F$  o'q parabolaning o'qi deb ataladi.

Parabolaning *ekstsentrisiteti*  $\varepsilon = \frac{NM}{MF} = 1$  ga teng bo'ladi, *direktrisasi*

$x = -\frac{p}{2}$  tenglama bilan aniqlanadi.

*4-misol.*  $y^2 = 6x$  parabola berilgan. Uning direktisasi tenglamasini toping va fokusini toping.

*Yechish.* Berilgan tenglamani parabolaning kanonik tenglamasi (3.10) bilan taqqoslab, ko'ramizki,  $2p = 6$  yoki  $p = 3$ .

(1) holda berilgan parabola uchun direktisasi tenglamasi

$x = -\frac{p}{2} = -\frac{3}{2}$  va fokusi  $F\left(\frac{p}{2}; 0\right) = \left(\frac{3}{2}; 0\right)$  bo'ladi.

Parabolaning boshqa kanonik tenglamalari shu kabi aniqlanadi.

**Uchi  $O(0;0)$  nuqtada bo'lgan parabolaning kanonik tenglamalari va asosiy xossalari**

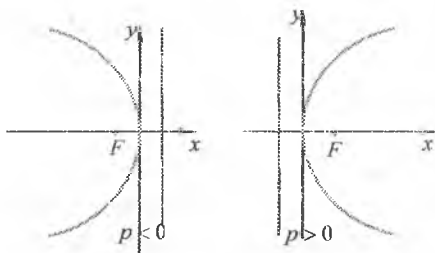
1.  $y^2 = 2px$

– fokus:  $F_1\left(\frac{p}{2}; 0\right)$ ;

– direktitsa:  $x = -\frac{p}{2}$

va  $Ox$  oqqa simmetrik;

– simmetriya o'qi –  $Ox$



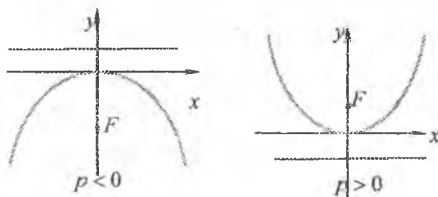
2.  $x^2 = 2py$

– fokus:  $F_1\left(0; \frac{p}{2}\right)$ ;

– direktitsa:  $y = -\frac{p}{2}$

va  $Oy$  oqqa simmetrik;

– simmetriya o'qi –  $Oy$



### 3.2.5. Ikkinchi tartibli chiziqlarning umumiy tenglamasi

Ikkita  $x$  va  $y$  o'zgaruvchining ikkinchi darajali tenglamasi umumiy ko'rinishda

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \quad A^2 + B^2 + C^2 \neq 0 \quad (2.11)$$

kabi yoziladi, bu yerda  $A, B, C, D, E, F$  – koeffitsiyentlar.

Oldingi bandlarda ta'riflari asosida ellips, giperbola va parabolalarning kanonik tenglamalarini keltirib chiqardik va xossalarini o'rgandik. Bunda chiziqlarning markazlarini koordinatalar boshiga joylashtirdik va ularning o'qlarini koordinata o'qlari bo'ylab yo'naltirdik.

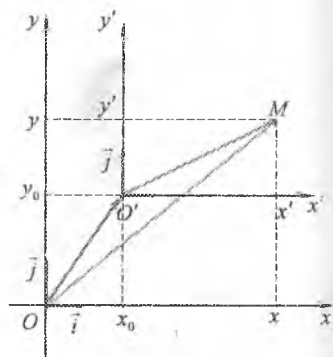
Ushbu bandda (2.11) tenglama koeffitsiyentlarining mos qiymatlarida konus kesimlardan birini, yoki mavhum konus kesimlardan birini, yoki bo'sh to'plamni aniqlashini ko'rsatamiz. Bunda konus kesimning markazi koordinatalar boshida yotmasligi va o'qlari koordinata o'qlariga nisbatan og'ishga ega bo'lishi qiyinchilik tug'dirishi mumkin. Bu qiyinchilikni bartaraf qilish uchun koordinatalar usulining ikki qurolidan – koordinatalar o'qlarini parallel ko'chirish va burishdan foydalanamiz.

#### *Koordinata o'qlarini parallel ko'chirish*

Tekislikda  $Oxy$  to'g'ri burchakli koordinatalar sistemasi berilgan bo'lsin.

Koordinata o'qlarini parallel ko'chirish – bu  $Oxy$  sistemadan uning o'qlari yo'nalishlarini va masshtablarini o'zgartirmasdan faqat koordinatalar boshining joylashishini o'zgartirish orqali yangi  $O'x'y'$  sistemaga o'tishdir.

Yangi  $O'x'y'$  sistemaning koordinatalar boshi  $O'$  eski  $Oxy$  sistemada  $(x_0; y_0)$  koordinatalarga ega bo'lsin, ya'ni  $O'(x_0; y_0)$ .



15-shakl.

Tekislik ixtiyoriy  $M$  nuqtasining  $Oxy$  sistemadagi koordinatalarini

15) bilan va  $Ox'y'$  sistemadagi koordinatalarini  $(x';y')$  bilan belgilaymiz (15-shakl).

U holda

$$\overline{OM} = x\vec{i} + y\vec{j}, \quad \overline{OO'} = x_0\vec{i} + y_0\vec{j}, \quad \overline{O'M} = x'\vec{i} + y'\vec{j}.$$

15-shakldan topamiz:  $\overline{OM} = \overline{OO'} + \overline{O'M}$ .

Hundan

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + x'\vec{i} + y'\vec{j}$$

yoki

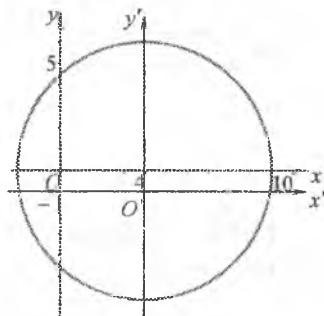
$$x = x_0 + x', \quad y = y_0 + y'. \quad (2.12)$$

(2.12) formulalar  $M$  nuqtaning  $Oxy$  sistemadagi  $(x,y)$  koordinatalarini  $Ox'y'$  sistemadagi  $(x';y')$  koordinatalar orqali topish imkonini beradi va aksincha.

*3-misol.*  $(x-4)^2 + (y+1)^2 = 36$  tenglamani  $Oxy$  koordinatalar sistemasini parallel ko'chirish orqali soddalashtiring.

*Yechish.* Berilgan tenglama  $Oxy$  koordinatalar sistemasida markazi  $O(4;-1)$  nuqtada yotuvchi va radiusi  $R=6$  ga teng aylananing ifodalaydi.

$Oxy$  koordinatalar sistemasini  $O'(x_0;y_0) = O'(4;-1)$  nuqtaga parallel ko'chirilsa, berilgan tenglama yangi  $Ox'y'$  sistemada ham aylana tenglamasini beradi.



16-shakl.

(2.12) formulalarni qo'llab, topamiz:

$$x' = x - x_0 = x - 4, \quad y' = y - y_0 = y + 1.$$

U holda berilgan tenglama  $Ox'y'$  sistemada

$$x'^2 + y'^2 = 36$$

ko'rinishini oladi, ya'ni markazi koordinatalar boshida bo'lgan va radiusi  $R=6$  ga teng aylananing ifodalaydi.

Aylana grafigini  $Oxy$  va  $Ox'y'$  sistemalarda chizamiz (16-shakl).

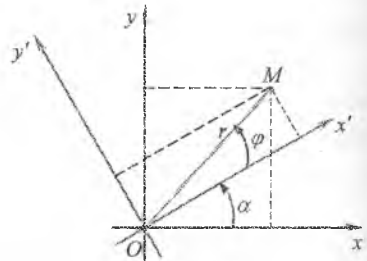


## Koordinata o'qlarini burish

Tekislikda  $Oxy$  to'g'ri burchakli koordinatalar sistemasi berilgan bo'lsin.

Koordinata o'qlarini burish — bu  $Oxy$  sistemadan uning koordinatalar boshini va o'qlari mashtablarini o'zgartirmasdan faqat koordinata o'qlarini biror burchakka burish orqali yangi  $Ox'y'$  sistemaga o'tishdir.

$Oxy$  sistemani  $O$  nuqta atrofida soat strelkasi yo'nalishiga teskari yo'nalishda  $\alpha$  burchakka burib,  $Ox'y'$  sistemaga o'tamiz. Tekislik ixtiyoriy  $M$  nuqtasining  $Oxy$  sistemadagi koordinatalarini  $(x; y)$



17-shakl.

bilan va  $Ox'y'$  sistemadagi koordinatalarini  $(x'; y')$  bilan belgilaymiz.  $M$  nuqta radius vektorining uzunligi  $r$  ga, uning  $Ox'$  o'q bilan tashkil qilgan burchagi  $\varphi$  ga teng bo'lsin (17-shakl).

17-shakldan topamiz:

$$x' = r \cos \varphi, \quad y' = r \sin \varphi, \quad (2.13)$$

$$x = r \cos(\varphi + \alpha), \quad y = r \sin(\varphi + \alpha). \quad (2.14)$$

(2.14) tengliklar ustida almashtirishlar bajaramiz va (2.13) tengliklarni hisobga olib, topamiz:

$$\begin{aligned} x &= r \cos(\varphi + \alpha) = r(\cos \varphi \cos \alpha - \sin \varphi \sin \alpha) = \\ &= (r \cos \varphi) \cos \alpha - (r \sin \varphi) \sin \alpha = x' \cos \alpha - y' \sin \alpha, \\ y &= r \sin(\varphi + \alpha) = r(\sin \varphi \cos \alpha + \cos \varphi \sin \alpha) = \\ &= r(\cos \varphi) \sin \alpha + r(\sin \varphi) \cos \alpha = x' \sin \alpha + y' \cos \alpha. \end{aligned}$$

Demak,

$$x = x' \cos \alpha - y' \sin \alpha, \quad y = x' \sin \alpha + y' \cos \alpha. \quad (2.15)$$

(2.15) formulalarga koordinata o'qlarini burish formulalari deyiladi. Bu formulalar  $M$  nuqtaning  $Oxy$  sistemadagi  $(x; y)$  koordinatalarini

... sistemadagi  $(x';y')$  koordinatalar orqali topish imkonini beradi  
 ... shaklida.

*Yechish.*  $xy = -2$  tenglamani koordinata o'qlarini  $45^\circ$  ga burish  
 ... kanonik shaklga keltiring.

*Yechish.* (2.15) tengliklardan  $\alpha = 45^\circ$  da topamiz:

$$x = x' \cos 45^\circ - y' \sin 45^\circ = \frac{\sqrt{2}}{2} (x' - y'),$$

$$y = x' \sin 45^\circ + y' \cos 45^\circ = \frac{\sqrt{2}}{2} (x' + y').$$

...  $xy = -2$  tenglamaga qo'yamiz:

$$\frac{\sqrt{2}}{2} (x' - y') \cdot \frac{\sqrt{2}}{2} (x' + y') = -2,$$

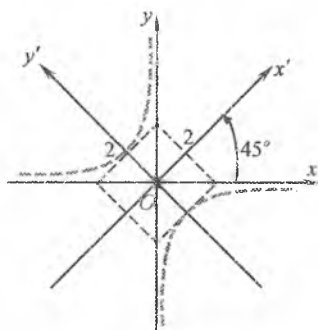
$$\frac{1}{2} (x'^2 - y'^2) = -2,$$

$$\frac{y'^2}{4} - \frac{x'^2}{4} = 1.$$

Bu tenglama giperbolaning kanonik  
 tenglamasi hisoblanadi, ya'ni  $xy = -2$   
 tenglama bilan aniqlanuvchi chiziq  $Ox'y'$

sistemada  $\frac{y'^2}{4} - \frac{x'^2}{4} = 1$  tenglama bilan  
 aniqlanuvchi giperbolani ifodalaydi.

Demak,  $y = -\frac{2}{x}$  funksiyaning grafigi  
 asimptotulari koordinata o'qlari bilan  
 ... ust tushadigan teng tomonli  
 giperboladan iborat (18-shakl).



18-shakl.

### **Ikkinchi tartibli umumiy tenglamani soddalashtirish**

Ikkinchi tartibli umumiy tenglamani qaraymiz:

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \quad (2.11)$$

Bunda  $B \neq 0$  bo'lsin. Koordinata o'qlarini  $\alpha$  burchakka buramiz,  
 ... (2.15) formulalar yordamida eski koordinatalarni yangi

koordinatalar orqali ifodalaymiz:

$$A(x' \cos \alpha - y' \sin \alpha)^2 + 2B(x' \cos \alpha - y' \sin \alpha)(x' \sin \alpha + y' \cos \alpha) + \\ + C(x' \sin \alpha + y' \cos \alpha)^2 + 2D(x' \cos \alpha - y' \sin \alpha) + 2E(x' \sin \alpha + y' \cos \alpha) + F = 0$$

yoki

$$A_1 x'^2 + 2B_1 x' y' + C_1 y'^2 + 2D_1 x' + 2E_1 y' + F_1 = 0,$$

bu yerda

$$A_1 = A \cos^2 \alpha + 2B \cos \alpha \sin \alpha + C \sin^2 \alpha;$$

$$B_1 = (C - A) \sin \alpha \cos \alpha + B(\cos^2 \alpha - \sin^2 \alpha);$$

$$C_1 = A \sin^2 \alpha - 2B \cos \alpha \sin \alpha + C \cos^2 \alpha;$$

$$D_1 = D \cos \alpha + E \sin \alpha; \quad E_1 = E \sin \alpha - D \cos \alpha; \quad F_1 = F.$$

$\alpha$  burchakni shunday tanlaymizki, yuqoridagi tenglamada  $x'y'$  oldidagi koeffitsiyent nolga aylansin, ya'ni

$$B_1 = (C - A) \sin \alpha \cos \alpha + B(\cos^2 \alpha - \sin^2 \alpha) = 0$$

tenglik bajarilsin.

Bundan

$$\operatorname{ctg} 2\alpha = \frac{A - C}{2B}. \quad (2.16)$$

Shunday qilib, koordinata o'qlarini (2.16) shartni qanoatlantiruvchi  $\alpha$  burchakka burish (2.11) tenglamani quyidagi tenglamaga keltiradi:

$$A_1 x'^2 + C_1 y'^2 + 2D_1 x' + 2E_1 y' + F_1 = 0, \quad A_1^2 + B_1^2 \neq 0. \quad (2.17)$$

**1-teorema.** (2.17) tenglama hamma vaqt yoki aylani ( $A_1 = C_1$  da), yoki ellipsni ( $A_1 \cdot C_1 > 0$  da), yoki giperbolani ( $A_1 \cdot C_1 < 0$  da), yoki parabolani ( $A_1 \cdot C_1 = 0$  da) aniqlaydi. Bunda ellips (aylana) uchun – nuqta yoki mavhum ellips (aylana), giperbola uchun – kesishuvchi to'g'ri chiziqlar juftligi, parabola uchun – parallel to'g'ri chiziqlar juftligi kabi buzilishlar bo'lishi mumkin.

*Isboti.*  $A_1 = C_1$  bo'lgan holni batafsil tahlil qilamiz.

$A_1 = C_1$  da (2.17) tenglik ustida almashtirishlar bajaramiz:

$$A_1 x'^2 + A_1 y'^2 + 2D_1 x' + 2E_1 y' + F_1 = 0,$$

$$x'^2 + y'^2 + 2\frac{D_1}{A_1}x' + 2\frac{E_1}{A_1}y' + \frac{F_1}{A_1} = 0,$$

$$x'^2 + 2\frac{D_1}{A_1}x' + \left(\frac{D_1}{A_1}\right)^2 + y'^2 + 2\frac{E_1}{A_1}y' + \left(\frac{E_1}{A_1}\right)^2 + \frac{F_1}{A_1} - \left(\frac{D_1}{A_1}\right)^2 - \left(\frac{E_1}{A_1}\right)^2 = 0,$$

$$\left(x' + \frac{D_1}{A_1}\right)^2 + \left(y' + \frac{E_1}{A_1}\right)^2 = \left(\frac{D_1}{A_1}\right)^2 + \left(\frac{E_1}{A_1}\right)^2 - \frac{F_1}{A_1}. \quad (2.18)$$

Hunda, (2.18) tenglama va mos ravishda (2.17) tenglama:

$$1) \left(\frac{D_1}{A_1}\right)^2 + \left(\frac{E_1}{A_1}\right)^2 - \frac{F_1}{A_1} > 0 \text{ bo'lganda markazi } O_1\left(-\frac{D_1}{A_1}, -\frac{E_1}{A_1}\right) \text{ nuqtada}$$

aylanishgan va radiusi  $R = \sqrt{\left(\frac{D_1}{A_1}\right)^2 + \left(\frac{E_1}{A_1}\right)^2 - \frac{F_1}{A_1}}$  ga teng aylananani aniqlaydi;

$$2) \left(\frac{D_1}{A_1}\right)^2 + \left(\frac{E_1}{A_1}\right)^2 - \frac{F_1}{A_1} = 0 \text{ bo'lganda } \left(x + \frac{D_1}{A_1}\right)^2 + \left(y + \frac{E_1}{A_1}\right)^2 = 0 \text{ ko'rinishga}$$

keladi. Bu tenglikni yagona  $O_1\left(-\frac{D_1}{A_1}, -\frac{E_1}{A_1}\right)$  nuqta koordinatalari

qamoutlantiradi. Bunda «aylana nuqtaga buzilgan» deyiladi;

$$3) \left(\frac{D_1}{A_1}\right)^2 + \left(\frac{E_1}{A_1}\right)^2 - \frac{F_1}{A_1} < 0 \text{ bo'lganda hech bir chiziqni aniqlamaydi.}$$

Bunda «aylana mavhum aylana buzilgan» deyiladi.

Qolgan hollarda teorema shu kabi tahlil qilinadi.

Shunday qilib, (2.17) tenglama (mos ravishda (2.11) tenglama) ikkinchi tartibli chiziqlardan birini aniqlaydi.

*7-misol.*  $4x^2 - 25y^2 - 24x + 50y - 89 = 0$  tenglama bilan berilgan ikkinchi tartibli chiziq ko'rinishini aniqlang.

*Yechish.* Berilgan tenglamada  $A = 4$ ,  $C = -25$ .

Bundan  $A \cdot C = 4 \cdot (-25) < 0$ .

Teoremaga ko'ra, berilgan tenglama giperbolani ifodalaydi.

Tenglamada almashtirishlar bajaramiz:

$$4(x^2 - 6x + 9) - 25(y^2 - 2y + 1) - 36 + 25 - 89 = 0,$$

$$4(x-3)^2 - 25(y-1)^2 = 100,$$

$$\frac{(x-3)^2}{25} - \frac{(y-1)^2}{4} = 1.$$

Demak, berilgan tenglama simmetriya markazi  $O(3;1)$  nuqtada joylashgan va yarim o'qlari  $a=5$ ,  $b=2$  ga teng bo'lgan giperbolani aniqlaydi.

### 3.2.6. Mashqlar

1. Aylananing tenglamasini tuzing: 1) markazi  $M_1(-1;3)$  nuqtada joylashgan va radiusi  $R=6$  ga teng bo'lgan; 2) markazi  $M_2(-3;5)$  nuqtada joylashgan va  $A(4;4)$  nuqtadan o'tgan; 3) diametrlaridan birining uchlari  $B(-1;3)$  va  $C(-3;5)$  nuqtalarda bo'lgan; 4)  $D(8;-4)$  nuqtadan o'tgan va koordinata o'qlariga uringan; 5) markazi  $M(2;-1)$  nuqtada joylashgan va urinmalaridan biri  $3x+4y+3=0$  to'g'ri chiziqda bo'lgan.

2. Aylanalarning markazi va radiusini toping:

1)  $x^2 + y^2 + 8x - 14y + 16 = 0;$

2)  $x^2 + y^2 + 4x - 6y - 3 = 0;$

3)  $x^2 + y^2 - x + 2y - 1 = 0;$

4)  $x^2 + y^2 + 3x - 7y - \frac{3}{2} = 0.$

3.  $A(1;-2)$  nuqta berilgan. Uning aylanalarning ichkarisida, tashqarisida yoki ustida yotishini aniqlang.

1)  $x^2 + y^2 = 5;$

2)  $x^2 + y^2 = 9;$

3)  $x^2 + y^2 - 8x - 4y - 5 = 0;$

4)  $x^2 + y^2 - 10x + 8y = 0.$

4.  $x^2 + y^2 - 2x + 4y - 20 = 0$  va  $x^2 + y^2 - 10y + 20 = 0$  tenglamalar bilan berilgan aylanalar markazlari orasidagi masofani toping.

5.  $\frac{x}{4} + \frac{y}{3} = 1$  to'g'ri chiziqdan koordinata o'qlari kesgan kesmani diametr qilib aylana chizilgan. Bu aylana tenglamasini tuzing.

6.  $A(2;-1)$ ,  $B(3;4)$  nuqtalardan o'tgan va markazi  $x - y - 4 = 0$  to'g'ri chiziqda joylashgan aylana tenglamasini tuzing.

7. Uchlari  $A(-2;2)$ ,  $B(0;-2)$ ,  $C(-1;-1)$  bo'lgan  $ABC$  uchburchakka tashqi chizilgan aylananing markazi va radiusini toping.

8.  $k$  ning qanday qiymatlarida  $y = kx$  to'g'ri chiziq  $x^2 + y^2 - 8x - 2y + 16 = 0$  aylanani kesadi, bu aylanaga urinadi?

9.  $(x-4)^2 + (y-2)^2 = 4$  aylanaga uringan va koordinatalar boshidan o'tgan to'g'ri chiziqlar tenglamalarini tuzing.

10. Aylana tenglamalarini parametrik ko'rishga keltiring:

1)  $x^2 + y^2 = 16x$ ;      2)  $x^2 + y^2 = 4y$ ;      3)  $x^2 + y^2 = 2x + 2y$ .

11. Fokuslari  $Oy$  o'qda  $O(0;0)$  nuqtaga nisbatan simmetrik joylashgan va quyidagi shartlarni qanoatlantiruvchi ellipsning kanonik tenglamasini tuzing:

1) kichik o'qi 12 ga va eksentrisiteti  $\frac{4}{5}$  ga teng; 2) fokuslari orasidagi masofa

10 ga va eksentrisiteti  $\frac{5}{7}$  ga teng; 3)  $M_1(6;0)$  va  $M_2(0;9)$  nuqtalardan o'tgan;

4) direktrisalari orasidagi masofa  $\frac{50}{3}$  ga va eksentrisiteti  $\varepsilon = \frac{3}{5}$  ga teng.

12.  $\frac{x^2}{12} + \frac{y^2}{4} = 1$  ellipsga tomonlari ellips o'qlariga parallel qilib kvadrat ichki

chizilgan. Kvadratning yuzini toping.

13.  $5x^2 + 20y^2 - 100 = 0$  ellipsning  $x + y - 20 = 0$  to'g'ri chiziqqa parallel bo'lgan o'timasi tenglamasini tuzing.

14.  $16x^2 + 25y^2 - 400 = 0$  ellipsning bir fokusidan uning kichik o'qiga parallel o'tgan vatari uzunligini toping.

15.  $\frac{x^2}{50} + \frac{y^2}{18} = 1$  ellipsning  $M(x; y)$  nuqtasidan uning o'ng fokusigacha bo'lgan masofa chap fokusigacha bo'lgan masofadan to'rt marta katta.  $M(x; y)$  nuqtani toping.

16.  $\frac{x^2}{9} + \frac{y^2}{8} = 1$  ellipsning  $M(x; y)$  nuqtasidan uning chap fokusigacha bo'lgan masofa o'ng fokusigacha bo'lgan masofadan ikki marta katta.  $M(x; y)$  nuqtani toping.

17. Ellipsning bir fokusidan uning katta o'qi oxirlarigacha bo'lgan masofalar 2 va 8 ga teng. Ellipsning kanonik tenglamasini tuzing.

18. Ellipsning tenglamalarini parametrik ko'rishga keltiring:

1)  $16x^2 + 25y^2 - 400 = 0$ ;      2)  $144x^2 + 25y^2 - 3600 = 0$ .

19.  $x + 2y - 7 = 0$  to'g'ri chiziq bilan  $x^2 + 4y^2 = 25$  ellipsning kesishish nuqtalarini toping.

20. Fokuslari ordinatalar o'qida joylashgan va quyidagi shartlarni qanoatlantiruvchi giperboianing kanonik tenglamasini tuzing: 1) direktrisalari

orasidagi masofa  $\frac{18}{5}$  ga va eksentrisiteti  $\frac{5}{3}$  ga teng; 2) direktrisalari orasidagi

masofa  $\frac{288}{13}$  ga teng va asimptotalari tenglamalari  $y = \pm \frac{12}{5}x$ ; 3) direktrisalari

orasidagi masofa  $\frac{32}{5}$  ga va haqiqiy o'qi 8 ga teng; 4) direktrisalari orasidagi masofa  $\frac{50}{7}$  ga va fokuslari orasidagi masofa 14 ga teng.

21. Giperbolaning nuqtalaridan biri va asimptotalarining tenglamalari berilgan. Giperbolaning kanonik tenglamasini tuzing:

1)  $M(6;2), y = \pm \frac{\sqrt{3}}{3}x$ ;

2)  $M(4;2), y = \pm \frac{\sqrt{2}}{2}x$ ;

3)  $M(4;3), y = \pm \frac{3}{2}x$ ;

4)  $M(6;3), y = \pm \frac{\sqrt{3}}{2}x$ .

22. Giperbolaning eksentrisiteti 2 ga teng. Uning asimptotalari orasidagi burchakni toping.

23. Giperbolaning asimptotasi haqiqiy o'q bilan  $\frac{\pi}{4}$  ga teng burchak tashkil qiladi. Giperbolaning eksentrisitetini toping.

24.  $5x^2 + 17y^2 - 85 = 0$  ellips berilgan. Ellips bilan bir xil fokuslarga ega bo'lgan teng tomonli giperbolaning kanonik tenglamasini tuzing.

25. Giperbola  $25x^2 + 9y^2 = 225$  ellips bilan bir xil fokuslarga ega. Giperbolaning eksentrisiteti 2 ga teng bo'lsa, uning kanonik tenglamasini tuzing.

26. Assimptotalari  $O(2;-3)$  nuqtada kesishuvchi va  $B(4;-1)$  nuqtadan o'tuvchi giperbola tenglamasini tuzing.

27. Giperbolaning  $y = \frac{4x-3}{3x+5}$  tenglamasini sodda ko'rinishga keltiring.

28. Berilgan fokusi va direktrisasi tenglamasiga ko'ra parabolaning kanonik tenglamasini tuzing:

1)  $F(-3;4), x-5=0$ ;

2)  $F(5;3), y+2=0$ .

29. Berilgan tenglamasiga ko'ra parabolaning uchini va simmetriya o'qining tenglamasini aniqlang:

1)  $y^2 - 2y + 16x + 65 = 0$ ;

2)  $2x^2 + y - 8x + 5 = 0$ .

30. Berilgan tenglamalar qanday chiziqlarni aniqlaydi?

1) 
$$\begin{cases} x = \frac{1}{2}(e^t + e^{-t}), \\ y = \frac{1}{2}(e^t - e^{-t}) \end{cases}$$

2) 
$$\begin{cases} x = \frac{2}{t^2}, \\ y = \frac{3}{t} \end{cases}$$

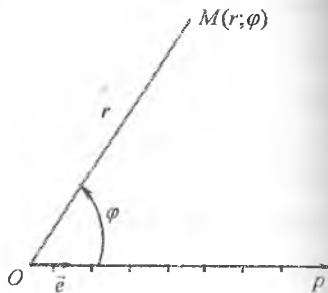
3)  $y = -2\sqrt{x^2 + 1}$ ;

4)  $x = -\sqrt{y^2 + 4}$ .

bo'lgan  $r$  masofa va  $Op$  qutb o'qi bilan  $\overline{OM}$  yo'nalgan kesma orasidagi  $\varphi$  burchak bilan aniqlanadi ( $Op$  nurdan boshlab burchak yo'nalishi soat strelkasi yo'alishiga teskari tanlanadi).

$r$  va  $\varphi$  sonlariga  $M$  nuqtaning qutb koordinatalari deyiladi va  $M(r; \varphi)$  deb yoziladi. Bunda  $r$  masofa qutb radiusi,  $\varphi$  burchak qutb burchagi deb ataladi (19-shakl).

Tekislikning barcha nuqtalarini aniqlash uchun  $r$  va  $\varphi$  kattaliklarni  $0 \leq r < +\infty$ ,  $-\pi < \varphi \leq \pi$  chegaralarda olish yetarli bo'ladi. Bunda tekislikning har bir nuqtasiga yagona  $r$  va  $\varphi$  sonlar jufti mos keladi, va aksincha, har bir  $(r; \varphi)$  sonlar juftiga tekislikdagi yagona nuqta mos keladi.



19-shakl.

Nuqtaning qutb va to'g'ri burchakli koordinatalari orasidagi bog'lanishni topamiz. Bunda to'g'ri burchakli koordinatalar sistemasining koordinatalari boshini qutb bilan va absissalar o'qini qutb o'qi bilan ustma-ust tushadigan qilib tanlaymiz (20-shakl).

$M$  nuqta  $x$  va  $y$  to'g'ri burchakli koordinatalarga,  $r$  va  $\varphi$  qutb koordinatalarga ega bo'lsin.

20-shakldan topamiz:

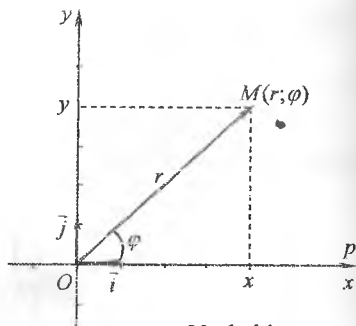
$$x = r \cos \varphi, \quad y = r \sin \varphi. \quad (3.1)$$

Bu tengliklar nuqtaning to'g'ri burchakli koordinatalarini uning qutb koordinatalari bilan bog'laydi.

(3.1) tengliklardan nuqtaning qutb koordinatalari bilan uning to'g'ri burchakli koordinatalari o'rtasida quyidagi bog'lanish hosil qilinadi:

$$r = \sqrt{x^2 + y^2}, \quad \operatorname{tg} \varphi = \frac{y}{x}. \quad (3.2)$$

Bunda  $\varphi$  burchakning qiymati nuqtaning joylashgan choragiga  $(x, y)$  larning ishoralari asosida) qarab,  $-\pi < \varphi \leq \pi$  oraliqda tanlanadi.



20-shakl.



U holda

$$r \cos(\alpha - \varphi) = p. \quad (3.4)$$

(3.4) tenglamaga to'g'ri chiziqning qutb tenglamasi deyiladi.

2-misol.  $M_1\left(5; \frac{\pi}{2}\right)$  va  $M_2(5; 0)$  nuqtalardan o'tuvchi to'g'ri chiziqning qutb tenglamasini tuzing.

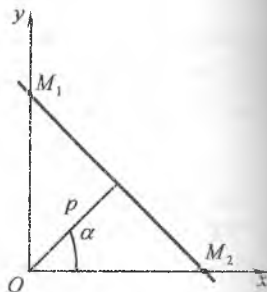
*Yechish.* To'g'ri chiziqning  $M_1$  va  $M_2$  nuqtalar orasidagi kesma katetlari 5 ga teng bo'lgan to'g'ri burchakli uchburchakning gipotenuzasi bo'ladi. Bunda qutbdan to'g'ri chiziqqa bo'lgan masofa to'g'ri burchak uchidan gipotenuzaga tushirilgan balandlikdan iborat (22-shakl).

Uning uzunligini ( $p$  ni) va yo'nalshini ( $\alpha$  ni) topamiz:

$$p = \frac{OM_1 \cdot OM_2}{\sqrt{OM_1^2 + OM_2^2}} = \frac{5 \cdot 5}{\sqrt{5^2 + 5^2}} = \frac{5\sqrt{2}}{2}, \quad \alpha = \frac{\pi}{4}.$$

U holda (3.4) formulaga ko'ra,

$$r \cos\left(\varphi - \frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}.$$



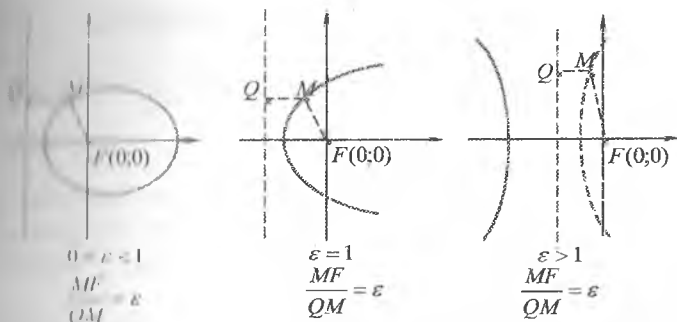
22-shakl.

### ***Konus kesimlarining qutb tenglamalari***

Konus kesimlarning (ellipsning, giperbolaning, parabolaning) qutb tenglamalarini keltirib chiqaramiz. Bunda chiziqlarning fokuslaridan biri qutbda yotsa, uning tenglamasi sodda ko'rinishni oladi.

Konus kesimlarning tenglamalarini keltirib chiqarishda ularning har bir nuqtasidan fiksirlangan nuqtagacha (fokusgacha) bo'lgan masofaning fiksirlangan to'g'ri chiziqqa (direktrisagacha) bo'lgan masofaga nisbati o'zgarmas kattalikka – ekstsentrisitetga teng bo'lishi xossasidan foydalaniladi. Bunda konus kesim  $\varepsilon < 1$  bo'lganda ellipsdan,  $\varepsilon = 1$  bo'lganda paraboladan va  $\varepsilon > 1$  bo'lganda giperboladan iborat bo'ladi (23-shakl).

$l$  – konus kesimlardan birining tarmog'i,  $F$  – bu tarmoqning fokusi,  $d$  – qutbdan chapda joylashgan vertikal direktrisasi,  $\varepsilon$  – ekstsentisitet bo'lsin. Fokusdan direktrisagacha bo'lgan masofani  $p$  bilan belgilaymiz (24-shakl).



23-shakl.

U holda ixtiyoriy  $M(r, \varphi)$  nuqtani olamiz.

U holda konus kesimlarning

qutb tenglamasi ko'ra,  $\frac{FM}{QM} = \varepsilon$  bo'ladi.

Shundan  $FM = \varepsilon QM$  yoki

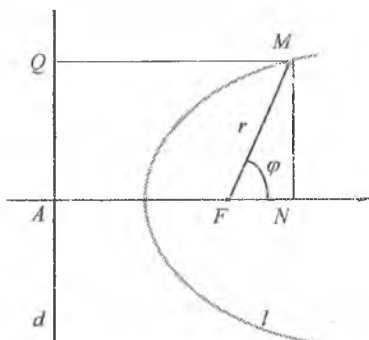
$$r = \varepsilon N = \varepsilon (AF + FN) = \varepsilon (p + r \cos \varphi).$$

O'sirgi tenglikni  $r$  ga nisbatan yechamiz:

$$r = \frac{\varepsilon \cdot p}{1 - \varepsilon \cos \varphi}. \quad (3.5)$$

(3.5) tenglamaga *konus kesimlarning qutb tenglamasi* deyiladi. Bu tenglamani  $\varepsilon < 1$  da ellipsni,  $\varepsilon > 1$  giperbolaning bir tarmog'ini va  $\varepsilon = 1$  da parabolaning bir tarmog'ini aniqlaydi.

**Izoh.** Konus kesimlarning qutb tenglamalari direktritsaning joylashishiga bog'liq bo'ladi:



24-shakl.

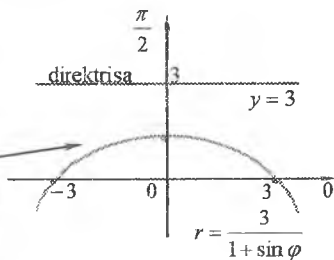
**Direktritsaning holati – tenglama**

vertikal va qutbdan chapda –  $r = \frac{\varepsilon \cdot p}{1 - \varepsilon \cos \varphi}$ ;

vertikal va qutbdan o'ngda –  $r = \frac{\varepsilon \cdot p}{1 + \varepsilon \cos \varphi}$ ;

gorizontal va qutbdan yuqorida –  $r = \frac{\varepsilon \cdot p}{1 + \varepsilon \sin \varphi}$ ;

gorizontal va qutbdan pastda –  $r = \frac{\varepsilon \cdot p}{1 - \varepsilon \sin \varphi}$ .



## Boshqa chiziqning qutb tenglamalari

Qutb tenglama bilan modellashtiriladigan masalalardan birini qaraymiz.

Uzunligi  $2a$  ( $a > 0$ )ga teng bo'lgan  $AB$  kesma uchlari bilan biror to'g'ri burchakning tomonlari bo'ylab sirpanib harakatlanayotgan bo'lsin. To'g'ri burchakning  $O$  uchidan bu kesmaga  $OM$  perpendikular tushirilgan (25-shakl).  $AB$  kesmaning harakati vaqtida  $OM$  perpendikular  $M$  asosining traektoriyasini topaylik.

$M$  asosning (nuqtaning) traektoriyasini qutb koordinatalar sistemasida tuzish uchun to'g'ri burchakning  $O$  uchini qutb,  $OA$  o'qni qutb o'qi deb olamiz.  $M$  nuqtaning qutb koordinatalarini  $r, \varphi$  deb belgilaymiz, ya'ni  $M(r; \varphi)$  deyimiz.

$AOM$  uchburchakdan topamiz:

$$OM = OA \cos \varphi,$$

$$r = OA \cos \varphi.$$

$AOB$  uchburchakdan topamiz:

$$OA = AB \sin \varphi,$$

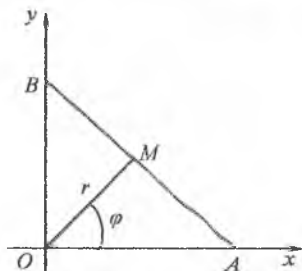
$$OA = 2a \sin \varphi.$$

Bundan  $M$  asos izlanayotgan traektoriyasining qutb koordinatalar sistemasidagi tenglamasini hosil qilamiz:

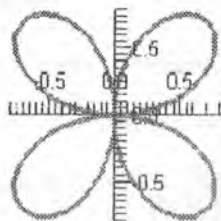
$$r = a \sin 2\varphi.$$

Bu tenglama bilan aniqlanuvchi chiziq to'rt yaproqli gul deb ataladi. Uning grafisini  $a=1$  da *Maple* paketi yordamida chizamiz (26-shakl).

```
> with(plots):  
> animatecurve([sin(2*t), t, t=-1*1..1*1], coords=olar,  
frames=60, numpoints=100);
```



25-shakl.



26-shakl.

Matematikaning keyingi bo'limlarining misol va masalalarini yechishda foydalaniladigan chiziqning grafiklari va ularning qutb va dekart parametrik tenglamalari 1-ilovada keltirilgan.

### 3.3.3. Mashqlar

1.  $A(\sqrt{3};1)$  va  $B(-\sqrt{3};-1)$  nuqtalarning qutb koordinatalarini toping.

2.  $A\left(2;-\frac{\pi}{3}\right)$  va  $B\left(1;\frac{2\pi}{3}\right)$  nuqtalarning to'g'ri burchakli koordinatalarini toping.

3.  $A\left(5;\frac{\pi}{4}\right)$  va  $B\left(8;-\frac{\pi}{12}\right)$  nuqtalar orasidagi masofani toping.

4. Uchlari  $O$  qutbda va  $A(r_1;\varphi_1)$ ,  $B(r_2;\varphi_2)$  nuqtalarda joylashgan  $OAB$  uchburchakning yuzini toping, bu yerda  $\varphi_2 > \varphi_1$ .

5. Ikkita qarama-qarshi uchlari  $A\left(2;-\frac{\pi}{6}\right)$ ,  $B\left(2;-\frac{2\pi}{3}\right)$  nuqtalarda bo'lgan kvadratning yuzini toping.

6. Kvadratning ikkita qo'shni uchlari berilgan:  $A\left(6;\frac{\pi}{3}\right)$ ,  $B\left(2;\frac{4\pi}{3}\right)$ . Kvadratning yuzini toping.

7. Berilgan tenglamalarni dekart koordinatalarida yozing:

1)  $r = 3\sin\varphi$ ;

2)  $r = 5\cos\varphi$ ;

3)  $r^2 = \sin 2\varphi$ ;

4)  $r = 4\cos 2\varphi$ ;

5)  $r = \frac{2}{1 + \sin\varphi}$ ;

6)  $r = \frac{3}{1 - \cos\varphi}$ ;

7)  $r = \frac{2}{5 + 3\cos\varphi}$ ;

8)  $r = \frac{32}{3 + 5\sin\varphi}$ .

8. Berilgan tenglamalarni qutb koordinatalarida yozing:

1)  $y = 5$ ;

2)  $y = 2x - 1$ ;

3)  $x^2 = 4y$ ;

4)  $x^2 - y^2 = a^2$ ;

5)  $x^2 + y^2 - 2ax = 0$ ;

6)  $xy = 4$ ;

7)  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ;

8)  $\frac{x^2}{36} - \frac{y^2}{4} = 1$ .

9. Berilgan tenglamasiga ko'ra chiziqlarning turini aniqlang:

$$1) r = \frac{6}{2 + \sin \varphi};$$

$$2) r = \frac{5}{-1 + 2 \cos \varphi};$$

$$3) r = \frac{3}{1 + \cos \varphi};$$

$$4) r = \frac{2}{2 - \cos \varphi}.$$

10. Berilgan chiziqlarning grafiklarini  $\varepsilon = 1$ ,  $\varepsilon = \frac{1}{2}$ ,  $\varepsilon = \frac{3}{4}$  larda chizing:

$$1) r = \frac{4\varepsilon}{1 + \varepsilon \cos \varphi};$$

$$2) r = \frac{4\varepsilon}{1 + \varepsilon \sin \varphi};$$

$$3) r = \frac{4\varepsilon}{1 - \varepsilon \sin \varphi};$$

$$4) r = \frac{4\varepsilon}{1 - \varepsilon \cos \varphi}.$$

### 3.4. TEKISLIK

#### 3.4.1. Fazoda sirt va chiziq

Umumiy boshlang'ich  $O$  nuqtaga va bir xil masshtab birligiga ega bo'lgan o'zaro perpendikulyar  $Ox$ ,  $Oy$  va  $Oz$  o'qlar fazoda to'g'ri burchakli  $Oxyz$  koordinatalar sistemasini hosil qiladi.

$Oxyz$  koordinatalar sistemasida uchta  $x$ ,  $y$  va  $z$  sonlari fazodagi har qanday  $M$  nuqtaning o'rnini to'liq aniqlaydi. Bunda nuqta  $M(x, y, z)$  kabi belgilanadi,  $x$  ga  $M$  nuqtaning *abssissasi*,  $y$  ga  $M$  nuqtaning *ordinatasi*,  $z$  ga  $M$  nuqtaning *applikatasi* deyiladi.

$Oxyz$  fazodagi *sirt tenglamasi* deb aynan shu sirt nuqtalarining  $x, y, z$  koordinatalari orasidagi bog'lanishni aniqlovchi uch noma'lumli

$$F(x, y, z) = 0$$

tenglamaga aytiladi.

Shu kabi, koordinatalari uch noma'lumli  $F(x, y, z) = 0$  tenglamani qanoatlantiruvchi  $Oxyz$  fazoning barcha  $M(x, y, z)$  nuqtalari to'plamiga *fazoda* shu tenglama bilan aniqlanuvchi *sirt* deyiladi.

Fazodagi sirt  $x = x(u, v)$ ,  $y = y(u, v)$ ,  $z = z(u, v)$ ,  $(u, v) \in D$  parametrik

tenglamalar bilan ham berilishi mumkin, bu yerda  $x(u, v), y(u, v), z(u, v)$  – holda berilgan sirt barcha nuqtalarining va faqat shu nuqtalarning koordinatalarini beruvchi ikki o'zgaruvchili funksiyalar.

Masalan,

$$x = R \sin u \cos v, \quad y = R \sin u \sin v, \quad z = R \cos u, \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

parametrik tenglamalar *sferani* ifodalaydi.

Fazodagi chiziqni ikki sirtning kesishish chizig'i yoki ikki sirt umumiy nuqtalarining geometrik o'rni deb qarash mumkin (27-shakl).

$l$  chiziqni aniqlovchi ikki sirt  $F(x, y, z) = 0$  va  $G(x, y, z) = 0$  tenglamalar bilan berilgan bo'lsin (27-shakl). U holda  $l$  chiziq ikkala tenglamani ham qanoatlantiruvchi  $M(x, y, z)$  nuqtalar to'plamidan tashkil topadi.

Koordinatalari

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0 \end{cases}$$

Tenglamalar sistemasi qanoatlantiruvchi  $Oxyz$  fazoning barcha  $M(x, y, z)$  nuqtalari to'plamiga *fazodagi* shu tenglamalar sistemasi bilan aniqlanuvchi *chiziq* deyiladi.

Shu kabi, *Oxyz fazodagi chiziq tenglamasi* deb aynan shu chiziq barcha nuqtalarining  $x, y, z$  koordinatalarini aniqlovchi

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0 \end{cases}$$

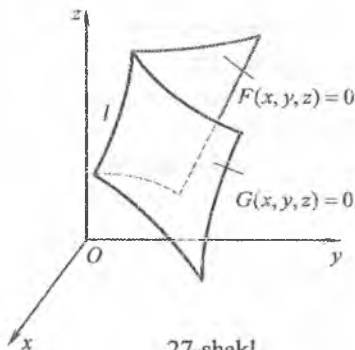
tenglamalar sistemasiga aytiladi.

Fazodagi chiziqni nuqtaning trayektoriyasi deb qarash mumkin. Bunda chiziq  $\vec{r} = \vec{r}(t)$  vektor tenglama bilan yoki  $x = x(t), y = y(t), z = z(t), t \in T$  parametrik tenglamalar bilan beriladi.

Masalan,

$$x = R \cos at, \quad y = R \sin at, \quad z = \frac{h}{2\pi} t$$

parametrik tenglamalar *vint chizig'ini* ifodalaydi.



27-shakl.

Fazodagi analitik geometriyada sirtini (yoki to'g'ri chiziqni) o'rganishda ikkita masala ko'riladi: geometrik xossalriga ko'ra, sirtning (yoki to'g'ri chiziqning) tenglamasini keltirib chiqarish; tenglamasiga asosan sirtning (yoki to'g'ri chiziqning) ko'rinishi va xossalari tekshirish.

### 3.4.2. Tekislik tenglamalari

Tekislikning fazodagi o'rni turli parametrlar bilan (masalan, tekislikning koordinata o'qlarida ajratgan kesmalari bilan) bir qiymatli aniqlanishi mumkin. Shu sababli parametrlariga ko'ra tekislikning turli tenglamalari keltirib chiqariladi.

*I. Tekislikda yotuvchi  $M_0(x_0; y_0; z_0)$  nuqta va to'g'ri chiziqqa perpendikular bo'lgan  $\vec{n} = \{A; B; C\}$  vektor berilgan.*

Tekislikka perpendikular bo'lgan har qanday vektorga tekislikning normal vektori deyiladi.

$\sigma$  tekislikning ixtiyoriy  $M(x; y; z)$  nuqtasini olamiz.  $M$  va  $M_0$  nuqtalarning radius vektorlari mos ravishda  $\vec{r}$  va  $\vec{r}_0$  bo'lsin.

U holda  $\overline{M_0M} = \vec{r} - \vec{r}_0$  bo'ladi.

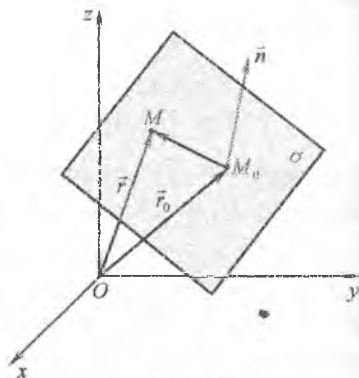
$M$  va  $M_0$  tekislik nuqtalari bo'lgani uchun  $\overline{M_0M}$  vektor tekislikda yotadi va tekislikning normal vektoriga perpendikular bo'ladi, ya'ni  $\vec{n} \perp \overline{M_0M}$  (28-shakl). Ikki vektorning perpendikularlik shartiga asosan tekislik tenglamasini topamiz:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0. \quad (4.1)$$

Bu tenglamaga tekislikning vektor tenglamasi deyiladi.

(4.1) tenglamaga normal vektor va radius vektorlarining koordinatalarini qo'yib, topamiz:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \quad (4.2)$$



28-shakl.

U tenglamaga *tekislikning skalyar tenglamasi* deyiladi.

Shuningdek, (4.2) tenglamaga *berilgan nuqtadan o'tuvchi va berilgan vektorga perpendikular tekislik tenglamasi* deyiladi.

*U misol.*  $M_0(3;4;5)$  nuqtadan o'tuvchi va normal vektori  $\vec{s} = (-1; -3; 2)$  bo'lgan tekislik tenglamasini tuzing.

*Yechish.* Masalaning shartiga ko'ra,

$$x_0 = 3, y_0 = 4, z_0 = 5, A = -1, B = -3, C = 2.$$

U holda (4.2) tenglamadan topamiz:

$$(-1) \cdot (x-3) + (-3) \cdot (y-4) + 2 \cdot (z-5) = 0$$

yoki

$$x + 3y - 2z - 5 = 0.$$

*II. Tekislikda yotuvchi uchta*  $M_1(x_1; y_1; z_1), M_2(x_2; y_2; z_2), M_3(x_3; y_3; z_3)$  nuqta berilgan.

U tekislikda yotuvchi ixtiyoriy  $M(x; y; z)$  nuqtani olamiz va

$$\overline{M_1M} = \{x - x_1; y - y_1; z - z_1\},$$

$$\overline{M_1M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\},$$

$$\overline{M_1M_3} = \{x_3 - x_1; y_3 - y_1; z_3 - z_1\}$$

vektorlarni yasaymiz.

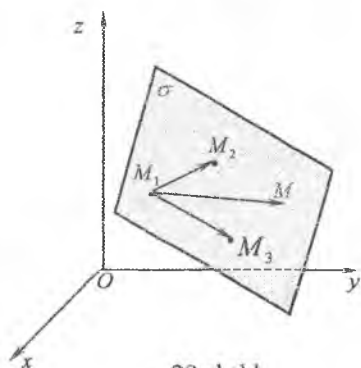
Bunda  $\overline{M_1M}, \overline{M_1M_2}, \overline{M_1M_3}$  vektorlar komplanar bo'ladi (29-shakl). Vektorlarning komplanarlik shartidan topamiz:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \quad (4.3)$$

(4.3) tenglamaga *berilgan uchta nuqtadan o'tuvchi tekislik tenglamasi* deyiladi.

(4.3) tenglamada

$$\vec{s} = \overline{M_1M_3} = \{p; q; r\}$$



29-shakl.



belgilash kiritib, topamiz:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ p & q & r \end{vmatrix} = 0. \quad (4.4)$$

(4.4) tenglamaga *berilgan ikkita nuqtadan o'tuvchi va berilgan vektorga parallel tekislik tenglamasi* deyiladi.

Shu kabi

$$\bar{s}_1 = \overline{M_1 M_2} = \{p_1; q_1; r_1\},$$

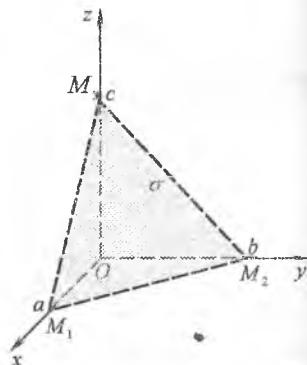
$$\bar{s}_2 = \overline{M_1 M_3} = \{p_2; q_2; r_2\}$$

belgilashlarda (4.3) tenglamadan topamiz:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0. \quad (4.5)$$

(4.5) tenglamaga *berilgan nuqtadan o'tuvchi va berilgan ikki vektorga parallel tekislik tenglamasi* deyiladi.

$M_1(x_1; y_1; z_1)$ ,  $M_2(x_2; y_2; z_2)$  va  $M_3(x_3; y_3; z_3)$  nuqtalar  $\sigma$  tekislikning mos ravishda  $Ox$ ,  $Oy$  va  $Oz$  o'qlarda yotuvchi nuqtalari, ya'ni  $M_1(a; 0; 0)$ ,  $M_2(0; b; 0)$  va  $M_3(0; 0; c)$  bo'lsin (30-shakl).



30-shakl.

U holda (4.3) formulaga ko'ra,

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

bo'ladi.

Bundan

$$bcx - abc + abz + acy = 0, \quad bcx + acy + abz = abc$$

yoki

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad (4.6)$$

10-shakldan ko'radiki,  $a, b, c$  mos ravishda  $\sigma$  tekislikning  $Ox, Oy$  va  $Oz$  o'qlarda ajratgan kesmalarini ifodalaydi.

Shu sababli (4.6) tenglamaga tekislikning kesmalarga nisbatan tenglamasi deyiladi.

2-misol.  $M_0(2; -1; 3)$  nuqtadan o'tuvchi,  $\vec{a} = \{3; 0; -1\}$  va  $\vec{b} = \{-3; 2; 2\}$  vektorlarga parallel tekislik tenglamasini tuzing.

Yechish. Izlanayotgan tekislik tenglamasini (4.5) formula bilan topamiz.

$$\begin{vmatrix} x-2 & y+1 & z-3 \\ 3 & 0 & -1 \\ -3 & 2 & 2 \end{vmatrix} = 0,$$

$$(x-2) \cdot 2 - (y+1) \cdot (6-3) + (z-3) \cdot 6 = 0,$$

$$2x - 3y + 6z - 25 = 0.$$

3-misol.  $Ox, Oy$  va  $Oz$  o'qlarda mos ravishda 2,  $(-4)$  va 6 ga teng kesmalar ajratuvchi tekislik tenglamasini tuzing.

Yechish. Masalaning shartiga ko'ra:  $a=2; b=-4; c=6$ .

Tekislikning kesmalarga nisbatan tenglamasidan topamiz:

$$\frac{x}{2} + \frac{y}{(-4)} + \frac{z}{6} = 1,$$

$$6x - 3y + 2z - 12 = 0.$$

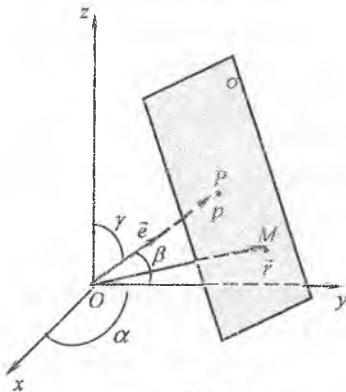
III. Tekislik  $\vec{n} = \overline{OP}$  normalining uzunligi  $p$  va birlik vektori  $\vec{e} = \{\cos\alpha; \cos\beta; \cos\gamma\}$  berilgan.

$\sigma$  tekislikda yotuvchi ixtiyoriy  $M(x; y; z)$  nuqtani olamiz. Bu nuqtaning radius vektori  $\vec{r} = \overline{OM} = \{x; y; z\}$  bo'lsin (11-shakl). Bunda  $\vec{r}$  radius vektorning  $\vec{e}$  vektor yo'nalishidagi proeksiyasi  $p$  ga teng bo'ladi, ya'ni

$$\text{Pr}_{\vec{e}} \vec{r} = p.$$

Bundan

$$\vec{r}\vec{e} = p, \quad \vec{r}\vec{e} - p = 0$$



31-shakl.

yoki

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0. \quad (4.7)$$

(4.7) tenglamaga *tekislikning normal tenglamasi* deyiladi.

Keltirib chiqarilgan (4.1)-(4.7) formulalar asosida ushbu xulosa kelib chaqadi:

$x, y, z$  o'zgaruvchilarning har qanday birinchi darajali tenglamasi fazodagi biror tekislikni ifodalaydi va aksincha, fazodagi har qanday tekislik  $x, y, z$  o'zgaruvchilarning biror birinchi darajali tenglamasi bilan aniqlanadi.

Demak, har bir  $\sigma$  tekislik tenglamasini

$$Ax + By + Cz + D = 0 \quad (4.8)$$

ko'rinishda yozish mumkin, bu yerda  $D$ —ozod had;  $A^2 + B^2 + C^2 \neq 0$ .

(4.8) tenglamada  $\vec{n} = \{A; B; C\}$  bo'lishini (4.2) tenglama yordamida (to'g'ri chiziqdagi kabi) ko'rsatish mumkin.

(4.8) tenglamaga *tekislikning umumiy tenglamasi* deyiladi.

(4.8) tenglamada: 1)  $A=0$  bo'lsa, tenglama  $By + Cz + D = 0$  ko'rinishga keladi. Bunda tekislikning  $\vec{n} = \{0; B; C\}$  normal vektori  $Ox$  o'qqa perpendikular bo'ladi. Shu sababli tekislik  $Ox$  o'qqa parallel bo'ladi. Shu kabi  $B=0$  da  $Ax + Cz + D = 0$  tenglama  $Oy$  o'qqa parallel tekislikni,  $C=0$  da  $Ax + By + D = 0$  tenglama  $Oz$  o'qqa parallel tekislikni ifodalaydi;  $Ax + By + D = 0$  tenglama  $Oz$  o'qqa parallel tekislikni ifodalaydi;

2)  $D=0$  bo'lsa, tenglama  $Ax + By + Cz = 0$  ko'rinishni oladi. Uni  $O(0;0;0)$  nuqta koordinatalari qanoatlantiradi va tekislik koordinatalar boshidan o'tadi;

3)  $A=0, D=0$  bo'lsa, tenglamadan  $By + Cz = 0$  kelib chiqadi. Bu tekislik  $Ox$  o'qdan o'tadi. Shu kabi  $Ax + Cz = 0$  tenglama  $Oy$  o'qdan o'tuvchi tekislikni,  $Ax + By = 0$  tenglama  $Oz$  o'qdan o'tuvchi tekislikni ifodalaydi;

4)  $A=0, B=0$  bo'lsa, tenglama  $Cz + D = 0$  yoki  $z = -\frac{D}{C}$  ko'rinishni oladi. Bu tekislik  $Oxy$  tekislikka parallel bo'ladi. Shu kabi  $By + D = 0$

tenglama  $Oxz$  tekislikka parallel tekislikni,  $Ax + D = 0$  tenglama  $Oyz$  tekislikka parallel tekislikni ifodalaydi;

3)  $A = 0$ ,  $B = 0$ ,  $D = 0$  bo'lsa, tenglama  $Cz = 0$  yoki  $z = 0$  ko'rinishga keladi. Bu tenglama  $Oxy$  tekislikni ifodalaydi. Shu kabi  $Oyz$  tekislik  $x = 0$  tenglama bilan,  $Oxz$  tekislik  $y = 0$  tenglama bilan aniqlanadi.

*Amisol.* Tekislik tenglamalarini tuzing: 1)  $Ox$  o'qdan va  $M_1(0; 2; 3)$  nuqtadan o'tuvchi; 2)  $Oy$  o'qqa parallel bo'lgan va  $M_1(0; 0; 4)$ ,  $M_2(5; -2; 3)$  nuqtalardan o'tuvchi; 3)  $Oxz$  tekislikka parallel bo'lgan va  $M_0(1; -2; 3)$  nuqtadan o'tuvchi.

*Yechish.* 1)  $Ox$  o'qdan o'tuvchi tekislik tenglamasi  $By + Cz = 0$  bo'ladi. Bu tenglamani  $M_0(0; -2; 3)$  nuqtaning koordinatalari qanoatlantiradi, chunki bu nuqta tekislikda yotadi.

$$\text{Demak, } (-2) \cdot B + 3C = 0 \text{ yoki } B = \frac{3}{2}C.$$

Hundan

$$\frac{3}{2}Cy + Cz = 0$$

yoki

$$3y + 2z = 0.$$

2)  $Oy$  o'qqa parallel tekislik tenglamasi  $Ax + Cz + D = 0$  bo'ladi. Uni  $M_1(0; 0; 4)$ ,  $M_2(5; -2; 3)$  nuqtalarning koordinatalari qanoatlantiradi, ya'ni

$$\begin{cases} 3A - 4C + D = 0, \\ 5A + 3C + D = 0. \end{cases}$$

Hundan

$$A = -\frac{7}{29}D \text{ va } C = \frac{2}{29}D.$$

3) holda

$$-\frac{7}{29}Dx + \frac{2}{29}Dz + D = 0$$

yoki

$$7x - 2z - 29 = 0.$$

3)  $Oxz$  tekislikka parallel tekislik tenglamasi  $By + D = 0$  bo'ladi. Bundan  $M_0(1; -2; 3)$  nuqtada  $-2B + D = 0$  yoki  $D = 2B$  keltiriladi.

U holda

$$By + 2B = 0$$

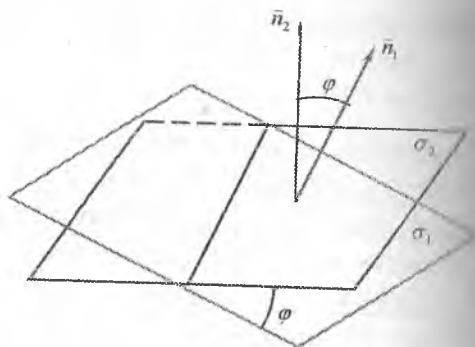
yoki

$$y + 2 = 0.$$

Tekislikning (4.1)-(4.8) tenglamalaridan har birini boshqalaridan keltirib chiqarish mumkin. Masalan, (4.8) tenglamani (4.7) tenglamaga o'tkazish uchun (4.8) tenglikning chap va o'ng tomonini *normallovchi ko'paytuvchi*

$$M = \pm \frac{1}{\sqrt{A^2 + B^2 + C^2}} \text{ songa}$$

ko'paytiriladi. Bunda  $M$  ko'paytuvchining ishorasi  $D$  koeffitsiyentning ishorasiga qarama-qarshi qilib tanlanadi.



32-shakl.

deb

ataluvchi

### 3.4.3. Fazoda ikki tekislikning o'zaro joylashishi

#### *Ikki tekislik orasidagi burchak*

Ikki tekislikning normal vektorlari orasidagi burchakka *ikki tekislik orasidagi burchak* deyiladi.

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0$$

tenglamalar bilan berilgan  $\sigma_1, \sigma_2$  tekisliklar orasidagi burchak  $\varphi$  ga teng bo'lsin (32-shakl).

Bunda  $\vec{n}_1 = \{A_1; B_1; C_1\}$ ,  $\vec{n}_2 = \{A_2; B_2; C_2\}$  va  $\varphi = (\sigma_1, \sigma_2) = (\vec{n}_1, \vec{n}_2)$ .

Ikki vektor orasidagi burchak kosinusi formulasidan topamiz:

$$\cos \varphi = \frac{\vec{n}_1 \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}}$$

$A, B, C$  koeffitsiyentlarni izlanayotgan tenglamaga qo'yamiz:

$$-\frac{7}{6}Dx + \frac{1}{3}Dy - \frac{1}{2}Dz + D = 0.$$

Bundan

$$7x - 2y + 3z - 6 = 0.$$

### *Ikki tekislikning parallellik sharti*

$\sigma_1$  va  $\sigma_2$  tekisliklar parallel bo'lsin. U holda  $\vec{n}_1 = \{A_1; B_1; C_1\}$  va  $\vec{n}_2 = \{A_2; B_2; C_2\}$  vektorlar kollinear bo'ladi. Ikki vektorning kollinearlik shartidan ikki tekislikning parallellik shartini topamiz:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \quad (4.11)$$

### *Ikki tekislikning ustma-ust tushishi*

$\sigma_1$  va  $\sigma_2$  tekisliklar ustma-ust tushsin. U holda birinchidan ular parallel bo'ladi. Ikki tekislikning parallellik shartidan topamiz:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \lambda$$

yoki

$$A_1 - \lambda A_2 = 0, \quad B_1 - \lambda B_2 = 0, \quad C_1 - \lambda C_2 = 0. \quad (4.12)$$

Ikkinchidan  $\sigma_1$  tekislikning har bir nuqtasi, jumladan  $M_0(x_0; y_0; z_0)$  nuqta  $\sigma_2$  tekislikda yotadi, ya'ni

$$A_1x_0 + B_1y_0 + C_1z_0 + D_1 = 0,$$

$$A_2x_0 + B_2y_0 + C_2z_0 + D_2 = 0.$$

Bu tengliklardan ikkinchisini  $\lambda$  ga ko'paytiramiz va birinchi tenglikdan ayiramiz:

$$(A_1 - \lambda A_2)x_0 + (B_1 - \lambda B_2)y_0 + (C_1 - \lambda C_2)z_0 + (D_1 - \lambda D_2) = 0.$$

Bundan (4.12) tengliklarni hisobga olsak  $D_1 - \lambda D_2 = 0$  yoki  $\frac{D_1}{D_2} = \lambda$  bo'ladi.

Demak,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}. \quad (4.13)$$

(4.13) tengliklar tekisliklarning ustma-ust tushish shartini ifodalaydi.

### 3.4.4. Nuqtadan tekislikkacha bo'lgan masofa

Nuqtadan tekislikka tushirilgan perpendikularning uzunligiga *nuqtadan tekislikkacha bo'lgan masofa* deyiladi.

$M_0(x_0; y_0; z_0)$  nuqta va  $Ax + By + Cz + D = 0$  tenglama bilan  $\sigma$  tekislik berilgan bo'lsin.  $M_0$  nuqtadan  $\sigma$  tekislikka tushirilgan perpendikularning asosini  $M_1(x_1; y_1; z_1)$  bilan belgilaymiz (33-shakl).

U holda  $M_0$  nuqtadan  $\sigma$  tekislikkacha bo'lgan masofa  $d = |\overrightarrow{M_0 M_1}|$  bo'ladi, bu yerda  $\overrightarrow{M_0 M_1} = \{x_0 - x_1; y_0 - y_1; z_0 - z_1\}$ .

Ikki vektor skalyar ko'paytmasining xossasiga ko'ra,

$$\begin{aligned} d &= \frac{|\overrightarrow{M_0 M_1} \cdot \vec{n}|}{|\vec{n}|} = \frac{|(x_0 - x_1)A + (y_0 - y_1)B + (z_0 - z_1)C|}{\sqrt{A^2 + B^2 + C^2}} = \\ &= \frac{|Ax_0 + By_0 + Cz_0 - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}}. \end{aligned}$$

$M_1(x_1; y_1; z_1)$  nuqta  $\sigma$  tekislikda yotgani sababli

$$Ax_1 + By_1 + Cz_1 + D = 0,$$

$$D = -Ax_1 - By_1 - Cz_1.$$

Bundan

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \quad (4.14)$$

Shunday qilib, *nuqtadan tekislikkacha bo'lgan masofa* (4.14) formula bilan topiladi.

*7-misol.*  $M_0(5; 4; -1)$  nuqtadan  $M_1(3; 0; 3)$ ,  $M_2(0; 4; 0)$  va  $M_3(0; 4; -3)$  nuqtalardan o'tuvchi tekislikkacha bo'lgan masofani toping.

*Yechish.* Avval berilgan uchta nuqtadan o'tuvchi tekislik tenglamasini tuzamiz:

$$\begin{vmatrix} x-3 & y & z-3 \\ 0-3 & 4 & 0-3 \\ 0-3 & 4 & -3-3 \end{vmatrix} = 0.$$

Bundan

$$-12 \cdot (x-3) - 9 \cdot y + 0 \cdot (z-3) = 0$$

yoki

$$4x + 3y - 12 = 0.$$

$$M_0(5;4;-1)$$

nuqtadan

$$4x + 3y - 12 = 0$$

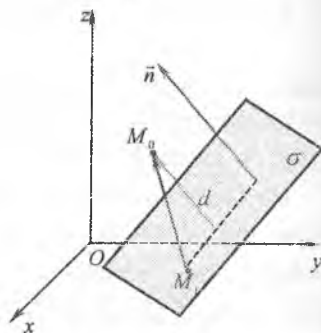
tekislikkacha

bo'lgan

masofani (4.14) formula bilan

hisoblaymiz:

$$d = \frac{|4 \cdot 5 + 3 \cdot 4 - 12|}{\sqrt{4^2 + 3^2 + 0^2}} = 4(b).$$



33-shakl.

### 3.4.5. Mashqlar

1.  $M_0(2;-1;3)$  nuqtadan o'tuvchi va shu nuqtaning radius vektoriga perpendikular bo'lgan tekislik tenglamasini tuzing.

2.  $\vec{n} = \{2;-3;4\}$  vektorga perpendikular bo'lgan va  $Oz$  manfiy yarim o'qida 5 ga teng kesma ajratuvchi tekislik tenglamasini tuzing.

3.  $M(2;3;-1)$  nuqtadan o'tuvchi  $2x - 3y + 5z - 4 = 0$  tekislikka parallel tekislik tenglamasini tuzing.

4.  $M(2;5;-1)$  nuqtadan o'tuvchi  $2x + 3y - 4z + 5 = 0$  tekislikka parallel tekislik tenglamasini tuzing.

5. Tekislik tenglamalarini tuzing: 1)  $M_0(1;3;-2)$  nuqtadan va  $Ox$  o'qidan o'tuvchi; 2)  $M_0(2;-1;3)$  nuqtadan o'tuvchi va  $Oy$  o'qqa perpendikular; 3)  $M_0(3;-2;4)$  nuqtadan o'tuvchi va  $Oxy$  tekislikka parallel; 4)  $M_1(2;-3;1)$ ,  $M_2(3;4;0)$  nuqtalardan o'tuvchi va  $Oy$  o'qqa parallel; 5) koordinatalar boshidan va  $M_1(3;-4;2)$ ,  $M_2(-1;3;4)$  nuqtalardan o'tuvchi.



8. Tekislik tenglamalarini tuzing: 1)  $M_0(2;-5;4)$  nuqtadan  $Oy$  o'qdan o'tuvchi; 2)  $M_0(3;7;-1)$  nuqtadan o'tuvchi va  $Ox$  o'qqa perpendikular; 3)  $M_0(1;0;0)$  nuqtadan o'tuvchi va  $Oxz$  tekislikka parallel; 4)  $M_1(2;1;-2), M_2(-7;-2;1)$  nuqtalardan o'tuvchi va  $Oy$  o'qqa parallel; 5) koordinatalar boshidan va  $M_1(-3;0;4)$  nuqtalardan o'tuvchi.

9.  $3x + y - 3z + 6 = 0$  tekislikning koordinata o'qlari bilan kesishish nuqtalarini toping.

10.  $2x + 3y - 5z + 30 = 0$  tekislik koordinata o'qlarida qanday kesmalar ajratadi?

11.  $M_1(2;-1;3), M_2(-1;3;2)$  nuqtalardan o'tuvchi va  $Ox, Oz$  o'qlarida teng kesmalar ajratuvchi tekislik tenglamasini tuzing.

12.  $M_1(2;5;-2)$  nuqtadan o'tuvchi va  $Ox, Oz$  o'qlarida  $Oy$  o'qqa nisbatan uch kesma ajratuvchi tekislik tenglamasini tuzing.

13. Berilgan uchta nuqtadan o'tuvchi tekislik tenglamasini tuzing:

1)  $M_1(1;-1), M_2(3;1;0), M_3(-1;2;-1)$ ; 2)  $M_1(1;-2;3), M_2(4;1;3), M_3(1;2;-1)$ .

14.  $M_0(3;3;3)$  nuqtadan koordinata tekisliklariga tushirilgan perpendikular kesmalar orqali o'tgan tekislik tenglamasini tuzing.

15.  $M_1(1;1;1), M_2(0;2;1)$  nuqtalardan o'tuvchi va  $\vec{a} = \{2;0;1\}$  vektorga parallel tekislik tenglamasini tuzing.

16.  $M_1(1;2;0), M_2(2;1;1)$  nuqtalardan o'tuvchi va  $\vec{a} = \{3;0;1\}$  vektorga parallel tekislik tenglamasini tuzing.

17.  $M_0(1;-2;3)$  nuqtadan o'tuvchi va  $\vec{a} = \{2;1;1\}, \vec{b} = \{3;1;-1\}$  vektorlarga parallel tekislik tenglamasini tuzing.

18.  $M_0(0;1;2)$  nuqtadan o'tuvchi va  $\vec{a} = \{2;0;1\}, \vec{b} = \{1;1;0\}$  vektorlarga parallel tekislik tenglamasini tuzing.

19.  $9x - 2y + 6z - 11 = 0$  tekislik tenglamasining kesmalariga nisbatan va normal ko'rinishlarini yozing.

20.  $5x + 7y - 34z + 5 = 0$  tekislik tenglamasining kesmalarga nisbatan va normal ko'rinishlarini yozing.

**19. Tekisliklar orasidagi burchakni toping:**

- 1)  $x-2y+2z+5=0$  va  $x-y-3=0$ ;
- 2)  $3x-y+2z+12=0$  va  $5x+9y-3z-1=0$ ;
- 3)  $2x-3y-4z+4=0$  va  $5x+2y+z-3=0$ ;
- 4)  $x+2y+3=0$  va  $y+2z-5=0$ .

**20.  $m$  va  $n$  ning qanday qiymatlarida tekisliklar parallel bo'ladi:**

- 1)  $3x-5y-nz-2=0$ ,  $mx+2y-3z+11=0$ ; 2)  $nx-6y-6z+4=0$ ,  $2x+my+3z-8=0$

**21.  $m$  ning qanday qiymatlarida tekisliklar perpendikular bo'ladi:**

- 1)  $4x-7y+2z-3=0$ ,  $-3x+2y+mz+5=0$ ; 2)  $x-my+z=0$ ,  $2x+3y+mz-4=0$

**22. Tekislik tenglamalarini tuzing:**

- 1)  $M_0(2;2;-2)$  nuqtadan o'tuvchi va berilgan tekislikka parallel:

a)  $x-2y-3z=0$ ;

b)  $2x+3y+z-1=0$ ;

- 2)  $M_0(-1;-1;2)$  nuqtadan o'tuvchi va berilgan ikki tekislikka perpendikular:

a)  $x+2y-2z+6=0$ ,  $x-2y+z+4=0$ ;

b)  $x+3y+z-1=0$ ,  $2x-y+z-2=0$

- 3)  $M_1(5;-4;3)$ ,  $M_2(-2;1;8)$  nuqtalardan o'tuvchi va berilgan tekislikka

perpendikular: a)  $Oxy$ ; b)  $Oyz$ ; c)  $Oxz$ .

**23.  $M(-2;1;3)$  nuqtadan va  $x-2y-2z+6=0$ ,  $2x+3y-z+3=0$  tekisliklarning kesishish chizig'idan o'tuvchi tekislik tenglamasini tuzing.**

**24.  $M(2;1;-2)$  nuqtadan o'tuvchi va  $x+3y+2z+1=0$ ,  $3x+2y-z+8=0$  tekisliklar kesishish chizig'iga perpendikular tekislik tenglamasini tuzing.**

**25.  $M_1(2;0;0)$ ,  $M_2(0;1;0)$  nuqtalardan o'tuvchi va  $Oxy$  tekislik bilan  $45^\circ$  li burchak tashkil qiluvchi tekislik tenglamasini tuzing.**

**26. Tekisliklarning kesishish nuqtasini toping:**

1)  $x+2y-z+2=0$ ,  $x-y-2z+7=0$ ,  $3x-y-2z+11=0$ ;

2)  $x-2y-4z=0$ ,  $x+2y-4z+4=0$ ,  $3x+y-z-4=0$ .

**27.  $M_0(5;-1;4)$  nuqtadan  $M_1(3;3;0)$ ,  $M_2(0;-3;4)$ ,  $M_3(0;0;4)$  nuqtalar yotuvchi tekislikkacha bo'lgan masofani toping.**

**28.  $Ox$  oqning  $2x+y-2z+6=0$ ,  $x+2y+2z-9=0$  tekisliklardan teng uzoqlikda yotuvchi nuqtasini toping.**

9.  $2x + y - 2z - 5 = 0$  tekislikka parallel bo'lgan va  $M_0(4;3;-2)$  nuqtadan  $d = 3$  masofada o'tuvchi tekislik tenglamasini tuzing.
10. Ikki yug'ri  $12x + 3y - 4z - 4 = 0$  va  $12x + 3y - 4z + 22 = 0$  tekisliklarda yotuvchi chiziqni toping.
11.  $M(-1;3;1)$  nuqtaning  $3x - 4y + 12z + 14 = 0$  tekislikdan chetlashishini toping.
12.  $M(3;0;1)$  nuqtaning  $2x + 9y - 6z + 33 = 0$  tekislikdan chetlashishini toping.
13.  $10x + 2y - 2z - 5 = 0$ ,  $5x + y - z - 1 = 0$  parallel tekisliklar orasidagi masofani toping.

### 3.5. FAZODAGI TO'G'RI CHIZIQ

#### 3.5.1. Fazodagi to'g'ri chiziq tenglamalari

##### *To'g'ri chiziqning kanonik tenglamasi*

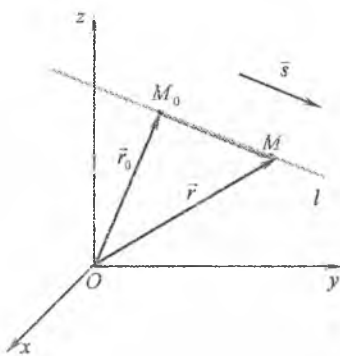
Analitik geometriyaning bir qancha masalalarini yechishda fazodagi to'g'ri chiziqning kanonik tenglamasi deb ataluvchi maxsus tenglama keng qo'llaniladi. Bu tenglamani keltirib chiqaramiz.

Fazoda biror to'g'ri chiziq berilgan bo'lsin. Bu to'g'ri chiziqqa parallel bo'lgan (yoki bu to'g'ri chiziqda yotuvchi) nolga teng bo'lmagan har qanday vektorga bu to'g'ri chiziqning *yo'naltiruvchi vektori* deyiladi.

Berilgan  $M_0(x_0; y_0; z_0)$  nuqtadan o'tuvchi va yo'naltiruvchi vektori  $\vec{s} = \{p; q; r\}$  bo'lgan  $l$  to'g'ri chiziq tenglamasini tuzamiz. Buning uchun

$l$  to'g'ri chiziqning ixtiyoriy  $M(x; y; z)$  nuqtasini olamiz va  $\vec{M_0M} = \{x - x_0; y - y_0; z - z_0\}$  vektorni yasaymiz (34-shakl).

Bunda  $\vec{s}$  va  $\vec{M_0M}$  vektorlar kollinear bo'ladi. Ikki vektorning



34-shakl.

kollinearlik shartidan topamiz:

$$\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r} \quad (5.1)$$

Bu tengliklarni  $l$  to'g'ri chiziqda yotuvchi har bir  $M(x, y, z)$  nuqtaning koordinatalari qanoatlantiradi, va aksincha agar  $M(x, y, z)$  nuqta  $l$  to'g'ri chiziqda yotmasa, u holda uning koordinatalari (5.1) tengliklarni qanoatlantirmaydi, chunki bunda  $\vec{s}$  va  $\overline{M_0M}$  vektorlar kollinear bo'lmaydi.

(5.1) tengliklarga *to'g'ri chiziqning kanonik tenglamasi* deyiladi.

Bunda ixtiyoriy  $\vec{s}$  yo'naltiruvchi vektorning  $p, q, r$  koordinatalari bu to'g'ri chiziqning yo'naltiruvchi parametrlari va  $\vec{s}$  vektorning yo'naltiruvchi kosinuslari bu to'g'ri chiziqning yo'naltiruvchi kosinuslari deb ataladi.

To'g'ri chiziqning kanonik tenglamasidan uning boshqa tenglamalarini keltirib chiqaramiz.

To'g'ri chiziqning kanonik tenglamasini

$$\begin{cases} \frac{x-x_0}{p} = \frac{y-y_0}{q}, \\ \frac{x-x_0}{p} = \frac{z-z_0}{r} \end{cases}$$

tenglamalar sistemasi deb qarash mumkin.

Bu tenglamalarning har ikkalasi birinchi darajali tenglamalar hisoblanadi, ya'ni tekislik tenglamalari bo'ladi. Bundan *fazodagi to'g'ri chiziq ikkita parallel bo'lmagan tekislikning kesishisidan hosil bo'lak* degan xulosaga kelish mumkin.

Masalan,  $\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$  to'g'ri chiziq  $2x+5y-2=0$  va  $3y-2z=0$  tekisliklarning kesishish chizig'i bo'ladi.

Shunday qilib, agar  $\sigma_1$  va  $\sigma_2$  tekisliklarning  $\vec{n}_1 = \{A_1; B_1; C_1\}$  va  $\vec{n}_2 = \{A_2; B_2; C_2\}$  normal vektorlari kollinear bo'lmasa, bu tekisliklarning kesishishidan hosil bo'lgan  $l$  to'g'ri chiziq quyidagi tenglamalar sistemasi bilan ifodalanadi:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases} \quad (5.2)$$

Bu tenglamalar sistemaga to'g'ri chiziqning umumiy tenglamalari beriladi.

(3.2) umumiy tenglamalari bilan berilgan to'g'ri chiziqning kanonik tenglamasi quyidagi tartibda topiladi.

1 To'g'ri chiziqning biror  $M_0(x_0; y_0; z_0)$  nuqtasi topiladi.

Buning uchun avval noma'lum

$x_0, y_0, z_0$  koordinatalardan biriga

qiymat beriladi va bu qiymat

(3.1) tenglamalardagi mos

o'zgaruvchi o'rniga qo'yiladi,

keyin boshqa koordinatalar

(3.1) sistemani yechish orqali

topiladi.

To'g'ri chiziqning  $\vec{s}$

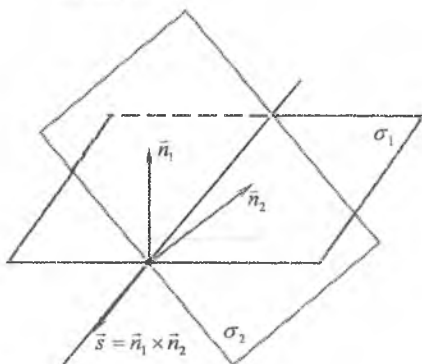
yo'naltiruvchi vektori topiladi.  $l$

to'g'ri chiziq  $\vec{n}_1$  va  $\vec{n}_2$

vektorlarga perpendekular

bo'lgani uchun  $\vec{s} \perp \vec{n}_1, \vec{s} \perp \vec{n}_2$

bo'ladi (35-shakl). Bundan



35-shakl.

$$\vec{s} = \vec{n}_1 \times \vec{n}_2 = \left\{ \left| \begin{array}{cc} B_1 & C_1 \\ B_2 & C_2 \end{array} \right|, \left| \begin{array}{cc} C_1 & A_1 \\ C_2 & A_2 \end{array} \right|, \left| \begin{array}{cc} A_1 & B_1 \\ A_2 & B_2 \end{array} \right| \right\} \quad (5.3)$$

vektor aniqlanadi.

U topilgan  $M_0$  nuqta va  $\vec{s}$  vektor asosida kanonik tenglama tuziladi.

*l-misol.*  $\begin{cases} x - 2y + 3z + 1 = 0, \\ 2x + y - 4z - 8 = 0 \end{cases}$  tenglamani kanonik ko'rinishga keltiring.

*Yechih.* Misol shartiga ko'ra:

$$A_1 = 1, B_1 = -2, C_1 = 3, A_2 = 2, B_2 = 1, C_2 = -4.$$

To'g'ri chiziqning  $M_0(x_0; y_0; z_0)$  nuqtasini topish uchun  $z_0 = 0$  deb olamiz.

U holda

$$\begin{cases} x_0 - 2y_0 = -1, \\ 2x_0 + y_0 = 8 \end{cases}$$

sistemadan  $x_0 = 3, y_0 = 2$  ekanini topamiz.

To'g'ri chiziqning yo'naltiruvchi vektorini (5.3) formuladan

topamiz:

$$\vec{s} = \left\{ \left| \begin{array}{cc|c} -2 & 3 & 3 \\ 1 & -4 & -4 \end{array} \right|, \left| \begin{array}{cc|c} 3 & 1 & 1 \\ -4 & 2 & 2 \end{array} \right|, \left| \begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 1 & 1 \end{array} \right| \right\} = \{5; 10; 5\}.$$

$M_0$  nuqta va  $\vec{s}$  vektorning koordinatalarini (5.1) tenglamaga qo'yamiz:

$$\frac{x-3}{5} = \frac{y-2}{10} = \frac{z}{5}$$

yoki

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z}{1}.$$

(5.1) tenglamada

$$\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r} = t, \quad t \in R$$

belgilash kiritamiz.

Bundan

$$\begin{aligned} x &= x_0 + pt, \\ y &= y_0 + qt, \\ z &= z_0 + rt \end{aligned} \quad (5.4)$$

tenglamalar kelib chiqadi, bu yerda  $t \in R$  – parametr.

(5.4) tenglamalarga *to'g'ri chiziqning parametrik tenglamalari* deyiladi.

Ma'lumki, fazodagi chiziqning uchta parametrik (skalyar) tenglamalarini bitta vektor tenglama bilan ifodalash mumkin.

Demak, (5.4) tenglamalarni

$$\vec{r} = \vec{r}_0 + t\vec{s} \quad (5.5)$$

ko'rinishda yozish mumkin, bu yerda  $\vec{r} = \{x; y; z\}$ ,  $\vec{r}_0 = \{x_0; y_0; z_0\}$  – mos ravishda  $M(x; y; z)$ ,  $M_0(x_0; y_0; z_0)$  nuqtalarning radius vektorlari;  $\vec{s} = \{p; q; r\}$  – to'g'ri chiziqning yo'naltiruvchi vektori (34-shakl).

(5.5) tenglamaga *to'g'ri chiziqning vektor tenglamasi* deyiladi.

Berilgan  $M_1(x_1; y_1; z_1)$  va  $M_2(x_2; y_2; z_2)$  nuqtalardan o'tuvchi  $l$  to'g'ri chiziq tenglamasini tuzamiz. Buning uchun  $l$  to'g'ri chiziqning

yoʻnaltiruvchi vektori sifatida

$$\vec{s} = \overline{M_1 M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$$

bu tengliklarni olamiz va (5.1) tengliklardan

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (5.6)$$

tengliklarni keltirib chiqaramiz.

bu tengliklarga berilgan ikki nuqtadan oʻtuvchi toʻgʻri chiziq tenglamasi deyiladi.

### 3.5.2. Fazoda ikki toʻgʻri chiziqning oʻzaro joylashishi

#### *Ikki toʻgʻri chiziq orasidagi burchak*

$\frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$  va  $\frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}$  tenglamalari bilan berilgan ikki  $l_1$  va  $l_2$  toʻgʻri chiziqlar orasidagi burchak  $\varphi$  boʻlsin.

Hunda toʻgʻri chiziqlarning yoʻnaltiruvchi vektorlari  $\vec{s}_1 = \{p_1; q_1; r_1\}$ ,  $\vec{s}_2 = \{p_2; q_2; r_2\}$  ga va ular orasidagi burchak toʻgʻri chiziqlar orasidagi burchaklardan biriga teng, yaʼni  $\varphi = (\hat{l}_1, \hat{l}_2) = (\vec{s}_1, \vec{s}_2)$  boʻladi.

$\varphi$  burchak kosinusini topamiz:

$$\cos \varphi = \frac{\vec{s}_1 \vec{s}_2}{|\vec{s}_1| |\vec{s}_2|} = \frac{p_1 p_2 + q_1 q_2 + r_1 r_2}{\sqrt{p_1^2 + q_1^2 + r_1^2} \sqrt{p_2^2 + q_2^2 + r_2^2}} \quad (5.7)$$

*2-misol.*  $x = 3t - 2$ ,  $y = 0$ ,  $z = -t + 3$  va  $x = 2t - 1$ ,  $y = 0$ ,  $z = t - 3$  toʻgʻri chiziqlar orasidagi oʻtmas burchakni toping.

*Yechish.* Toʻgʻri chiziqlar tenglamalarini kanonik shaklga keltiramiz:

$$\frac{x + 2}{3} = \frac{y}{0} = \frac{z - 3}{-1}, \quad \frac{x + 1}{2} = \frac{y}{0} = \frac{z + 3}{1}$$

U holda (5.7) formuladan topamiz:

$$\cos \varphi = \frac{3 \cdot 2 + 0 \cdot 0 + (-1) \cdot 1}{\sqrt{3^2 + 0^2 + (-1)^2} \cdot \sqrt{2^2 + 0^2 + 1^2}} = \pm \frac{\sqrt{2}}{2}$$

O'tmas burchak uchun  $\cos\varphi = -\frac{\sqrt{2}}{2}$  bo'ladi. Bundan  $\varphi = 135^\circ$ .

### *Ikki to'g'ri chiziqning perpendikularlik sharti*

$l_1 \perp l_2$  bo'lsin. U holda  $\cos\varphi = 0$  va (5.7) tenglikdan topamiz:

$$p_1 p_2 + q_1 q_2 + r_1 r_2 = 0. \quad (5.8)$$

Bu tenglik ikki to'g'ri chiziqning perpendikularlik shartini ifodalaydi.

### *Ikki to'g'ri chiziqning parallellik sharti*

$l_1$  va  $l_2$  to'g'ri chiziqlar parallel bo'lsin. U holda  $\vec{s}_1 = \{p_1; q_1; r_1\}$  va  $\vec{s}_2 = \{p_2; q_2; r_2\}$  vektorlar kollinear bo'ladi. Ikki vektorning kollinearlik shartidan ikki to'g'ri chiziqning parallellik shartini keltirib chiqaramiz:

$$\frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}. \quad (5.9)$$

### *Ikki to'g'ri chiziqning bir tekislikda yotishi*

$l_1$  va  $l_2$  to'g'ri chiziqlar bitta  $\sigma$  tekislikda yotsin.  $M_1(x_1; y_1; z_1) - l_1$  to'g'ri chiziqning nuqtasi va  $M_2(x_2; y_2; z_2) - l_2$  to'g'ri chiziqning nuqtasi bo'lsin.

U holda  $\vec{s}_1 = \{p_1; q_1; r_1\}$ ,  $\vec{s}_2 = \{p_2; q_2; r_2\}$ ,  $\overline{M_1 M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$  vektorlar  $\sigma$  tekislikda yotadi, ya'ni bu vektorlar komplanar bo'ladi.

Vektorlarning komplanarlik shartiga ko'ra topamiz:

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0. \quad (5.10)$$

Bu tenglik ikki to'g'ri chiziqning bir tekislikda yotishi shartini ifodalaydi.

### *Ikki to'g'ri chiziqning ayqash bo'lishi*

$l_1$  va  $l_2$  to'g'ri chiziqlar ayqash bo'lsa,  $\vec{s}_1 = \{p_1; q_1; r_1\}$ ,  $\vec{s}_2 = \{p_2; q_2; r_2\}$ ,  $\overline{M_1 M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$  vektorlar bir tekislikda yotmaydi, ya'ni



komplanar bo'lmaydi.

Shu sababli

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \neq 0. \quad (5.11)$$

bo'ladi. Bu shart ikki to'g'ri chiziqning ayqash bo'lishini belgilaydi.

### ***Ikki to'g'ri chiziqning ustma-ust tushishi***

$l_1$  va  $l_2$  to'g'ri chiziqlar ustma-ust tushsin. U holda bu to'g'ri chiziqlar birinchidan, parallel bo'ladi va ikkinchidan,

$$\vec{MM}_1 = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$$

vektor bu to'g'ri chiziqlardan birida, masalan  $l_1$  da yotadi.

Shu sababli

$$\begin{cases} \frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}, \\ \frac{x_2 - x_1}{p_1} = \frac{y_2 - y_1}{q_1} = \frac{z_2 - z_1}{r_1}. \end{cases} \quad (5.12)$$

(5.12) tengliklar ikki to'g'ri chiziqning ustma-ust tushish shartini ifodalaydi.

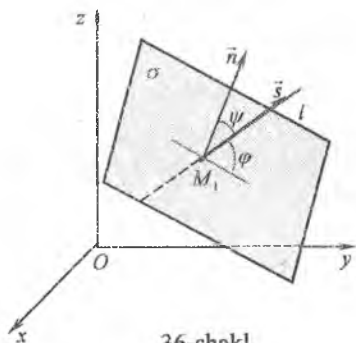
### **3.5.3. Fazoda to'g'ri chiziq bilan tekislikning o'zaro joylashishi**

#### ***To'g'ri chiziq bilan tekislik orasidagi burchak***

To'g'ri chiziq bilan uning tekislikdagi proeksiyasi orasidagi burchakka *to'g'ri chiziq bilan tekislik orasidagi burchak* deyiladi.

$$l: \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r} \text{ to'g'ri}$$

chiziq bilan  $\sigma: Ax + By + Cz + D = 0$  tekislik orasidagi burchak  $\varphi$  bo'lsin. U holda to'g'ri chiziqning yo'naltiruvchi vektori  $\vec{s} = \{p; q; r\}$  bilan tekislikning normal vektori



36-shakl.

$\vec{n} = \{A; B; C\}$  orasidagi burchak  $\psi = 90^\circ - \varphi$  bo'ladi (36-shakl).

$\cos\psi = \sin\varphi$  tenglikni hisobga olib, topamiz:

$$\sin\varphi = \frac{|Ap + Bq + Cz|}{\sqrt{A^2 + B^2 + C^2} \sqrt{p^2 + q^2 + r^2}} \quad (5.11)$$

3-misol.  $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-5}{-2}$  to'g'ri chiziq bilan  $2x - y - z + 9 = 0$

tekislik orasidagi o'tkir burchakni toping.

*Yechish.* Izlanayotgan burchakni (5.13) formula bilan topamiz:

$$\sin\varphi = \frac{|2 \cdot 1 + (-1) \cdot 1 + (-1) \cdot (-2)|}{\sqrt{2^2 + (-1)^2 + (-1)^2} \cdot \sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{2}$$

Bundan  $\varphi = 35^\circ$ .

### To'g'ri chiziq bilan tekislikning perpendikularlik sharti

$l \perp \sigma$  bo'lsin. U holda to'g'ri chiziqning yo'naltiruvchi vektori  $\vec{s} = \{p; q; r\}$  va tekislikning normal vektori  $\vec{n} = \{A; B; C\}$  kollinear bo'ladi.

Bundan

$$\frac{A}{p} = \frac{B}{q} = \frac{C}{r} \quad (5.14)$$

to'g'ri chiziq bilan tekislikning perpendikularlik sharti kelib chiqadi.

### To'g'ri chiziq bilan tekislikning parallellik sharti

$l \parallel \sigma$  bo'lsin. U holda,  $\vec{s} \perp \vec{n}$  bo'ladi.

Bundan

$$Ap + Bq + Cr = 0. \quad (5.15)$$

Bu tenglik to'g'ri chiziq bilan tekislikning parallellik shartini ifodalaydi.

4-misol.  $M_0(-1; 2; -3)$  nuqtadan o'tuvchi va  $2x - 3y + 6z - 1 = 0$  tekislikka perpendikular to'g'ri chiziq tenglamasini tuzing.

*Yechish.* To'g'ri chiziq bilan tekislikning perpendikularlik shartiga ko'ra,

$$\frac{2}{p} = \frac{-3}{q} = \frac{6}{r}$$

Bundan  $q = -\frac{3}{2}p$ ,  $r = 3p$ .

Damuk,  $M_0(-1;2;-3)$ ,  $\vec{s} = \left\{ p; -\frac{3}{2}p; 3p \right\}$ .

U holda (5.1) tenglamaga ko'ra:

$$\frac{x+1}{p} = \frac{y-2}{-\frac{3}{2}p} = \frac{z+3}{3p}$$

$$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z+3}{6}.$$

Bu masalani boshqacha yechish mumkin. To'g'ri chiziq tekislikka perpendikular bo'lgani sababli tekislikning normal vektori to'g'ri chiziqning

normalituvchi vektori bo'ladi, ya'ni  $\vec{s} = \{2; -3; 6\}$ .

U holda  $M_0(-1;2;-3)$  nuqtadan o'tuvchi to'g'ri chiziqning kanonik tenglamasi:

$$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z+3}{6}$$

*5-misol.*  $m$  ning qanday qiymatida  $\frac{x+2}{3} = \frac{y-1}{m} = \frac{z+3}{m+1}$  to'g'ri chiziq va  $3x + y - 3z - 1 = 0$  tekislik parallel bo'ladi?

*Yechish.*  $m$  ning izlanayotgan qiymatini to'g'ri chiziq va tekislikning parallellik shartidan topamiz:  $3 \cdot 3 + 1 \cdot m + (-3) \cdot (m+1) = 0$ .  
Shundan  $m = 3$ .

### To'g'ri chiziq bilan tekislikning kesishishi

Agar  $l \parallel \sigma$  bo'lmasa, u holda to'g'ri chiziq va tekislik kesishadi. Shu sababli

$$Ap + Bq + Cr \neq 0 \quad (5.16)$$

bo'ladi.

Bu shart to'g'ri chiziq bilan tekislikning kesishishini belgilaydi.

Shunday qilib, (5.16) shart bajarilsa, to'g'ri chiziq bilan tekislik qandaydir nuqtada kesishadi. Bu nuqta  $M_1(x_1, y_1, z_1)$  bo'lsin. U holda  $M_1(x_1, y_1, z_1)$  nuqtaning koordinatalari to'g'ri chiziq va tekislikning tenglamalarini qanoatlantiradi:

$$\frac{x_1 - x_0}{p} = \frac{y_1 - y_0}{q} = \frac{z_1 - z_0}{r}, \quad (5.17)$$

$$Ax_1 + By_1 + Cz_1 + D = 0. \quad (5.18)$$

Bu tenglamalardan  $M_1(x_1, y_1, z_1)$  nuqtani topish quyidagi tartibda amalga oshiriladi:

1°. (5.17) tenglama parametrik ko'rinishga keltiriladi:

$$x_1 = x_0 + pt, \quad y_1 = y_0 + qt, \quad z_1 = z_0 + rt; \quad (5.19)$$

2°.  $x_1, y_1$  va  $z_1$  lar (5.18) tenglamaga qo'yiladi va u  $t$  ga nisbatan yechiladi;

3°.  $t$  ning topilgan qiymati (5.19) tenglamalarga qo'yiladi va  $M_1(x_1, y_1, z_1)$  nuqta aniqlanadi.

6-misol.  $\frac{x+2}{-1} = \frac{y+1}{-2} = \frac{z-1}{3}$  to'g'ri chiziq bilan  $2x + 3y - z - 3 = 0$

tekislikning kesishish nuqtasini toping.

*Yechish.*

1°.  $\frac{x_1+2}{-1} = \frac{y_1+1}{-2} = \frac{z_1-1}{3} = t, x_1 = -2-t, y_1 = -1-2t, z_1 = 1+3t;$

2°.  $2(-2-t) + 3(-1-2t) - (1+3t) - 3 = 0 \Rightarrow t = -1;$

3°.  $x_1 = -2 - (-1) = -1, y_1 = -1 - 2 \cdot (-1) = 1, z_1 = 1 + 3 \cdot (-1) = -2.$

Demak,  $M_1(-1; 1; -2).$

### To'g'ri chiziqning tekislikda yotishi

$l$  to'g'ri chiziq  $\sigma$  tekislikda yotsin.

U holda birinchidan,  $\vec{s} \perp \vec{n}$  bo'ladi va ikkinchidan, to'g'ri chiziqning  $M_0(x_0; y_0; z_0)$  nuqtasi tekislikda ham yotadi.

Shu sababli

$$\begin{cases} Ap + Bq + Cr = 0, \\ Ax_0 + By_0 + Cz_0 + D = 0. \end{cases} \quad (5.20)$$

(5.20) shart to'g'ri chiziqning tekislikda yotishini belgilaydi.

### 3.5.4. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa

$M_1(x_1; y_1; z_1)$  nuqta va  $\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$  tenglama bilan  $l$  to'g'ri chiziq berilgan bo'lsin.  $l$  to'g'ri chiziq  $M_0(x_0; y_0; z_0)$  nuqtadan o'tadi va

$M_1(x_1; y_1; z_1)$  nuqtadan  $M_0$  ga  $d$  masofa bo'lgan masofa  $d$  bo'lsin.

Ushbu masofa  $\overline{M_0M_1}$  va  $\vec{s}$  vektorlarga qurilgan parallelogramm balandligining uzunligiga teng bo'ladi (37-shakl).

Ushbu parallelogrammning

maydoni  $|M_0M_1 \times \vec{s}|$  ga teng.

Hundan

$$d = \frac{|M_0M_1 \times \vec{s}|}{|\vec{s}|} \quad (5.21)$$

ko'rib chiqadi.

**Misol.**  $M_1(1; -1; -2)$  nuqtadan

$M_0(1; -2; -8)$  to'g'ri chiziqqacha

bo'lgan masofani toping.

**Yechish.** Misolning shartiga ko'ra:

$M_1(1; -1; -2)$ ,  $M_0(-3; -2; 8)$ ,  $\vec{s} = \{3; 2; -2\}$ .

Hundan

$$\overline{M_0M_1} = \{1 - (-3); -1 - (-2); -2 - 8\} = \{4; 1; -10\}.$$

Hundan

$$\begin{aligned} \overline{M_0M_1} \times \vec{s} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & -10 \\ 3 & 2 & -2 \end{vmatrix} = \\ &= (-2 + 20)\vec{i} - (-8 + 30)\vec{j} + (8 - 3)\vec{k} = 18\vec{i} - 22\vec{j} + 5\vec{k}, \end{aligned}$$

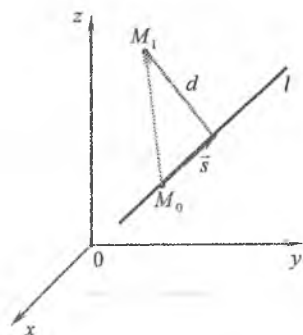
$$|\overline{M_0M_1} \times \vec{s}| = \sqrt{18^2 + (-22)^2 + 5^2} = 7\sqrt{17}, \quad |\vec{s}| = \sqrt{3^2 + 2^2 + (-2)^2} = \sqrt{17}.$$

(5.21) formula bilan topamiz:

$$d = \frac{7\sqrt{17}}{\sqrt{17}} = 7(u.b).$$

### 3.5.5. Mashqlar

- 1) Berilgan to'g'ri chiziqning kanonik tenglamasini tuzing: 1)  $M_1(1; 3; -2)$  nuqtadan o'tuvchi va  $\vec{s} = \{2; 3; -1\}$  vektorga parallel; 2)  $M_2(2; -3; -1)$  nuqtadan o'tuvchi va  $Oy$  o'qqaga parallel; 3)  $M_3(2; -1; -2)$  nuqtadan o'tuvchi va  $4x + 2y - 4z - 5 = 0$  tekislikka perpendikular.



37-shakl.

2.  $M_0(2;-3;5)$  nuqtadan o'tuvchi berilgan to'g'ri chiziq'larga parallel to'g'ri chiziq tenglamasini tuzing:

$$1) \frac{x+1}{4} = \frac{y+1}{1} = \frac{z-2}{3}; \quad 2) x=3+2t, y=-1+3t, z=1-t; \quad 3) \begin{cases} x+3y+z+6=0 \\ 2x-y-4z+1=0 \end{cases}$$

3.  $M(-3;6;2)$  nuqtadan o'tuvchi va  $Oz$  o'qni to'g'ri burchak ostida kesuvchi to'g'ri chiziq tenglamasini tuzing.

4.  $M(-1;2;-3)$  nuqtadan o'tuvchi va koordinata o'qlari bilan  $\alpha = \frac{\pi}{3}$ ,  $\beta = \frac{\pi}{4}$ ,  $\gamma = \frac{2\pi}{3}$  burchaklar tashkil qiluvchi to'g'ri chiziq tenglamalarini tuzing.

5. Berilgan nuqталardan o'tuvchi to'g'ri chiziqning umumiy tenglamasini tuzing:

$$1) M_1(-1;2;2), M_2(3;1;-2); \quad 2) M_1(1;-2;1), M_2(3;1;-1).$$

6.  $M(2;2;-1)$  nuqtadan o'tuvchi va  $\vec{a} = \{1;1;2\}$ ,  $\vec{b} = \{-1;3;1\}$  vektorlarga perpendikular to'g'ri chiziqning umumiy tenglamasini tuzing.

7. To'g'ri chiziq tenglamasini parametrik ko'rinishga keltiring:

$$1) \begin{cases} 5x+y-3z+5=0, \\ 8x-4y-z+6=0; \end{cases} \quad 2) \begin{cases} x+y-z-1=0, \\ x-y+2z+1=0. \end{cases}$$

8. To'g'ri chiziq tenglamasini kanonik ko'rinishga keltiring:

$$1) \begin{cases} 4x-y-z+12=0, \\ y-z-2=0; \end{cases} \quad 2) \begin{cases} x+y-z+3=0, \\ 2x-y-1=0. \end{cases}$$

9. Uchburchakning uchlari berilgan:  $A(-1;2;3)$ ,  $B(-1;-2;1)$ ,  $C(3;4;5)$ .  $A$  uchdan o'tkazilgan mediana tenglamasini tuzing.

10.  $ABCD$  parallelogramming ikki uchi  $A(-1;2;0)$ ,  $B(4;1;3)$  va diagonalning kesishish nuqtasi  $O(-2;1;2)$  berilgan. Parallelogramm  $CD$  tomonining tenglamasini tuzing.

11. To'g'ri chiziqlar orasidagi o'tkir burchakni toping:

$$1) x=-2+3t, y=0, z=3-t \text{ va } x=-1+2t, y=0, z=-3+t;$$

$$2) \frac{x}{2} = \frac{y-2}{-1} = \frac{z+2}{3} \text{ va } \begin{cases} 2x+y-z-1=0, \\ 2x-y+3z+5=0. \end{cases}$$

12.  $M(-2;3;-1)$  nuqtadan o'tuvchi va berilgan to'g'ri chiziq'larga perpendikular to'g'ri chiziq tenglamasini tuzing:

$$1) \frac{x}{2} = \frac{y}{1} = \frac{z-2}{3}, \frac{x+1}{1} = \frac{y+1}{-1} = \frac{z-2}{2}; \quad 2) \frac{x-5}{3} = \frac{y+1}{1} = \frac{z-3}{-2}, \frac{x+2}{2} = \frac{y}{-5} = \frac{z+1}{4}$$

14.  $(1; 0; -1)$  nuqtadan o'tuvchi va  $\frac{x-2}{1} = \frac{y+1}{3} = \frac{z-2}{2}$  to'g'ri chiziqqa

perpendikular bo'lgan chiziq tenglamasini tuzing.

14. To'g'ri chiziqning o'zaro joylashishini aniqlang:

1)  $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-3}{2}$ ,  $x=2+8t$ ,  $y=6t$ ,  $z=-3-4t$ ;

2)  $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-1}{1}$ ,  $\frac{x}{-2} = \frac{y-1}{3} = \frac{z+2}{-1}$ .

15. To'g'ri chiziq bilan tekislik orasidagi burchakni toping:

1)  $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z+1}{-2}$ ,  $2x+2y-9=0$ ;      2)  $\begin{cases} x-2y-1=0, \\ y-z-2=0, \end{cases} \quad x+2y-z+6=0.$

16. To'g'ri chiziq bilan tekislikning o'zaro joylashishini aniqlang:

1)  $\begin{cases} 2x+y+4z-6=0, \\ 3x+y+z+3=0, \end{cases} \quad 3x-y+6z-12=0$ ;      2)  $\frac{x+1}{2} = \frac{y-2}{8} = \frac{z+2}{3}$ ,  $2x+y-4z-8=0$ .

17. To'g'ri chiziq bilan tekislikning kesishish nuqtasini toping:

1)  $\frac{x-1}{1} = \frac{y-7}{5} = \frac{z-5}{4}$ ,  $x-3y-2z+5=0$ ;      2)  $\frac{x}{2} = \frac{y+13}{17} = \frac{z+7}{13}$ ,  $5x-z-4=0$ .

18.  $M(4;5;-6)$  nuqtadan berilgan tekislikka tushirilgan perpendikular tenglamasini tuzing:

1)  $x-2y-3=0$ ;      2)  $x-y+z-5=0$ .

19.  $m$  va  $n$  ning qanday qiymatlarida  $\frac{x-3}{-4} = \frac{y-1}{4} = \frac{z+3}{-1}$  to'g'ri chiziq:  
1)  $m+2y-4z+n=0$  tekislikda yotadi;      2)  $mx+ny+3z-5=0$  tekislikka perpendikular bo'ladi;      3)  $2x+3y+2mz-n=0$  tekislikka parallel bo'ladi.

20.  $M(1;-1;-1)$  nuqtadan o'tuvchi va berilgan to'g'ri chiziqqa perpendikular tekislik tenglamasini tuzing:

1)  $\frac{x+1}{2} = \frac{y+2}{-3} = \frac{z+2}{4}$ ;      2)  $\frac{x+3}{4} = \frac{y-1}{-1} = \frac{z-5}{-2}$ .

21.  $M(0;1;2)$  nuqtadan va  $\begin{cases} x-3y+5=0 \\ 2x+y+z-2=0 \end{cases}$  to'g'ri chiziqdan o'tuvchi tekislik tenglamasini tuzing.

22.  $\frac{x-1}{3} = \frac{y+1}{-1} = \frac{z-2}{5}$  to'g'ri chiziq bilan  $x+y-2z-4=0$  tekislikning

kesishish nuqtasini toping.

23.  $M(2;3;4)$  nuqtaning  $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$  to'g'ri chiziqdagi proeksiyasini toping.

24.  $\begin{cases} 8x+2y+3z+6=0, \\ 2x+4y+z+1=0 \end{cases}$  to'g'ri chiziqdan o'tuvchi va  $\frac{x+1}{2} = \frac{y-4}{3} = \frac{z-1}{-2}$  to'g'ri

chiziqqa parallel tekislik tenglamasini tuzing.

25.  $\begin{cases} 3x+y-4z+5=0, \\ x-y+2z-1=0 \end{cases}$  to'g'ri chiziqdan va  $M(1;-1;2)$  nuqtadan o'tgan tekislik

tenglamasini tuzing.

26.  $M(2;-3;-1)$  nuqtadan berilgan to'g'ri chiziqqacha bo'lgan masofani toping:

1)  $\frac{x-3}{4} = \frac{y+2}{3} = \frac{z+1}{5}$ ;

2)  $\frac{x+1}{2} = \frac{y+2}{-1} = \frac{z+1}{2}$ .

### 3.6. IKKINCHI TARTIBLI SIRTLAR

$Oxyz$  koordinatalar sistemasida  $x, y, z$  o'zgaruvchilarning ikkinchi darajali tenglamasi bilan aniqlanuvchi sirt *ikkinchi tartibli sirt* deyiladi.

Uchta  $x, y$  va  $z$  o'zgaruvchining ikkinchi darajali tenglamasi umumiy ko'rinishda

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Kz + L = 0, \quad (6.1)$$

kabi yoziladi, bu yerda  $A, B, C, D, E, F, G, H, K, L$  o'zgarmaslar,  $A^2 + B^2 + C^2 \neq 0$ .

Har qanday (6.1) ko'rinishdagi tenglamani koordinata o'qlarini parallel ko'chirish va burish orqali *kanonik ko'rinishga* keltirish mumkin. Kanonik tenglamada har bir o'zgaruvchi faqat bir marta, bitta (yo nolinchi, yo birinchi, yo ikkinchi) darajada qatnashadi. (6.1) tenglama koordinatalar sistemasining o'qlari sirtning simmetriya o'qlari bilan ustma-ust tushganida va koordinatalar boshi maxsus tanlanganida (masalan, markaziy-simmetrik sirtlarda simmetriya markazi tanlanadi) kanonik ko'rinishni oladi.

Shu bilan birga ikkinchi tartibli sirt

$$F(x, y) = 0 \quad (G(x, y) = 0, \quad H(x, z) = 0) \quad (6.2)$$

tenglama bilan berilishi mumkin. Bunday tenglama bilan aniqlanuvchi sirtlar silindrik sirtlar deyiladi.



Ellipsoidning  $Oxy$  tekislikka parallel tekisliklar bilan kesimlarini qaraymiz. Bu tekisliklarning har biri  $z=h$  tenglamaga ega bo'ladi. Bunda  $h$  – birorta son.

Kesimda hosil bo'lgan chiziq

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{h^2}{c^2}, \\ z = h \end{cases} \quad (6.5)$$

tenglamalar sistemasi bilan aniqlanadi.

(6.5) sistema tenglamalarini tekshiramiz.

$|h| > c$  bo'lganda  $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 0$  bo'ladi va (6.5) sirtning  $z=h$  tekislik bilan kesishish nuqtasi mavjud bo'lmaydi.

$|h| = c$ , ya'ni  $h = \pm c$  bo'lganda  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$  bo'ladi. Bunda sirtlar  $(0;0;c)$  va  $(0;0;-c)$  nuqtalarga kesishadi va  $z=c$  va  $z=-c$  tekisliklar berilgan sirtga urinadi.

$|h| < c$  bo'lganda (6.5) tenglamalarni quyidagicha yozish mumkin:

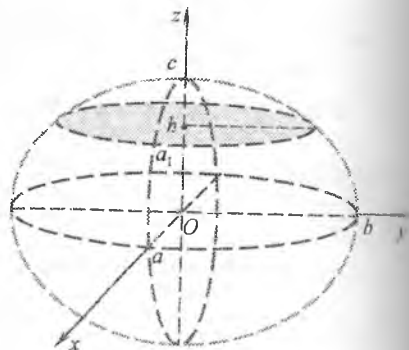
$$\begin{cases} \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \\ z = h, \end{cases}$$

bu yerda  $a_1 = a\sqrt{1 - \frac{h^2}{c^2}}$ ,  $b_1 = b\sqrt{1 - \frac{h^2}{c^2}}$ .

Demak, kesimda yarim o'qlari  $a_1$  va  $b_1$  bo'lgan ellips hosil bo'ladi (38-shakl). Bunda  $|h|$  qancha kichik bo'lsa, yarim o'qlar shuncha katta bo'ladi.  $h=0$  da ular o'zlarining eng katta qiymatlariga erishadi:  $a_1 = a$ ,  $b_1 = b$ .

(6.4) sirtning  $x=h$  va  $y=h$  tekisliklar bilan kesimlari ham ellipslardan iborat bo'ladi.

Shunday qilib, qaralgan kesimlar (6.4) tenglama bilan aniqlanuvchi sirt 38-shaklda keltirilgan ellipsoiddan iborat bo'lishini ko'rsatadi.



38-shakl.

$a, b, c$  kattaliklarga ellipsoidning yarim o'qlari deyiladi. Yarim o'qlar har xil bo'lganda ellipsoid uch o'qli ellipsoid bo'ladi. Yarim o'qlardan istalgan ikkitasi bir-biriga teng bo'lganda ellipsoid aylanish ellipsoidi bo'ladi.

Yarim o'qlarning uchalasi teng bo'lganda ellipsoid tenglamasi

$$x^2 + y^2 + z^2 = R^2, \quad R = a = b = c \quad (6.6)$$

bo'lgan sferaga aylanadi.

*2 misol.*  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  ellipsning  $Ox$  va  $Oy$  oqlari atrofida aylanishidan hosil bo'lgan sirtlarning tenglamalarini toping.

*Yechish.* Agar ikkinchi tartibli chiziq  $F(x, y) = 0$  tenglama bilan berilgan bo'lsa, u holda bu sirtning  $Ox$  oqi atrofida aylanishidan hosil bo'lgan sirt  $F(x; \pm\sqrt{y^2 + z^2}) = 0$  tenglama bilan,  $Oy$  oqi atrofida aylanishidan hosil bo'lgan sirt esa  $F(\pm\sqrt{x^2 + z^2}; y) = 0$  tenglama bilan aniqlanadi.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  ellipsning  $Ox$  oqi atrofida aylanishidan hosil bo'lgan sirt tenglamasini topamiz:

$$\frac{x^2}{a^2} + \frac{(\pm\sqrt{y^2 + z^2})^2}{b^2} = 1 \quad \text{yoki} \quad \frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$

Ellipsning  $Oy$  oqi atrofida aylanishidan hosil bo'lgan sirt tenglamasini shu kabi topiladi:

$$\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hosil bo'lgan tenglamalarning har ikkalasi aylanish ellipsoidini aniqlaydi.

### 3.6.3. Giperboloidlar

$Oxyz$  koordinatalar sistemasida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (6.7)$$

kanonik tenglama bilan aniqlanuvchi sirtga bir pallali giperboloid deyiladi.

Bu sirtni  $Oxy$  tekislikka parallel  $z=h$  tekisliklar bilan kesamiz. Kesimda

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}, \\ z = h \end{cases}$$

yoki

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1+\frac{h^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1+\frac{h^2}{c^2}}\right)^2} = 1, \\ z = h, \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi chiziq hosil bo'ladi. Bu chiziq yarim o'qlari  $a_1 = a\sqrt{1+\frac{h^2}{c^2}}$  va  $b_1 = b\sqrt{1+\frac{h^2}{c^2}}$  bo'lgan ellipsdan iborat.

Yarim o'qlar  $h=0$  da eng kichik qiymatlariga erishadi:  $a_1 = a, b_1 = b$ .  $|h|$  ning o'sishi bilan ular o'sib boradi.

Sirtning  $Oxz$  va  $Oyz$  tekisliklar bilan kesimlarni

$$\begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \\ y = 0 \end{cases}, \quad \begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \\ x = 0 \end{cases}$$

tenglamalar sistemalari bilan aniqlanuvchi giperbolalardan iborat bo'ladi.

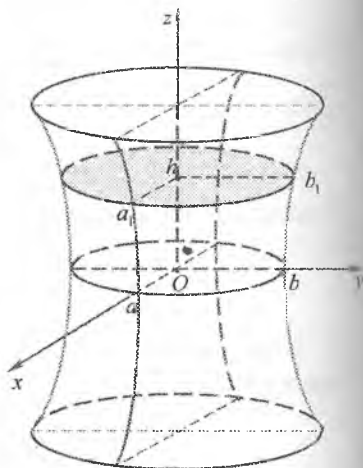
Kesimlarning tahlili shuni ko'rsatadiki, (6.7) tenglama bilan aniqlanuvchi giperboloid musbat va manfiy yo'nalishlarida chegaralanmagan holda kengayuvchi «trubka» ko'rinishdagi sirtan iborat bo'ladi (39-shakl).

$a=b$  bo'lganda (6.7) tenglama bir pallali aylanish giperboloidni ifodalaydi.

$Oxyz$  kordinatlar sistemasida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \quad (6.8)$$

kanonik tenglama bilan aniqlanuvchi sirtga *ikki pallali giperboloid* deyiladi.



39-shakl.

Bu sirtning  $Oxy$  tekislikka parallel tekisliklar bilan kesish chizig'i

$$\begin{cases} x^2 + y^2 = \frac{h^2}{c^2} - 1, \\ z = h \end{cases} \quad (6.9)$$

tenglamalar sistemasi bilan aniqlanadi. Bunda  $|h| < c$  bo'lganda  $z = h$  tekislik sirtni kesmaydi,  $|h| = c$  bo'lganda  $z = c$  va  $z = -c$  tekisliklar sirtga  $(0,0;c)$  va

$(0,0;-c)$  nuqtalarga urinadi,  $|h| > c$  bo'lganda  $z = h$  tekislik sirtni kesadi.

$|h| > c$  bo'lganda (6.9) tenglamalarni quyidagicha yozish mumkin:

$$\begin{cases} \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \\ z = h, \end{cases}$$

bu yerda  $a_1 = a\sqrt{\frac{h^2}{c^2} - 1}$ ,  $b_1 = b\sqrt{\frac{h^2}{c^2} - 1}$ .

Bu chiziq  $|h|$  ning o'lishi bilan yarim ellips o'zgarishi o'zgarishi beradi.

Sirtning  $Oxz$  va  $Oyz$  tekisliklar bilan kesimlari

$$\begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1, \\ y = 0 \end{cases}, \quad \begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \\ x = 0 \end{cases}$$

tenglamalar sistemalari bilan aniqlanuvchi giperbolalar bo'ladi.

Bu kesimlar (40-shakl) (6.8)

sirtning ikki pallali giperboloid deb atalishiga sabab bo'ladi.  $a = b$  bo'lganda (6.8) tenglama ikki pallali aylanish giperboloidni aniqlaydi.

3-misol.  $x^2 - 4y^2 + 4z^2 + 2x + 8y - 7 = 0$  tenglama bilan aniqlanuvchi sirt turini toping.

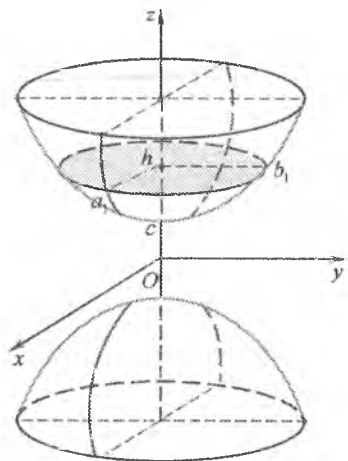
*Yechish.* Tenglamani chap tomonini to'la kvadratlarga ajratamiz:

$$x^2 + 2x + 1 - 4(y^2 + 2y + 1) + 4z^2 - 1 + 4 - 7 = 0,$$

$$(x+1)^2 - 4(y-1)^2 + 4z^2 = 4.$$

Hundan

$$\frac{(x+1)^2}{2^2} + \frac{z^2}{1^2} - \frac{(y-1)^2}{1^2} = 1.$$



40-shakl.

$x' = x + 1, y' = y - 1, z' = z$  deb,  $Oxyz$  sistema markazini  $O'(-1; 1; 0)$  nuqtaga parallel ko'chirish orqali  $O'x'y'z'$  sistemaga o'tamiz.

Bu sistemada tenglama

$$\frac{x'^2}{2^2} + \frac{z'^2}{1^2} - \frac{y'^2}{1^2} = 1$$

ko'rinishni oladi.

Bu tenglama  $O'y'$  oq bo'ylab yo'nalgan bir pallali giperboloidni aniqlaydi.

### 3.6.4. Konuslar

$Oxyz$  koordinatalar sistemasida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad (6.10)$$

kanonik tenglama bilan aniqlanuvchi sirt *ikkinchi tartibli konus* deyiladi.

(6.10) sirtning  $Oxy$  tekislikka parallel tekisliklar bilan kesishish chizig'i  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{h^2}{c^2}, z = h$  bo'ladi.

U  $h = 0$  da  $O(0; 0; 0)$  nuqtaga aylanadi.

$h \neq 0$  bo'lsa, kesimda

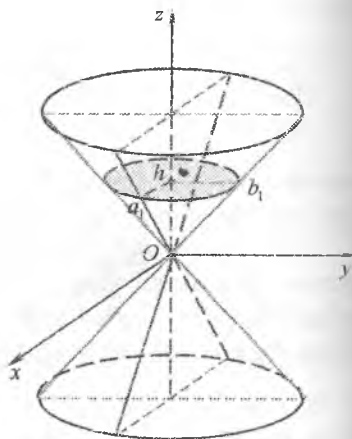
$$\begin{cases} \frac{x^2}{\left(\frac{ah}{c}\right)^2} + \frac{y^2}{\left(\frac{bh}{c}\right)^2} = 1, \\ z = h, \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi ellips hosil bo'ladi. Bu ellipsning yarim o'qlari  $|h|$  ning o'sishi bilan o'sadi.

Sirtning  $Oxz$  va  $Oyz$  tekisliklar bilan kesimlari

$$\begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = 0, \\ y = 0 \end{cases}, \quad \begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \\ x = 0 \end{cases}$$

sistemalar bilan aniqlanuvchi ikkita kesishuvchi to'g'ri chiziqlardan iborat bo'ladi (41-shakl).



41-shakl.

### 3.6.5. Paraboloidlar

Oxy koordinatalar sistemasida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}, \quad a > 0, b > 0, c > 0 \quad (6.11)$$

Ushbu tenglama bilan aniqlanuvchi sirt *elliptik paraboloid* deyiladi.

Uy sirtning *Oxy* tekislikka parallel tekisliklar bilan kesimi ushbu

$$\begin{cases} \frac{x^2}{(ah)^2} + \frac{y^2}{(bh)^2} = 1, \\ z = h, h > 0 \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi ellips bo'ladir. Uning yarim o'qlari  $|h|$  ning o'sishi bilan o'sadi.

Sirtning *Oxz* va *Oyz* tekisliklar

bilan kesimlarida  $z = \frac{x^2}{a^2}$  va  $z = \frac{y^2}{b^2}$

parabolalar hosil bo'ladir (42-shakl). Ushbu sababli (6.11) tenglama bilan aniqlanuvchi sirt *elliptik paraboloid* deyiladi.

$a = b$  bo'lganda (6.11) tenglama *aylanish paraboloidini* aniqlaydi.

*Umisol.*  $M_1(0; b; 0)$  nuqtadan va  $y = -b$  tekislikdan teng uzoqlikda yotuvchi nuqtalarning geometrik o'rnini toping.

*Yechish.*  $M(x; y; z)$  izlanayotgan

nuqta bo'lsin.

Masala shartiga ko'ra

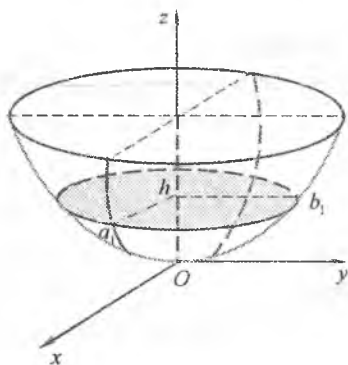
$$|M_1M| = |y + b|$$

yoki

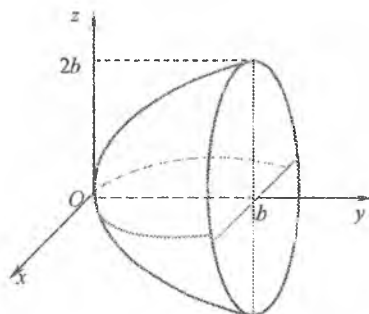
$$\sqrt{x^2 + (y - b)^2 + z^2} = |y + b|.$$

Shundan

$$\begin{aligned} \sqrt{x^2 + (y - b)^2 + z^2} &= y^2 + 2yb + b^2, \\ x^2 + z^2 &= 4by \end{aligned}$$



42-shakl.



43-shakl.

yoki

$$\frac{x^2}{4b} + \frac{z^2}{4b} = y.$$

Sirtning  $Oxz$  tekislikka parallel tekislik bilan kesimi ushbu

$$\begin{cases} x^2 + y^2 = 4bh, \\ y = h, \quad h > 0 \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi aylanalardan iborat. Sirtning  $Oxy$  va  $Oyz$  tekisliklar bilan kesimlarida  $y = \frac{x^2}{4b}$  va  $y = \frac{z^2}{4b}$  parabolalar hosil bo'ladi.

Bu sirt aylanish paraboloidini aniqlaydi (43-shakl).

$Oxyz$  koordinatalar sistemasida

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z, \quad a > 0, \quad b > 0 \quad (6.12)$$

kanonik tenglama bilan aniqlanuvchi sirt *giperbolik paraboloid* deyiladi.

Sirtning  $Oxy$  tekislikka parallel tekisliklar bilan kesimi

$$\begin{cases} \frac{x^2}{(ah)^2} - \frac{y^2}{(bh)^2} = 1, \\ z = h, \quad h > 0 \end{cases}$$

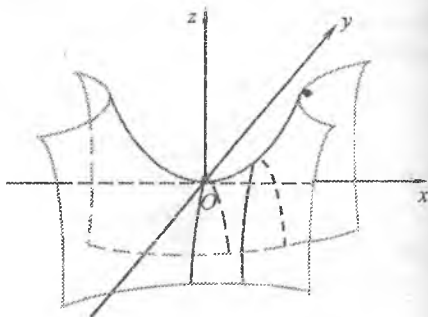
tenglamalar sistemasi bilan aniqlanuvchi giperboladan,  $Oxz$  va  $Oyz$  tekisliklar bilan kesimlari

$z = \frac{x^2}{a^2}$  va  $z = \frac{y^2}{b^2}$  parabolalardan

iborat bo'ladi.

Shunday qilib, (6.12) tenglama bilan aniqlanuvchi

sirtning ko'rinishi «egar» shaklida bo'ladi (44-shakl). Bu sirt giperbolik paraboloid deb ataladi.



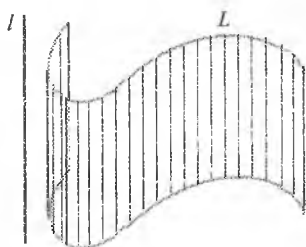
44-shakl.

### 3.6.6. Silindrik sirtlar

Tekislikda  $L$  chiziq va bu tekislikka perpendikular  $l$  to'g'ri chiziq berilgan bo'lsin.

$l$  chiziqning har bir nuqtasi orqali  $l$  to'g'ri chiziqqa parallel qilib o'tkazilgan to'g'ri chiziqlar to'plamidan hosil bo'lgan sirtga *silindrik sirt* deyiladi. Bunda  $L$  chiziq *silindrik sirtning yo'naltiruvchisi*,  $l$  to'g'ri chiziqqa parallel bo'lgan to'g'ri chiziqlar *silindrik sirtning yasovchilari* deb ataladi (45-shakl).

$Oxy$  koordinatalar sistemasini o'z o'q  $l$  yasovchiga parallel va  $L$  yo'naltiruvchi  $Oxy$  tekislikda yotdirilgan qilib tanlaymiz.

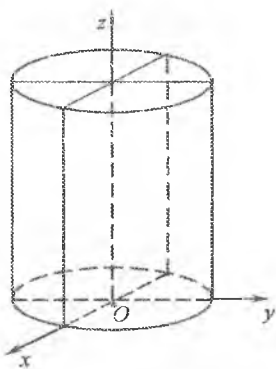


45-shakl.

$l$  yo'naltiruvchining  $Oxy$  tekislikdagi tenglamasi  $F(x, y) = 0$  bo'lsin. U holda  $F(x, y) = 0$  tenglama yasovchilari  $Oz$  o'qqa parallel bo'lgan silindrik sirtни ifodalaydi.

Silindrik sirtning nomlanishi va tenglamasi  $l$  yo'naltiruvchining shakli asosida aniqlanadi.

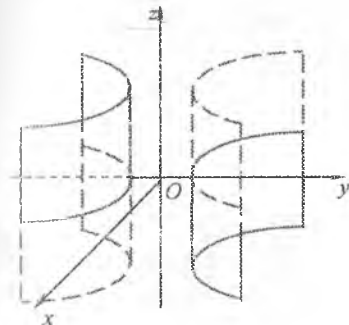
Agar  $Oxy$  tekislikdagi yo'naltiruvchi ellipsoiddan iborat bo'lganda  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  tenglama *elliptik silindrni* ifodalaydi (46-shakl).



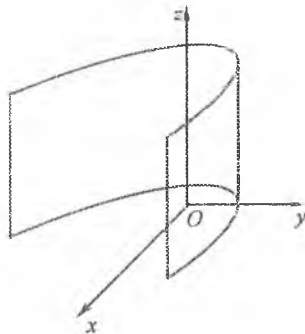
46-shakl.

$x^2 + y^2 = R^2$  tenglama bilan aniqlanuvchi *davriy silindr* elliptik silindrning xususiy holı bo'ladi.

Shu kabi  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  tenglama *giperbolik silindrni* (47-shakl),  $y^2 = 2px$  tenglama *parabolik silindrni* ifodalaydi (48-shakl).



47-shakl.



48-shakl.



5-misol.  $x^2 = 2z$  tenglama bilan aniqlanuvchi sirt shaklini chizing.

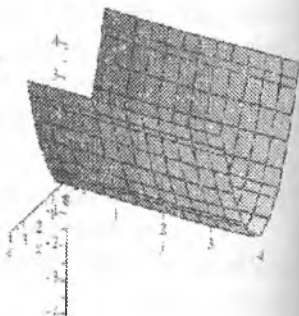
*Yechish.* Berilgan tenglamada  $y$  qatnashmaydi va  $x^2 = 2z$  chiziq  $Oxz$  tekislikda yotuvchi parabolanı ifodalaydi.

Shu sababli  $\begin{cases} x^2 = 2z, \\ y = 0 \end{cases}$  tenglama

yasovchilari  $Oy$  o'qqa parallel bo'lgan parabolik silindrni ifodalaydi.

Parabola  $y=0$  tekislikda  $Oz$  o'qqa nisbatan simmetrik bo'ladi, uchi  $O(0;0;0)$  nuqtada yotadi va  $M_1(-2;0;2)$ ,  $M_2(2;0;2)$  nuqtalardan o'tadi. Chiziq shaklini *Maple* paketida chizamiz (49-shakl):

```
> with(plots):
> implicitplot3d([x^2=2z], x=-4..4, y=-0..4,
z=-4..4, grid=[13,13,13]);
```



49-shakl.

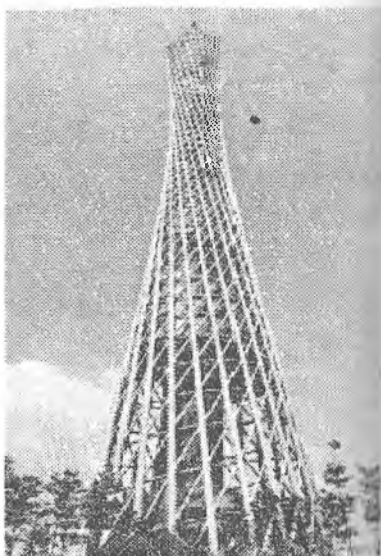
### 3.6.7. Ikkinchi tartibli sirtlarning to'g'ri chiziqli yasovchilari

To'g'ri chiziqlarning harakatidan hosil bo'ladigan sirtlarga *to'g'ri chiziqli sirtlar* deyiladi. Bu sirtlarning yasovchilari *to'g'ri chiziqli yasovchilar* deb ataladi.

To'g'ri chiziqli sirtlarga konus sirtlar va silindrik sirtlarlar misol bo'la oladi. Bundan tashqari, bir pallali giperboloid va giperbolik paraboloid ham to'g'ri chiziqli sirtlar bo'lishi isbotlangan.

Bir pallali giperboloidning har bir nuqtasi orqali

$$\begin{cases} \frac{x}{a} + \frac{z}{c} = k \left(1 + \frac{y}{b}\right), \\ \frac{x}{a} - \frac{z}{c} = l \left(1 - \frac{y}{b}\right) \end{cases} \quad (6.13)$$



50-shakl.

tenglamalar bilan berilgan to'g'ri chiziqlar oilasidan faqat bitta chiziq o'tishi ko'rsatilgan, bu yerda  $a, b, c$  – bir pallali giperboloidning yarim o'qlari,  $k, l$  – nolga teng bo'lmagan sonlar.

Shunday qilib, (6.13) tenglamalar  $k$  va  $l$  ning turli qiymatlarida bir pallali giperboloidda yotuvchi va uni to'liq qoplovchi cheksiz to'g'ri chiziqlar sistemasini tashkil qiladi. Bu to'g'ri chiziqlarga bir pallali giperboloidning *to'g'ri chiziqli yasovchilari* deyiladi.

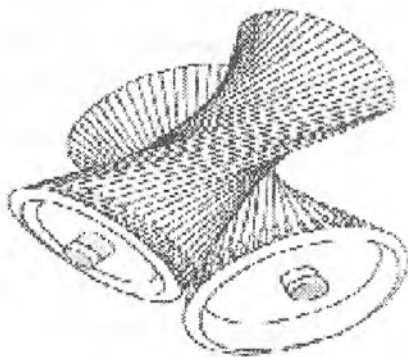
Shu kabi

$$\begin{cases} \frac{x}{a} + \frac{z}{b} = kz, \\ \frac{x}{a} - \frac{z}{b} = \frac{1}{k} \end{cases} \text{ va } \begin{cases} \frac{x}{a} + \frac{z}{b} = \frac{1}{l}, \\ \frac{x}{a} - \frac{z}{b} = lz \end{cases} \quad (6.14)$$

tenglamalar bilan berilgan to'g'ri chiziqlar giperbolik paraboloidning to'g'ri chiziqli yasovchilari bo'ladi.

To'g'ri chiziqli yasovchilardan bir pallali giperboloid sirtlarining yasovchilari qurilish va tasnifining konstruksiyalarida keng foydalaniladi.

*Masalan*, bir pallali giperboloidning to'g'ri chiziqli yasovchilari bo'yicha metall balkalar joylashtirilgan radiomachta, suv va yadro qurilmalarining minoralari, teleminoralar (masalan, Guanjou (Xitoy) teleminorasi, 50-shakl), tepaklar va uzatmalar (51-shakl) kabi konstruksiyalar ishlab chiqilgan. Bunday konstruksiyalar yengil va mustahkam bo'lgani sababli keng tatbiq etiladi.



51-

### 3.6.8. Mashqlar

1. Sferaning tenglamasini tuzing: 1) diametrlaridan birining uchlari  $M_1(1; 3)$  va  $M_2(2; -3; 5)$  nuqtalarda yotgan; 2) markazi  $M_0(3; -5; -2)$  nuqtada yotgan va  $2x - y - 3z + 11 = 0$  tekislikka uringan.

2. Sfera markazining koordinatalarini va radiusini toping:

1)  $x^2 + y^2 + z^2 + x - y + z = 0;$

2)  $x^2 + y^2 + z^2 - 6x + 8y + 10z + 25 = 0.$

3. Har bir nuqtasidan  $M_1(-3;0;0)$  va  $M_2(3;0;0)$  nuqtalargacha bo'lgan masofalar kvadratlarining yig'indisi 36 ga teng bo'lgan fazoviy nuqtalarning geometrik o'rnini toping.

4.  $M\left(0; \frac{5}{2}; 0\right)$  nuqtadan va  $y = -\frac{5}{2}$  tekislikdan teng uzoqlikda yotgan fazoviy nuqtalarning geometrik o'rnini toping.

5. Har bir nuqtasidan  $M_1(0;0;-4)$  va  $M_2(0;0;4)$  nuqtalargacha bo'lgan masofalar yig'indisi 10 ga teng bo'lgan sirtning tenglamasini tuzing.

6. Har bir nuqtasidan  $M_1(-5;0;0)$  va  $M_2(5;0;0)$  nuqtalargacha bo'lgan masofalar ayirmasining moduli 6 ga teng bo'lgan sirtning tenglamasini tuzing.

7. Berilgan sirtning ko'rsatilgan o'qlar atrofida aylanishidan hosil bo'lgan sirt tenglamasini tuzing:

1)  $z = -\frac{x^2}{2}, O_x$  va  $O_z$ ;      2)  $\frac{x^2}{16} - \frac{y^2}{25} = 1, O_x$  va  $O_y$ ;      3)  $\frac{y^2}{64} + \frac{z^2}{16} = 1, O_y$  va  $O_z$ .

8. Markazi koordinatalar boshida yotgan va yo'naltiruvchilari  $x^2 - 2z + 1 = 0$ ,  $y - z + 1 = 0$  tenglamalar bilan berilgan konus tenglamasini tuzing.

9.  $m$  ning qanday qiymatlarida  $x + my - 2 = 0$  tekislik  $\frac{x^2}{2} + \frac{z^2}{3} = y$  elliptik paraboloidni 1) ellips bo'yicha; 2) parabola bo'yicha kesadi?

10. Berilgan sirtlarning kesishish chizig'ini aniqlang:

1)  $\frac{x^2}{3} + \frac{y^2}{6} = 2z, 3x - y + 6z - 14 = 0$ ;      2)  $\frac{x^2}{4} - \frac{y^2}{3} = 2z, 3x - y + 6z - 14 = 0$ ;  
3)  $\frac{(x-1)^2}{4} - \frac{(y+1)^2}{3} = 2z, x - 2y - 1 = 0$ ;      4)  $\frac{x^2}{3} + \frac{y^2}{9} - \frac{z^2}{25} = -1, 5x + 2z - 5 = 0$ .

11. Har bir nuqtasidan  $M(3;0;0)$  nuqtagacha va  $x=1$  tekislikkacha bo'lgan masofalar nisbati  $\sqrt{3}$  ga teng bo'lgan fazoviy nuqtalarning geometrik o'rnini toping.

12. Berilgan sirt va to'g'ri chiziqning kesishish nuqtasini toping:

1)  $\frac{x^2}{81} + \frac{y^2}{36} + \frac{z^2}{9} = 1, \frac{x-3}{3} = \frac{y-4}{-6} = \frac{z+2}{4}$ ;      2)  $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{4} = 1, \frac{x}{4} = \frac{y}{-3} = \frac{z+2}{4}$ .

13. Berilgan tenglama bilan aniqlanuvchi sirt turini aniqlang:

1)  $36x^2 + 64y^2 - 144z^2 + 576 = 0$ ;      2)  $x^2 + y^2 + z^2 - 2(x + y + z) - 22 = 0$ ;  
3)  $3x^2 + 2y^2 - 12z = 0$ ;      4)  $16x^2 + 3y^2 + 16z^2 - 64x - 6y + 19 = 0$ ;  
5)  $25x^2 - 9y^2 - 225 = 0$ ;      6)  $9x^2 - 4y^2 - 36z = 0$ ;  
7)  $4x^2 + 3y^2 - 5z^2 + 60 = 0$ ;      8)  $x^2 + y^2 - 2x - 3 = 0$ .

- Haqiqiy sonlar
- Sonli ketma-ketliklar
- Bir o'zgaruvchining funksiyasi
- Funksiyaning limiti
- Funksiyaning uzluksizligi



*Ogusten Lui Koshi  
(1789–1857) –  
fransuz matematigi  
va mexanigi.*

*Matematik analiz asolarini ishlab chiqqan, matematikaning matematik analiz, algebra, matematik fizika va boshqa bo'limlarining rivojiga katta hissa qo'shgan.*

*O.L. Koshi birinchi bo'lib limit, uzluksizlik, hosila, differensial, integral, qat'orning yaqinlashishi tushunchalariga qat'iy ta'rif bergan.*

# MATEMATIK ANALIZGA KIRISH

Matematik analiz tarixdan «Cheksiz kichiklar tahlili»ga mos bo'limlar majmuasi bo'lib, u differensial va integral hisobni birlashtiradi. Matematik analiz sistemali bo'lim sifatida XVII–XVIII asrlarda I.Nuyton, G.Leybnis, L.Eyler, J.Lagranj va boshqa olimlar asarlarida yuzaga keldi. Uning asosi bo'lgan limitlar nazariyasi XIX asrda O.Koshi tomonidan ishlab chiqildi. Matematik analizning boshlang'ich tushunchalari XIX–XX asrlarda to'plamlar nazariyasi, o'lchamlar nazariyasi, haqiqiy o'zgaruvchi funksiyalari nazariyalarining rivojiga asoslanib chuqur tahlil qilindi va umumlashtirildi.

Matematikaning «Matematik analizga kirish» bo'limida matematik analizning asosiy tushunchalari bo'lgan bir o'zgaruvchining funksiyasi, sonli ketma-ketliklar, limitlar, cheksiz kichik funksiyalar va uzluksizlik tushunchalari o'rganiladi.

## 4.1. HAQIQIY SONLAR

### 4.1.1. To'plam

*To'plam* matematikaning boshlang'ich (ta'riflanmaydigan) va muhim tushunchalaridan biri hisoblanadi. To'plam deganda tayin xossaga ega bo'lgan ixtiyoriy tabiatli obyektlar majmuasi tushuniladi. Masalan, guruhdagi talabalar to'plami, butun sonlar to'plami, berilgan nuqtadan o'tuvchi to'g'ri chiziqlar to'plami.

To'plamni tashkil etuvchi obyektarga to'plamning *elementlari* deyiladi. To'plam, odatda, lotin alifbosining bosh harflari bilan, uning elementlari esa shu alifboning kichik harflari bilan belgilanadi.

$A$  to'plamning  $a, b, c, d$  elementlardan tashkil topganligi  $A = \{a, b, c, d\}$  kabi yoziladi. Ba'zan to'plam sonlar, belgilar, so'zlar yoki formulalar yordamida beriladi.

$a$  elementning  $A$  to'plamga tegishli ekanligi  $a \in A$  deb yoziladi.  $b$  elementning  $A$  to'plamga tegishli emasligi  $b \notin A$  (yoki  $b \bar{\in} A$ ) kabi belgilanadi. Masalan,  $A = \{2, 4, 6, 8\}$  to'plam uchun  $4 \in A$  va  $5 \notin A$ .

$A$  to'plam chekli sondagi elementlardan tashkil topgan bo'lsa  $A$  to'plamga *chekli to'plam*, aks holda *cheksiz to'plam* deyiladi. Masalan,  $A = \{x : 6 < x < 20, x \in \mathbb{N}\}$  chekli to'plam,  $B = \{x : x > 15, x \in \mathbb{N}\}$  cheksiz to'plam bo'ladi.

Bitta ham elementga ega bo'lmagan to'plam *bo'sh to'plam* deb ataladi va  $\emptyset$  kabi belgilanadi. Masalan,  $A = \{x : x^2 + 1 = 0, x \in \mathbb{R}\}$  bo'sh to'plam, chunki  $x^2 + 1 = 0$  tenglama haqiqiy sonlar to'plami  $\mathbb{R}$  da yechimga ega emas.

Agar  $A$  to'plamning har bir elementi  $B$  to'plamning ham elementi bo'lsa  $A$  to'plamga  $B$  to'plamning *qismi (qisman to'plami)* deyiladi va  $A \subset B$  (yoki  $B \supset A$ ) kabi belgilanadi. Masalan,  $A = \{2, 3, 4\}$  va  $B = \{1, 2, 3, 4, 5\}$  bo'lsa  $A \subset B$  bo'ladi.

Agar  $A \subset B$  va  $B \subset A$  bo'lsa  $A$  va  $B$  to'plamlarga *teng to'plamlar* deyiladi va  $A = B$  kabi yoziladi. Demak,  $A = B$  tenglik  $A$  va  $B$  to'plamlarning bir xil elementlardan tashkil topganini bildiradi.

$A$  va  $B$  to'plamlarning har ikkalasiga tegishli bo'lgan element bu to'plamlarning *umumiy elementi* deyiladi.

**1-ta'rif.**  $A$  va  $B$  to'plamlarning *birlashmasi* (yoki *yig'indisi*) deb ularning kamida bittasiga tegishli bo'lgan elementlardan tashkil topgan to'plamga aytiladi va  $A \cup B$  (yoki  $A + B$ ) kabi belgilanadi.

Demak,  $A \cup B = \{x : x \in A \text{ yoki } x \in B\}$ .

**2-ta'rif.**  $A$  va  $B$  to'plamlarning *kesishmasi* (yoki *ko'paytmasi*) deb ularning barcha umumiy elementlaridan tashkil topgan to'plamga aytiladi va  $A \cap B$  (yoki  $A \cdot B$ ) kabi belgilanadi.

Demak,  $A \cap B = \{x : x \in A \text{ va } x \in B\}$ .

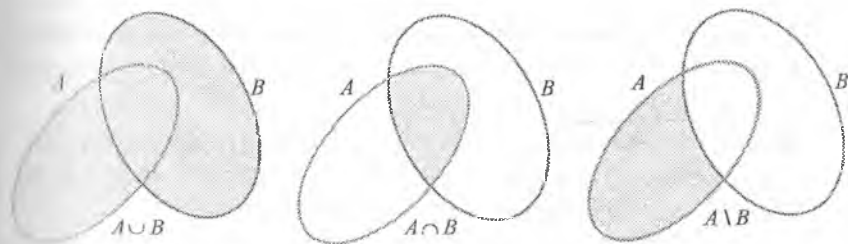
**3-ta'rif.**  $A$  to'plamdan  $B$  to'plamning *ayirmasi* deb  $A$  to'plamning  $B$  to'plamga kirmagan elementlaridan tashkil topgan to'plamga aytiladi va  $A \setminus B$  kabi belgilanadi.

Demak,  $A \setminus B = \{x : x \in A \text{ va } x \notin B\}$ .

Masalan,  $A = \{1,3,5,7\}$  va  $B = \{2,5,7,9\}$  to'plamlar uchun  $A \cup B = \{1,2,3,5,7,9\}$ ,  $A \cap B = \{5,7\}$ ,  $A \setminus B = \{1,3\}$ ,  $B \setminus A = \{2,9\}$  bo'ladi.

$B$  to'plam  $A$  to'plamning qisman to'plami bo'lsa,  $A \setminus B$  ayirma  $B$  to'plamning  $A$  to'plamga to'ldiruvchisi deyiladi.

1.3 ta'riflarning chizmadagi ifodasi 1-shaklda keltirilgan. Bunda  $A \cap B$ ,  $A \cup B$ ,  $A \setminus B$  bo'yab ko'rsatilgan.



1-shakl.

#### 4.1.2. Sonli to'plamlar

##### *Haqiqiy sonlar va ularning asosiy xossalari*

Elementlari sonlardan iborat bo'lgan to'plam *sonli to'plam* deyiladi.

Son matematik analizning asosiy tushunchalaridan biri bo'lib, uzoq tarixiy rivojlanish yo'liga ega. Narsalarni, buyumlarni sanash zaruriyati tufayli natural sonlar paydo bo'lgan. Natural sonlar to'plamiga ularga qarshilik qaratilgan sonlarni va nol sonini qo'shish bilan butun sonlar to'plami hosil qilingan. Matematikaning taraqqiyoti ratsional sonlarning va keyinchalik irratsional sonlarning kiritilishini taqozo etgan. Ratsional sonlar to'plami va irratsional sonlar to'plami haqiqiy sonlar to'plami deb atalgan.

Shunday qilib,  $N \subset Z \subset Q \subset R$  sonli to'plamlar hosil qilingan, bu yerda  $N = \{1,2,3,\dots,n,\dots\}$  – barcha natural sonlar to'plami;  $Z = \{0, \pm 1, \pm 2, \dots, \pm n, \dots\}$  – barcha butun sonlar to'plami;  $Q = \left\{ \frac{p}{q}, p \in Z, q \in N \right\}$  – barcha ratsional sonlar to'plami;  $R$  – barcha haqiqiy sonlar to'plami.

Har qanday ratsional son yoki chekli oʻnli kasr bilan yoki cheksiz davriy oʻnli kasr bilan ifodalanadi. Masalan,  $\frac{3}{2} = 1,5 = (1,500\dots)$ ,  $\frac{1}{3} = 0,333\dots$  – ratsional sonlar.

Ratsional boʻlmagan haqiqiy sonlarga irratsional sonlar deyiladi. Irratsional son cheksiz davriy boʻlmagan oʻnli kasr bilan ifodalanadi. Masalan,  $\sqrt{2} = 1,4142356\dots$ ,  $\pi = 3,1415926\dots$  – irratsional sonlar.

Shunday qilib, *haqiqiy sonlar* toʻplamini barcha cheksiz oʻnli kasrlar toʻplami deyish va  $R = \{x: x = a, \alpha, \alpha, \alpha, \dots\}$  kabi yozish mumkin, bu yerda  $a \in \mathbb{Z}, \alpha_i \in \{0, 1, 2, \dots, 9\}, i = 1, 2, \dots$

Haqiqiy sonlar toʻplami  $R$  quyidagi asosiy xossalarga ega boʻladi.

1°.  $R$  toʻplam tartiblangan toʻplamdir, yaʼni istalgan ikkita har xil  $a$  va  $b$  sonlar uchun  $a < b$  (yoki  $b < a$ ) tengsizlik bajariladi.

2°.  $R$  toʻplam zichdir, yaʼni istalgan ikkita har xil  $a$  va  $b$  sonlar orasida  $a < x < b$  tengsizlikni qanoatlantiruvchi cheksiz koʻp  $x$  haqiqiy sonlar mavjud boʻladi;

3°.  $R$  toʻplam uzluksizdir.

### **Son oʻqi. Sonlarning sodda toʻplamlari**

Haqiqiy sonlarning uzluksizligi xossasi asosida barcha haqiqiy sonlar toʻplami bilan toʻgʻri chiziq nuqtalari toʻplami orasida bir qiymatli moslik oʻrnatiladi.

Buning uchun biror toʻgʻri chiziqda (u gorizontal yoʻnalgan boʻlsin (2-shakl)) musbat yoʻnalishni,  $O$  hisob boshini va masshtab birligini tanlaymiz. Musbat  $x$  sonini ifodalash uchun bu toʻgʻri chiziqda  $O$  hisob boshidan oʻng tomonda tanlangan masshtab birligida berilgan  $x$  songa teng masofada yotuvchi  $M$  nuqtani olamiz; manfiy  $x$  sonini ifodalash uchun esa bu toʻgʻri chiziqda  $O$  hisob boshidan chap tomonda  $|x|$  songa (bu son haqida keyingi badda tushuncha beriladi) teng masofada yotuvchi  $M$  nuqtani olamiz;  $x = 0$  soniga  $O$  hisob boshi mos keladi.



2-shakl.

Barcha nuqtalari uchun barcha haqiqiy sonlar toʻplami bilan koʻrsatilgan bir qiymatli moslik oʻrnatilgan toʻgʻri chiziqqa *son oʻqi* (yoki *sonli toʻgʻri chiziq*) deyiladi.

Shunday qilib, har bir haqiqiy songa son o'qining yagona  $M$  nuqtasi mos qo'yiladi va aksincha, bu son o'qining har bir  $M$  nuqtasiga yagona haqiqiy son mos keladi. Bunda haqiqiy son va son o'qining nuqtasi buta  $x$  belgi bilan ifodalanadi. Shu sababli « $x$  son» so'zi o'miga ko'p hollarda « $x$  nuqta» so'zi ishlatiladi.

Son o'qi haqiqiy sonlarning joylashishi to'g'risida ko'rgazmali ma'lumot beradi.  $x_1 < x_2$  tengsizlik  $x_1$  nuqta  $x_2$  nuqtaga nisbatan chapda yotishini anglatadi,  $x_1 < x_2 < x_3$  tengsizlik  $x_2$  nuqta  $x_1$  va  $x_3$  nuqtalar orasida yotishini bildiradi.

$a \in R, b \in R, a < b$  bo'lsin. Haqiqiy sonlar to'plamining quyidagi qism to'plamlariga *oraliqlar (intervallar)* deyiladi:

$[a; b] = \{x: a \leq x \leq b\}$  – kesma (yopiq oraliq, sigment);

$(a; b) = \{x: a < x < b\}$  – interval (ochiq oraliq);

$[a; b) = \{x: a \leq x < b\}, (a; b] = \{x: a < x \leq b\}$  – yarim ochiq intervallar;

$(-\infty; b] = \{x: x \leq b\}, (-\infty; b) = \{x: x < b\}, [a; +\infty) = \{x: x \geq a\},$

$(a; +\infty) = \{x: x > a\}, (-\infty; +\infty) = \{x: -\infty < x < +\infty\}$  – cheksiz intervallar.

Bunda  $a$  va  $b$  sonlar mos ravishda bu oraliqlarning chap va o'ng

chegaralarini aniqlaydi,  $-\infty$  va  $+\infty$  belgilar son o'qi nuqtalarining  $O$  nuqtadan

chapga va o'ngga qarab cheksiz uzoqlashishini simvolik tasvirlaydi.

$x_0 (x_0 \in R)$  nuqtani o'z ichiga olgan har qanday  $(a; b)$  intervalga  $x_0$  nuqtaning atrofi deyiladi. Xususan,  $(x_0 - \varepsilon; x_0 + \varepsilon)$  interval  $x_0$  nuqtaning  $\varepsilon$  atrofi deb ataladi. Bunda  $x_0$  soniga bu atrofning markazi,  $\varepsilon (\varepsilon > 0)$  soniga bu atrofning radiusi deyiladi.

Agar  $x_0 \in (x_0 - \varepsilon; x_0 + \varepsilon)$  bo'lsa, u holda  $x_0 - \varepsilon < x < x_0 + \varepsilon$  yoki  $|x - x_0| < \varepsilon$  tengsizlik bajariladi. Bu tengsizlikning bajarilishi  $x$  nuqta  $x_0$  nuqtaning  $\varepsilon$  atrofiga tushishini bildiradi.

### Sonning absolut qiymati

4-ta'rif.  $x (x \in R)$  sonining *absolut qiymati* (yoki *moduli*) deb  $x \geq 0$  bo'lganida  $x$  sonining o'ziga,  $x$  manfiy bo'lganida  $(-x)$  soniga aytiladi.

$x$  sonining absolut qiymati  $|x|$  belgi bilan belgilanadi.

Demak, ta'rifga ko'ra,

$$|x| = \begin{cases} x, & \text{agar } x \geq 0, \\ -x, & \text{agar } x < 0. \end{cases}$$



Sonning absolut qiymati quyidagi xossalarga ega.

$$1^{\circ}. x \in R \text{ da } |x| \geq 0, |-x| = |x|, -|x| \leq x \leq |x|;$$

2<sup>o</sup>.  $a > 0$  da  $|x| \leq a$  tengsizlik  $-a \leq x \leq a$  tengsizlikka ekvivalent bo'ladi;

$$3^{\circ}. x \in R, y \in R \text{ da}$$

$$|x+y| \leq |x| + |y|, |x-y| \geq |x| - |y|, |x \cdot y| = |x| \cdot |y|, \frac{|x|}{|y|} = \frac{|x|}{|y|}, (y \neq 0).$$

Bu xossalarning isboti bevosita sonning absolut qiymati ta'rifidan kelib chiqadi. Ulardan birini, masalan,  $|x+y| \leq |x| + |y|$  bo'lishini isbotlaymiz.

*Isboti.*  $x+y > 0$  bo'lsin.

U holda  $|x+y| = x+y$ ,  $x \leq |x|$ ,  $y \leq |y|$  bo'ladi.

Bundan

$$|x+y| = x+y \leq |x| + |y|.$$

$x+y < 0$  bo'lsin.

U holda

$$|x+y| = -(x+y) = (-x) + (-y), -x \leq |x|, -y \leq |y|$$

bo'ladi.

Bundan

$$|x+y| = (-x) + (-y) \leq |x| + |y|.$$

### Sonli to'planning aniq chegaralari

**5-ta'rif.** Agar shunday  $M$  soni topilsa va istalgan  $x \in X$  uchun  $x \leq M$  tengsizlik bajarilsa,  $X$  haqiqiy sonlar to'plami *yuqoridan chegaralangan* deyiladi.

Masalan,  $X = (-\infty, 1]$  to'plam yuqoridan chegaralangan.

**6-ta'rif.** Agar shunday  $m$  soni topilsa va istalgan  $x \in X$  uchun  $x \geq m$  tengsizlik bajarilsa,  $X$  haqiqiy sonlar to'plami *quyidan chegaralangan* deyiladi.

Masalan, barcha natural sonlar to'plami quyidan chegaralangan.

**7-ta'rif.** Agar  $X$  to'plam ham quyidan ham yuqoridan chegaralangan bo'lsa, y'ani shunday  $m$  va  $M$  sonlari topilsa va istalgan  $x \in X$  uchun  $m \leq X \leq M$  tengsizlik bajarilsa,  $X$  haqiqiy sonlar to'plamiga *chegaralangan* deyiladi.

Bu ta'rifdan agar  $X$  to'plamning elementlari biror kesmada joylashsa, u holda bu to'plam chegaralangan bo'ladi degan xulosa kelib chiqadi.

Yuqoridan (quyidan) chegaralanmagan to'plamga *yuqoridan (quyidan) chegaralanmagan* deyiladi. Masalan, barcha natural sonlar to'plami yuqoridan chegaralanmagan (ammo quyidan chegaralangan) bo'lsa, barcha manfiy sonlar to'plami quyidan chegaralanmagan (ammo yuqoridan chegaralangan). Barcha butun sonlar to'plami, barcha ratsional sonlar to'plami, shuningdek, barcha haqiqiy sonlar to'plami ham quyidan, ham yuqoridan chegaralanmagan.

Agar  $X$  to'plam yuqoridan  $M$  soni bilan chegaralangan bo'lsa, bu songa  $X$  to'plamning *yuqori chegarasi* deyiladi. Bunda  $M$  sonidan katta bo'lgan ixtiyoriy  $M'$  son ham  $X$  to'plamning yuqori chegarasi bo'ladi.

**8-ta'rif.** Agar istalgan  $x \in X$  uchun  $x \leq M$  bo'lsa va yetarlicha kichik ixtiyoriy  $\varepsilon > 0$  musbat son uchun shunday  $x^* \in X$  soni topilsa va  $M - \varepsilon < x^* < M$  tengsizlik bajarilsa,  $M$  soniga  $X$  to'plamning *aniq yuqori chegarasi* deyiladi.

Boshqacha aytganda,  $X$  to'plamning aniq yuqori chegarasi  $X$  ning barcha yuqori chegaralarining eng kichigi bo'ladi.

$X$  to'plamning aniq yuqori chegarasi  $M = \sup X$  (yoki  $M = \sup\{x\}$ ) bilan belgilanadi (lotin tilida supremum – eng yuqori so'zidan olingan).

Yuqoridan chegaralanmagan  $X$  to'plam uchun ta'rif asosida  $\sup X = +\infty$  deb olinadi.

Agar  $X$  to'plam quyidan  $m$  soni bilan chegaralangan bo'lsa, bu songa  $X$  to'plamning *quyi chegarasi* deyiladi. Bunda  $m$  sonidan kichik bo'lgan ixtiyoriy  $m'$  son ham  $X$  to'plamning quyi chegarasi bo'ladi.

**9-ta'rif.** Agar istalgan  $x \in X$  uchun  $x \geq m$  bo'lsa va yetarlicha kichik ixtiyoriy  $\varepsilon > 0$  musbat son uchun shunday  $x^* \in X$  soni topilsa va  $m - \varepsilon < x^* < m + \varepsilon$  tengsizlik bajarilsa,  $m$  soniga  $X$  to'plamning *aniq quyi chegarasi* deyiladi.

Shunday qilib,  $X$  to'plamning aniq quyi chegarasi  $X$  ning barcha quyi chegaralarining eng kattasi bo'ladi.

$X$  to'plamning aniq quyi chegarasi  $M = \inf X$  (yoki  $M = \inf\{x\}$ )

bilan belgilanadi (lotin tilida infimum – eng quyi so‘zidan oliingan).

Quyidan chegaralanmagan  $X$  to‘plam uchun ta’rif asosida  $\inf X = -\infty$  deb olinadi. Masalan,  $X = \left\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\}$  to‘plam uchun  $\inf X = 0$ ,  $\sup X = 1$ .

$X$  to‘plamning aniq chegaralari uchun quyidagiu tasdiq o‘rinli bo‘ladi.

**1-teorema.** Har qanday yuqoridan (quyidan) chegaralangan bo‘sh bo‘lmagan haqiqiy sonlar to‘plami aniq yuqori (aniq quyi) chegaraga ega bo‘ladi.

### 4.1.3. Matematik mantiq elementlari

#### *Mantiqiy belgilar*

Ta’riflarning, teoremlarning ifodalanishida va bosqa matematik tasdiqlarda ko‘pincha ayrim so‘zlar va butun ifodalar takrorlanib keladi. Bunday hollarda ularning yozilishida mantiqiy belgilarni qo‘llash qulay bo‘ladi.

Matematik mantiqda *mulohaza* deb rost yoki yolg‘onligi bir qiymatli aniqlanadigan darak gaplarga aytiladi. Masalan, «Yer quyosh atrofida aylanadi», «6 oddiy son» gaplari mulohaza bo‘lsa, «kech kirmoqda», «matematika qiyin fan» gaplari mulohaza bo‘lmaydi.

$\alpha$  mulohazaning *inkori*  $\bar{\alpha}$  – « $\alpha$  emas» yoki « $\alpha$  bo‘lishi to‘g‘ri emas» deb o‘qiladi.

$\alpha$  va  $\beta$  mulohazalarning *konyunksiyasi*  $\alpha \wedge \beta$  – « $\alpha$  va  $\beta$ » deb o‘qiladi.

$\alpha$  va  $\beta$  mulohazalarning *dizyunksiyasi*  $\alpha \vee \beta$  – « $\alpha$  yoki  $\beta$ » deb o‘qiladi.

$\alpha$  va  $\beta$  mulohazalarning *implikatsiyasi*  $\alpha \Rightarrow \beta$  – «agar  $\alpha$  bo‘lsa, u holda  $\beta$  bo‘ladi» (yoki « $\alpha$  dan  $\beta$  kelib chiqadi») mulohazasini bildiradi.

$\alpha$  va  $\beta$  mulohazalarning *ekvivalensiyasi*  $\alpha \Leftrightarrow \beta$  – « $\alpha$  va  $\beta$  ekvivalent» (yoki « $\alpha$  dan  $\beta$  kelib chiqadi va  $\beta$  dan  $\alpha$  kelib chiqadi») mulohazasini bildiradi.

*Mavjudlik kvantori*  $\exists$  – «mavjudki», «topiladiki» so‘zlarini bildiradi.

*Umumiylik kvantori*  $\forall$  – «har qanday», «ixtiyoriy», «barcha» soʻzlarini ifodalaydi.

1) «oʻrinli boʻladi», «bajariladi» soʻzlarini anglatadi.

2) «moslik» ni bildiradi.

Mantiqiy belgilar yordamida yozilgan tasdiqlarni tushunish va ifodani osonlashtirish uchun belgilarning har biriga tegishli boʻlganlari alohida qavslarga olinadi.

Masalan,

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \neq x_0, |x - x_0| < \delta): |f(x) - A| < \varepsilon$$

yozuv «ixtiyoriy  $\varepsilon > 0$  son uchun shunday  $\delta > 0$  son topiladiki,  $x$  ning  $x_0$  ga teng boʻlmagan va  $|x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha qiymatlarida  $|f(x) - A| < \varepsilon$  tengsizlik bagariladi» deb oʻqiladi.

### *Zarur va yetarli shartlar*

$\beta$  – birorta mulohaza boʻlsin.  $\beta$  mulohaza kelib chiqadigan har qanday  $\alpha$  mulohazaga  $\beta$  mulohaza uchun *yetarli shart* deyiladi.

$\beta$  mulohazadan kelib chiqadigan har qanday  $\alpha$  mulohazaga  $\beta$  mulohaza uchun *zarur shart* deyiladi.

Masalan,  $\alpha$ : « $x$  soni nolga teng» va  $\beta$ : « $xy$  koʻpaytma nolga teng»

mulohazalari boʻlsin. Bunda  $\alpha$  mulohaza  $\beta$  mulohaza uchun yetarli shart boʻladi. Haqiqatan ham,  $xy$  koʻpaytma nolga teng boʻlishi uchun  $x$  nolga teng boʻlishi yetarli.  $x$  nolga teng boʻlishi uchun  $xy$  koʻpaytma nolga teng boʻlishi zarur. Ammo,  $\beta$  mulohaza  $\alpha$  mulohaza uchun yetarli shart boʻlmaydi, chunki  $xy$  koʻpaytma nolga teng boʻlishidan  $x$  sonining, albatta, nolga teng boʻlishi kelib chiqmaydi.

«Agar  $\alpha$  mulohaza rost boʻlsa, u holda  $\beta$  mulohaza rost boʻladi» teoremani  $\alpha \Rightarrow \beta$  koʻrinishda yozish va quyidagi ifodalardan biri bilan berish mumkin: « $\alpha$  mulohaza  $\beta$  mulohaza uchun yetarli shart boʻladi»; « $\beta$  mulohaza  $\alpha$  mulohaza uchun zarur shart boʻladi».

Agar  $\alpha$  va  $\beta$  mulohazalarning har biridan ikkinchisi kelib chiqsa, uningni  $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$  boʻlsa, u holda  $\alpha$  va  $\beta$  mulohazalarning har biri ikkinchisi uchun zarur va yetarli shart boʻladi va  $\alpha \Leftrightarrow \beta$  deb yoziladi.

Bu yozuv boshqacha quyidagicha oʻqilishi mumkin:

1)  $\alpha$  oʻrinli boʻlishi uchun  $\beta$  oʻrinli boʻlishi zarur va yetarli;

2)  $\alpha$  faqat va faqat  $\beta$  bajarilsa oʻrinli boʻladi;

3)  $\alpha$  faqat va faqat  $\beta$  rost bo'lganida rost bo'ladi.

### Matematik induksiya metodi

Matematik induksiya metodi muhim matematik isbotlash usullaridan biri hisoblanadi. Bu usul  $n$  natural songa bog'liq tasdiqlarni isbotlash uchun qo'llaniladi.

Uni umumiy holda ifodalaymiz:  $n$  natural songa bog'liq biror tasdiqni (masalan, formulani) isbotlash quyidagi tartibda amalga oshiriladi:

1) tasdiqning to'g'riligi  $n=1$  da tekshiriladi (agar  $n=1$  da tasdiq ma'noga ega bo'lmasa, tekshirish tasdiq ma'noga ega bo'ladigan eng kichik  $n$  dan boshlanadi);

2) tasdiq biror  $n$  ( $n > 1$ ) da to'g'ri deb faraz qilinadi va uning  $n+1$  da to'g'ri bo'lishi isbotlanadi. Keyin bu tasdiqning istalgan  $n$  natural son bajarilishi haqida xulosa chiqariladi.

Matematik induksiya metodi bilan *Nuyton binomi formulasi* deb ataluvchi

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + \dots + C_n^k a^{n-k} b^k + \dots + C_n^n b^n \quad (1.1)$$

formulani isbotlaymiz, bu yerda  $n$  - natural son;  $0 \leq k \leq n$ .

(1.1) formulada qatnashayotgan

$$C_n^k = \frac{n!}{k!(n-k)!}$$

koeffitsiyentlarga *binominal koeffitsiyentlar* (yoki  $n$  ta elementdan  $k$  tadan guruhlashlar soni) deyiladi, bu yerda  $n!$  (*en faktorial*) belgi orqali birinchi  $n$  ta natural son ko'paytmasi belgilanadi. Binominal koeffitsiyentlar va faktorial uchun quyidagi bog'lanishlar o'rinli bo'ladi:

$$C_n^{k+1} + C_n^k = C_{n+1}^k$$

$(n+1)! = n!(n+1)$  (bunda  $0! = 1$  deb olinadi).

(1.1) formulani isbotlaymiz. Buning uchun: 1) formula to'g'ri bo'lishini  $n=1$  da tekshiramiz:

$$(a+b)^1 = C_1^0 a^1 + C_1^1 a^0 b = \frac{1!}{0! \cdot 1!} a + \frac{1!}{1! \cdot 0!} b = a + b;$$

2) (1.1) formula biror  $n$  da to'g'ri bo'ladi deb faraz qilamiz va

(1.1) da shu kabi formula o'rinli bo'lishini ko'rsatamiz, ya'ni

$$(a+b)^{n+1} = C_{n+1}^0 a^{n+1} + C_{n+1}^1 a^n b + \dots + C_{n+1}^{k+1} a^{n-k} b^{k+1} + \dots + C_{n+1}^n a b^n + C_{n+1}^{n+1} b^{n+1} \quad (1.2)$$

formulani isbotlaymiz.

Haqiqatan ham,

$$\begin{aligned} (a+b)^{n+1} &= (C_n^0 a^n + C_n^1 a^{n-1} b + \dots + C_n^k a^{n-k} b^k + \dots + C_n^n b^n)(a+b) = \\ &= C_n^0 a^{n+1} + C_n^1 a^n b + \dots + C_n^{k+1} a^{n-k} b^{k+1} + \dots + C_n^n a b^n + C_n^0 a^n b + \dots + \\ &+ C_n^k a^{n-k} b^{k+1} + \dots + C_n^{n-1} a b^n + C_n^n b^{n+1} = C_n^0 a^{n+1} + (C_n^0 + C_n^1) a^n b + \dots + \\ &+ (C_n^k + C_n^{k+1}) a^{n-k} b^{k+1} + \dots + (C_n^{n-1} + C_n^n) a b^n + C_n^n b^{n+1}. \end{aligned}$$

Binominal koeffitsiyentlar uchun

$$C_n^0 = 1 = C_{n+1}^0, \quad C_n^0 + C_n^1 = C_{n+1}^1, \quad C_n^k + C_n^{k+1} = C_{n+1}^{k+1},$$

$$C_n^{n-1} + C_n^n = C_{n+1}^n, \quad C_n^n = 1 = C_{n+1}^{n+1}$$

bo'lishi inobatga olinsa, oxirgi tenglikdan (1.2) tenglik kelib chiqadi.

Demak, matematik induksiya metodining 1) va 2) bandlari bajarilgani uchun

Newton binomi formulasi istalgan  $n$  natural soni uchun to'g'ri bo'ladi.

(1.1) formula qisqacha

$$(a+b)^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

ko'rinishda yoziladi.

Xususan, (1.1) formuladan  $n=2$  va  $n=3$  da tanish qisqa ko'paytirish formulalari kelib chiqadi:

$$(a+b)^2 = C_2^0 a^2 + C_2^1 a b + C_2^2 b^2 = a^2 + 2ab + b^2;$$

$$(a+b)^3 = C_3^0 a^3 + C_3^1 a^2 b + C_3^2 a b^2 + C_3^3 b^3 = a^3 + 3a^2 b + 3ab^2 + b^3.$$

#### 4.1.4. Mashqlar

1.  $A$  va  $B$  to'plamlar berilgan.  $A \cap B$ ,  $A \cup B$ ,  $A \setminus B$ ,  $B \setminus A$  to'plamlarni toping.

1)  $A = \{1, 2, 3, 4\}$ ,  $B = \{4, 5, 6\}$ ;

2)  $A = \{1, 3, 4, 8\}$ ,  $B = \{1, 2, 4, 5, 7, 8, 9\}$ ;

3)  $A = \{x \in \mathbb{R} : x^2 + x - 20 = 0\}$ ,  $B = \{x \in \mathbb{R} : x^2 - x + 12 = 0\}$ .

2.  $A$  - musbat juft sonlar to'plami va  $B$  - musbat toq sonlar to'plami bo'lsa,  $A \cap B$  va  $A \cup B$  to'plamlarni toping.

3.  $A$  – barcha 2 ga bo‘linadigan sonlar to‘plami va  $B$  – barcha 5 ga bo‘linadigan sonlar to‘plami bo‘lsa,  $A \cap B$  to‘plamni toping.

4.  $\lg 5$  irratsional son ekanini ko‘rsating.

5.  $\sqrt{2} - \sqrt{5} < \sqrt{3} - 2$  ekanini ko‘rsating.

6. Berilgan to‘plam elementlarini toping.

1)  $A = \{x \in \mathbb{N} : x^2 - 3x - 4 \leq 0\}$ ;

2)  $A = \left\{ x \in \mathbb{N} : \log_{\frac{1}{2}} \frac{1}{x} < 2 \right\}$ .

7. Berilgan tenglamalarni yeching.

1)  $|3x - 4| = \frac{1}{2}$ ;

2)  $| -x^2 + 2x - 3 | = 1$ ;

3)  $\sqrt{x^2} + x^3 = 0$ ;

4)  $\sqrt{(x-2)^2} = -x + 2$ .

8. Berilgan tengsizliklarni yeching.

1)  $|x - 2| \geq 1$ ;

2)  $|x^2 - 7x + 12| > x^2 - 7x + 12$ ;

3)  $x^2 + 2\sqrt{(x+3)^2} - 10 \leq 0$ ;

4)  $\sqrt{(x+1)^2} \leq -x - 1$ .

9. Berilgan  $X$  to‘plam uchun  $\sup X$  va  $\inf X$  larni toping.

1)  $X = \{x \in \mathbb{Z} : -5 \leq x < 0\}$ ;

2)  $X = \{x \in \mathbb{R} : x < 0\}$ .

10. Tengliklarni matematik induksiya metodi bilan isbotlang:

1)  $1 + 3 + 6 + \dots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}, \forall n \in \mathbb{N}$ ;

2)  $1 + 3 + 5 + \dots + (2n-1) = n^2$ .

## 4.2. SONLI KETMA-KETLIKLAR

### 4.2.1. Sonli ketma-ketliklar

**1-ta’rif.**  $1, 2, 3, \dots, n, \dots$  natural sonlar qatorining har bir  $n$  natural soniga mos qo‘yilgan  $x_1, x_2, x_3, \dots, x_n, \dots$  haqiqiy sonlar to‘plamiga *sonli ketma-ketlik* (yoki *ketma-ketlik*) deyiladi va  $\{x_n\}$  bilan belgilanadi.

Bunda  $x_1, x_2, x_3, \dots, x_n, \dots$  sonlar  $\{x_n\}$  ketma-ketlikning hadlari,  $x_n$  bu ketma-ketlikning umumiy hadi,  $n$  uning nomeri deb ataladi.

Agar ketma-ketlikning har bir hadini topish mumkin bo‘lsa, ketma-ketlik berilgan deyiladi. Ketma-ketlik analitik yoki rekurrent usullarda berilishi mumkin.

*Analitik usulda* ketma-ketlikning umumiy hadi formula ko‘rinishida beriladi. Bunda  $n$  ga 1,2,3,4,... qiymatlar beriladi va ketma-ketlikning o‘sha hadlari topiladi.

Masalan,  $x_n = (-1)^n \cdot n$  formula  $\{x_n\} = \{-1, 2, -3, \dots, (-1)^n \cdot n, \dots\}$  ketma-ketlikni beradi.

*Rekurrent usulda* ketma-ketlikning birinchi (yoki bir nechta birinchi) hadi beriladi va keyingi hadni (yoki bir nechta keyingi hadlarni) birinchi (yoki bir nechta birinchi) had (hadlar) asosida topish formulasi beriladi. Masalan,  $x_1 = a_1, x_{n+1} = x_n + d$  rekurrent formula arifmetik progressiyani,  $x_1 = b_1, x_{n+1} = x_n \cdot q$  rekurrent formula geometrik progressiyani beradi.

Agar  $\forall n \in \mathbb{N}$  uchun  $x_n = c$  ( $c \in \mathbb{R}$ ) bo‘lsa,  $\{x_n\}$  ketma-ketlikka *o‘zgarmas ketma-ketlik* deyiladi.

**2-ta’rif.** Agar shunday  $M$  soni ( $m$  soni) topilsa va  $\forall n \in \mathbb{N}$  uchun  $x_n \leq M$  ( $x_n \geq m$ ) tengsizlik bajarilsa,  $\{x_n\}$  ketma-ketlikka *yuqoridan chegaralangan* (*quyidan chegaralangan*) deyiladi.

**3-ta’rif.** Agar  $\{x_n\}$  ketma-ketlik ham quyidan ham yuqoridan chegaralangan bo‘lsa, ya’ni shunday  $m, M$  sonlari topilsa va  $\forall n \in \mathbb{N}$  uchun  $m \leq x_n \leq M$  tengsizlik bajarilsa,  $\{x_n\}$  ketma-ketlikka *chegaralangan* deyiladi.

$A = \max\{m, |M|\}$  bo‘lsin. U holda ketma-ketlikning chegaralanganlik shartini  $|x_n| \leq A$  ko‘rinishda yozish mumkin.

**4-ta’rif.** Agar  $\forall A > 0$  son uchun  $\{x_n\}$  ketma-ketlikning  $|x_n| > A$  ( $x_n > A$  yoki  $x_n < -A$ ) tengsizlikni qanoatlantiruvchi hadi topilsa,  $\{x_n\}$  ketma-ketlikka *chegaralanmagan* deyiladi.

Ta’riflardan ko‘rinadiki, ketma-ketlikning barcha elementlari u: yuqoridan chegaralangan bo‘lsa,  $(-\infty; M]$  oraliqqa, quyidan chegaralangan bo‘lsa  $[m; +\infty)$  oraliqqa, ham quyidan ham yuqoridan chegaralangan bo‘lsa  $[m; M]$  oraliqqa tegishli bo‘ladi. Chegaralanmagan ketma-ketlik yuqoridan yoki quyidan chegaralangan bo‘lishi mumkin.

Chegaralangan va chegaralanmagan ketma-ketliklarga misollar keltiramiz.

1.  $\{x_n\} = \{n^2 + 1\} = \{2, 5, 10, \dots, n^2 + 1, \dots\}$  – quyidan chegaralangan ( $m = 2$ ), ammo yuqoridan chegaralanmagan.

2.  $\{y_n\} = \{-n^2\} = \{-1, -4, -9, \dots, -n^2, \dots\}$  – yuqoridan chegaralangan ( $M = -1$ ), ammo quyidan chegaralanmagan.



$$3. \{z_n\} = \left\{ \frac{n-1}{n} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-1}{n}, \dots \right\} - \text{chegaralangan } (m=0, M=1).$$

$$4. \{u_n\} = \{(-1)^n n\} = \{-1, 2, -3, 4, \dots, (-1)^n n, \dots\} - \text{chegaralanmagan}.$$

**5-ta'rif.** Agar  $\forall n \in N$  uchun:  $x_n < x_{n+1}$  bo'lsa,  $\{x_n\}$  ketma-ketlikka *qat'iy o'suvchi* deyiladi;  $x_n > x_{n+1}$  bo'lsa,  $\{x_n\}$  ketma-ketlikka *qat'iy kamayuvchi* deyiladi;  $x_n \leq x_{n+1}$  bo'lsa,  $\{x_n\}$  ketma-ketlikka *kamaymaydigan* deyiladi;  $x_n \geq x_{n+1}$  bo'lsa,  $\{x_n\}$  ketma-ketlikka *o'smaydigan* deyiladi.

Barcha bunday ketma-ketliklar umumiy bitta *monoton ketma-ketlik* nomi bilan birlashtiriladi. Bunda o'suvchi va kamayuvchi ketma-ketliklarga *qat'iy monoton* ketma-ketliklar deyiladi.

Monoton ketma-ketliklarga misollar keltiramiz.

$$1. \{x_n\} = \left\{ \frac{n}{n+1} \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\} - \text{o'suvchi va chegaralangan ketma-ketlik}.$$

$$2. \{y_n\} = \{n, n\} = \{1, 1, 2, 2, \dots, n, n, \dots\} - \text{kamaymaydigan va chegaralanmagan ketma-ketlik}.$$

$$3. \{z_n\} = \left\{ \frac{1}{n^2} \right\} = \left\{ 1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots \right\} - \text{kamayuvchi va chegaralangan ketma-ketlik}.$$

$$4. \{u_n\} = \left\{ \frac{1}{n}, \frac{1}{n} \right\} = \left\{ 1, 1, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{n}, \frac{1}{n}, \dots \right\} - \text{o'smaydigan va chegaralangan ketma-ketlik}.$$

Ikki  $\{x_n\}$  va  $\{y_n\}$  ketma-ketlikning *yig'indisi*, *ayirmasi*, *ko'paytmasi*, *bo'linmasi* (bunda  $y_n \neq 0$ ) deb, har bir hadi bu ketma-ketliklar mos hadlarining yig'indisidan, ayirmasidan, ko'paytmasidan va bo'linmasidan iborat bo'lgan ketma-ketlikka aytiladi.

Ko'rsatilgan amallar simvolik tarzda quyidagicha yoziladi:

$$\{x_n\} + \{y_n\} = \{x_n + y_n\}, \quad \{x_n\} - \{y_n\} = \{x_n - y_n\},$$

$$\{x_n\} \cdot \{y_n\} = \{x_n \cdot y_n\}, \quad \frac{\{x_n\}}{\{y_n\}} = \left\{ \frac{x_n}{y_n} \right\}, y_n \neq 0.$$

Xususan,  $\{x_n\}$  ketma-ketlikning  $m$  songa ko'paytmasi  $m \cdot \{x_n\}$  deb, har bir hadi  $\{x_n\}$  ketma-ketlik mos hadining shu songa ko'paytmasidan iborat bo'lgan  $\{m \cdot x_n\}$  ketma-ketlikka aytiladi.

## 4.2.2. Cheksiz katta va cheksiz kichik ketma-ketliklar

**6-ta'rif.** Agar  $\forall A > 0$  son uchun shunday  $N$  nomer topilsa va  $\forall n > N$  uchun  $|x_n| > A$  tengsizlik bajarilsa,  $\{x_n\}$  ketma-ketlikka *cheksiz katta* ketma-ketlik deyiladi.

Har qanday cheksiz katta ketma-ketlik chegaralanmagan bo'ladi. Ammo chegaralanmagan ketma-ketlik cheksiz katta bo'lmasligi mumkin. Masalan,  $\left\{n \sin \frac{n\pi}{2}\right\}$  shunday ketma-ketliklardan biridir.

**7-ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $N = N(\varepsilon)$  nomer topilsa va  $\forall n > N$  uchun  $|x_n| < \varepsilon$  tengsizlik bajarilsa,  $\{x_n\}$  ketma-ketlikka *cheksiz kichik* ketma-ketlik deyiladi.

*1-misol.*  $\{\alpha_n\} = \left\{\frac{1}{n}\right\}$  ketma-ketlik cheksiz kichik ekanini ko'rsating.

*Yechish.*  $\forall \varepsilon > 0$  son olamiz.  $|\alpha_n| = \left|\frac{1}{n}\right| < \varepsilon$  tengsizlikdan  $n > \frac{1}{\varepsilon}$

tengsizlik kelib chiqadi.  $N$  ni  $\frac{1}{\varepsilon}$  ning butun qismi, ya'ni  $N = \left[\frac{1}{\varepsilon}\right]$

desak, u holda  $\forall n > N$  uchun  $|\alpha_n| < \varepsilon$  bo'ladi. Demak, ta'rifiga ko'ra,

$\left\{\frac{1}{n}\right\}$  ketma-ketlik cheksiz kichik bo'ladi.

Keltirilgan misolda  $N = \left[\frac{1}{\varepsilon}\right]$  nomer  $\varepsilon$  ga bog'liq bo'ladi, ya'ni

$\varepsilon$  ning turli qiymatida har xil qiymatlarni qabul qiladi. Masalan,  $\varepsilon = 0,1$  da  $N = 10$ ,  $\varepsilon = 0,01$  da  $N = 100$ . Shu sababli cheksiz kichik ketma-ketlikning ta'rifida  $N = N(\varepsilon)$  deb yozilgan.

**1-teorema.** Agar  $\{x_n\}$  ketma-ketlik cheksiz katta va  $x_n \neq 0, n \in N$  bo'lsa, u holda  $\left\{\frac{1}{x_n}\right\}$  ketma-ketlik cheksiz kichik bo'ladi, va aksincha

$\{\alpha_n\}$  – cheksiz kichik va  $\alpha_n \neq 0, n \in N$  bo'lsa,  $\left\{\frac{1}{\alpha_n}\right\}$  – cheksiz katta

bo'ladi.

*Isboti.*  $\{x_n\}$  cheksiz katta ketma-ketlik,  $x_n \neq 0$  bo'lsin.  $\forall \varepsilon > 0$  son olamiz va  $A = \frac{1}{\varepsilon}$  deymiz. 6-ta'rifga ko'ra bu  $A$  soni uchun shunday  $N$  nomer topiladi va  $\forall n > N$  uchun  $|x_n| > A$  tengsizlik bajariladi. Bundan  $\forall n > N$  uchun  $\left| \frac{1}{x_n} \right| = \frac{1}{|x_n|} < \frac{1}{A} = \varepsilon$  bo'ladi. Bu esa  $\left\{ \frac{1}{x_n} \right\}$  ketma-ketlikning cheksiz kichik bo'lishini bildiradi. Teoremaning ikkinchi qismi ham shu kabi isbotlanadi.

**Cheksiz kichik ketma-ketliklar quyidagi xossalarga ega.**

1°. Ikkita (chekli sondagi) cheksiz kichik ketma-ketlikning algebraik yig'indisi cheksiz kichik ketma-ketlik bo'ladi.

2°. Ikkita (chekli sondagi) cheksiz kichik ketma-ketlikning ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

3°. Cheksiz kichik ketma-ketlikning chegaralangan ketma-ketlikka ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

4°. Cheksiz kichik ketma-ketlikning chekli songa ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

Xossalardan birining, masalan, 3-xossaning isbotini keltirish bilan chegaralanamiz.

$\{x_n\}$  chegaralangan ketma-ketlik,  $\{\alpha_n\}$  cheksiz kichik ketma-ketlik bo'lsin.  $\{x_n \cdot \alpha_n\}$  ketma-ketlik cheksiz kichik bo'lishini isbotlash kerak.

$\{x_n\}$  ketma-ketlik chegaralangan. Shu sababli biror  $A > 0$  son va  $\forall m$  uchun  $|x_n| \leq A$  bo'ladi.

$\forall \varepsilon > 0$  son olamiz.  $\{\alpha_n\}$  cheksiz kichik bo'lgani sababli  $\frac{\varepsilon}{A} > 0$  son uchun shunday  $N$  nomer topiladi va  $\forall n > N$  uchun  $|\alpha_n| < \frac{\varepsilon}{A}$  bo'ladi.

U holda  $\forall n > N$  da  $|x_n \cdot \alpha_n| = |x_n| \cdot |\alpha_n| < A \cdot \frac{\varepsilon}{A} = \varepsilon$  bo'ladi.

Demak,  $\{x_n \cdot \alpha_n\}$  – cheksiz kichik ketma-ketlik.

### 4.2.3. Ketma-ketlikning limiti

**8-ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $N = N(\varepsilon)$  nomer topilsa va  $\forall n > N$  uchun  $|x_n - a| < \varepsilon$  tengsizlik bajarilsa,  $a$  soniga  $\{x_n\}$  ketma-ketlikning limiti deyiladi va bu  $\lim_{n \rightarrow \infty} x_n = a$  kabi yoziladi.

2-misol. Limit ta'rifidan foydalanib,  $\lim_{n \rightarrow \infty} \frac{3n-2}{3n+1} = 1$  bo'lishini ko'rsating.

*Yechish.*  $\forall \varepsilon > 0$  olamiz. Misolning shartidan topamiz:

$$|x_n - 1| = \left| \frac{3n-2}{3n+1} - 1 \right| = \left| \frac{-3}{3n+1} \right| = \frac{3}{3n+1}$$

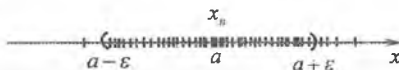
$|x_n - a| < \varepsilon$  tengsizlikni qanoatlantiruvchi  $n$  ning qiymatlarini topish uchun  $\frac{3}{3n+1} < \varepsilon$  tengsizlikni yechamiz:  $n > \frac{3-\varepsilon}{3\varepsilon}$ .

$N$  nomer sifatida  $\frac{3-\varepsilon}{3\varepsilon}$  sonining butun qismini, ya'ni  $N(\varepsilon) = \left[ \frac{3-\varepsilon}{3\varepsilon} \right]$  sonini olish mumkin. Bunda  $\forall \varepsilon > 0$  son olinganda ham  $\forall n > N$  uchun  $|x_n - 1| < \varepsilon$  bo'ladi.

Shu sababli ketma-ketlik limitining ta'rifiga asosan,

$$\lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} = 1.$$

Ma'lumki,  $|x_n - a| < \varepsilon$  tengsizlik  $x_n$  had  $a$  nuqtaning  $\varepsilon$  atrofiga tushushini bildiradi. Shu sababli ketma-ketlikning limiti ta'rifini quyidagicha talqin qilish mumkin: agar  $\lim_{n \rightarrow \infty} x_n = a$  bo'lsa, u holda  $a$  nuqtaning istalgan  $\varepsilon$  atrofi uchun shunday  $N$  nomer topiladi va  $n > N$  nomerli barcha  $x_n$  nuqtalar  $a$  nuqtaning  $\varepsilon$  atrofiga, ya'ni  $(a-\varepsilon; a+\varepsilon)$  intervalda yotadi va bu intervaldan tashqarida berilgan ketma-ketlikning chekli sonidagi nuqtalari joylashadi (3-shakl). Bu jumla ketma-ketlik limitining geometrik ma'nosini anglatadi.



3-shakl.

### 4.2.4. Yaqinlashuvchi ketma-ketliklar

**8-ta'rif.** Chekli limitga ega bo'lgan ketma-ketlikka *yaqinlashuvchi* ketma-ketlik deyiladi.

Yaqinlashuvchi bo'lmagan ketma-ketlikka *uzoqlashuvchi* ketma-ketlik deyiladi.

$\{x_n\}$  ketma-ketlik yaqinlashuvchi va  $a$  limitga ega bo'lsin. U holda  $\{x_n - a\} = \{\alpha_n\}$  cheksiz kichik ketma-ketlik bo'ladi, chunki  $\forall \varepsilon > 0$  son uchun shunday  $N(\varepsilon)$  nomer topiladi va  $\forall n > N$  uchun  $|\alpha_n| = |x_n - a| < \varepsilon$  bo'ladi. Shu sababli yaqinlashuvchi va  $a$  limitga ega bo'lgan ixtiyoriy  $\{x_n\}$  ketma-ketlikning umumiy hadini  $x_n = a + \alpha_n$  ko'rinishda yozish mumkin bo'ladi va aksincha,  $\{x_n\}$  ketma-ketlikning istalgan hadini  $x_n = a + \alpha_n$  ko'rinishida ifodalash mumkin bo'lsa, u holda  $\lim_{n \rightarrow \infty} x_n = a$  bo'ladi, bu yerda  $\{\alpha_n\}$  – cheksiz kichik ketma-ketlik.

Shunday qilib, cheksiz kichik ketma-ketlikning limiti nolga teng bo'ladi. Cheksiz katta ketma-ketlik limitga ega bo'lmaydi. Uning limiti  $\infty$  deb belgilanadi.

Yaqinlashuvchi ketma-ketliklar quyidagi xossalarga ega.

1°. Yaqinlashuvchi ketma-ketlik yagona limitga ega bo'ladi.

2°. Yaqinlashuvchi ketma-ketlik chegaralangan bo'ladi.

3°. Agar  $\{x_n\}$  va  $\{y_n\}$  ketma-ketliklar yaqinlashuvchi bo'lsa, u holda  $\{x_n \pm y_n\}$  ketma-ketlik yaqinlashuvchi va  $\lim_{n \rightarrow \infty} (x_n \pm y_n) = \lim_{n \rightarrow \infty} x_n \pm \lim_{n \rightarrow \infty} y_n$  bo'ladi.

4°. Agar  $\{x_n\}$  va  $\{y_n\}$  ketma-ketliklar yaqinlashuvchi bo'lsa, u holda  $\{x_n \cdot y_n\}$  ketma-ketlik yaqinlashuvchi va  $\lim_{n \rightarrow \infty} x_n \cdot y_n = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$  bo'ladi.

5°. Agar  $\{x_n\}$  va  $\{y_n\}$  ketma-ketliklar yaqinlashuvchi va  $\lim_{n \rightarrow \infty} y_n \neq 0$  bo'lsa, u holda  $\left\{ \frac{x_n}{y_n} \right\}$  ketma-ketlik yaqinlashuvchi va  $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$  bo'ladi.

6°. Agar  $\{x_n\}$  ketma-ketlik yaqinlashuvchi bo'lsa, u holda  $\lim_{n \rightarrow \infty} c \cdot x_n = c \cdot \lim_{n \rightarrow \infty} x_n$  ( $c \in R$ ) bo'ladi.

7°. Agar  $\{x_n\}$  va  $\{y_n\}$  ketma-ketliklar yaqinlashuvchi va  $\forall n \in N$  uchun  $x_n \leq y_n$  bo'lsa, u holda  $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$  bo'ladi.

8°. Agar  $\{x_n\}$  va  $\{z_n\}$  ketma-ketliklar yaqinlashuvchi,  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = a$  va  $\forall n \in N$  uchun  $x_n \leq y_n \leq z_n$  bo'lsa, u holda  $\lim_{n \rightarrow \infty} y_n = a$  bo'ladi.

Xossalardan birining, masalan, 3-xossaning isbotini keltirish bilan chegaralanamiz.

$\lim_{n \rightarrow \infty} x_n = a$ ,  $\lim_{n \rightarrow \infty} y_n = b$  bo'lsin. U holda  $x_n = a + \alpha_n$  va  $y_n = b + \beta_n$  bo'ladi, bu yerda  $\{\alpha_n\}$ ,  $\{\beta_n\}$  cheksiz kichik ketma-ketliklar.

Bundan

$$x_n \pm y_n = (a \pm b) + (\alpha_n \pm \beta_n)$$

kelib chiqadi.

Cheksiz kichik ketma-ketlikning 1-xossasiga asosan  $\{\alpha_n \pm \beta_n\}$  cheksiz kichik ketma-ketlik. Shu sababli  $\{x_n \pm y_n\}$  ketma-ketlik  $a \pm b$  limitga ega bo'ladi, ya'ni

$$\lim_{n \rightarrow \infty} (x_n \pm y_n) = \lim_{n \rightarrow \infty} x_n \pm \lim_{n \rightarrow \infty} y_n.$$

3-misol.  $\{x_n\} = \left\{ \left( \frac{n+2}{n^2} \right)^n \right\}$  ketma-ketlikning yaqinlashuvchi ekanini

ko'rsating.

*Yechish.* Birinchidan,  $x_n = \frac{n+2}{n^2} \leq \frac{n+2n}{n^2} = \frac{3n}{n^2} = \frac{3}{n}$ .

Bunda  $\forall n \geq 6$  uchun  $x_n \leq \frac{1}{2}$  bo'ladi. Ikkinchidan,  $\forall n \in \mathbb{N}$  uchun

$x_n = \frac{n+2}{n^2} \geq \frac{1+2}{n^2} = \frac{3}{n^2} > 0$  bo'ladi. Agar  $y_n = 0$ ,  $z_n = \frac{1}{2^n}$  belgilash kiritsak,

$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0$  va  $\forall n \geq 6$  uchun  $y_n \leq x_n \leq z_n$  bo'ladi. U holda 8-xossaga ko'ra,

$\lim_{n \rightarrow \infty} x_n = 0$ , ya'ni berilgan ketma-ketlik yaqinlashuvchi bo'ladi.

Har qanday ketma-ketlik ham limitga ega bo'lmaydi. Ketma-ketlik limiti mavjud bo'lishi haqidagi teorema bilan tanishamiz.

**2-teorema (Veershtross teoremasi).** Har qanday chegaralangan monoton ketma-ketlik limitiga ega bo'ladi.

*Isboti.* Monoton kamaymaydigan ketma-ketlikni qaraymiz.

$x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$  va shunday  $M$  soni topilsin va  $x_n \leq M$  bo'lsin. Elementlari bu ketma-ketlikning hadlaridan tashkil topgan  $X$  to'plamni qaraymiz.

Shartga ko'ra, bu to'plam bo'shmas va yuqoridan chegaralangan. U holda

4.2 banddagi 1-teoremaga asosan,  $X$  to'plam aniq yuqori chegaraga ega bo'ladi. Bu chegarani  $a$  bilan belgilaymiz va  $a$  soni berilgan ketma-ketlikning limiti bo'lishini ko'rsatamiz.

$a$  soni  $x_n$  ketma-ketlik elementlarining aniq yuqori chegarasi bo'lgani uchun uning xossasiga ko'ra,  $\forall \varepsilon > 0$  son uchun shunday  $N$  nomer topiladi va  $n > N$  da  $x_n > a - \varepsilon$  bo'ladi. Ikkinchidan, aniq yuqori chegaraning ta'rifiga asosan, barcha  $n$  lar uchun  $x_n \leq a + \varepsilon$  bo'ladi. Shunday qilib,  $n > N$  uchun  $a - \varepsilon < x_n < a + \varepsilon$ , ya'ni  $|x_n - a| < \varepsilon$  tengsizlik kelib chiqdi. Bu tengsizlik  $a$  soni  $\{x_n\}$  ketma-ketlikning limiti bo'lishini bildiradi.

Monoton o'smaydigan ketma-ketlik uchun teorema shu kabi isbotlanadi.

*Izoh.* Monoton ketma-ketlikning chegaralanganligi uning yaqinlashuvchi bo'lishining zarur va yetarli shartini beradi.

Haqiqatan ham, agar monoton ketma-ketlik chegaralangan bo'lsa, u holda 2-teoreмага ko'ra, u yaqinlashadi; agar monoton ketma-ketlik yaqinlashuvchi bo'lsa, yaqinlashuvchi ketma-ketlikning 2-xossasiga asosan, u chegaralangan bo'ladi.

2- teoremadan ichma-ich qo'yilgan kesmalar haqidagi lemma kelib chiqadi.

$[a_1; b_1], [a_2; b_2], \dots [a_n; b_n], \dots$  kesmalar ketma-ketligi berilgan bo'lib, bunda  $[a_1; b_1] \supset [a_2; b_2] \supset \dots \supset [a_n; b_n] \supset \dots$ , ya'ni har bir kesma o'zidan oldingi kesmaning ichiga joylashgan va  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$  bo'lsin. Bu kesmalarga ichma-ich qo'yilgan kesmalar deyiladi.

**Lemma (Kantor lemmasi).** Ixtiyoriy ichma-ich qo'yilgan kesmalar ketma-ketligi uchun bu ketma-ketlikning har biriga tegishli bo'lgan  $c$  nuqta mavjud va yagona bo'ladi.

Bu lemma haqiqiy sonlar to'plamining uzluksizligi yoki son o'qining to'laligi haqidagi muhim xossani ifodalaydi.

Yaqinlashuvchi ketma-ketlikning xossalari va limitlar haqidagi teoremlar nafaqat nazariy, balki amaliy jihatdan ham muhim ahamiyatga ega. Ulardan, masalan, ketma-ketliklarning limitini hisoblashda keng foydalaniladi.

4-misol.  $\lim_{n \rightarrow \infty} \frac{3n+1}{8n-1}$  limitni toping.

*Yechish.* Kasrning surati va maxrajidagi ketma-ketliklar limitga ega emas, chunki ular chegaralanmagan. Shu sababli bo'linmaning limiti haqidagi 5-xossani to'g'ridan-to'g'ri qo'llab bo'lmaydi. Bu xossani qo'llash uchun avval ketma-ketlikning surat va maxrajini  $n$  ga bo'lamiz va keyin yaqinlashuvchi ketma-ketlikning xossalarini

qo'llaymiz:

$$\lim_{n \rightarrow \infty} \frac{3n+1}{8n-1} = \lim_{n \rightarrow \infty} \frac{3 + \frac{1}{n}}{8 - \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} \left(3 + \frac{1}{n}\right)}{\lim_{n \rightarrow \infty} \left(8 - \frac{1}{n}\right)} = \frac{\lim_{n \rightarrow \infty} 3 + \lim_{n \rightarrow \infty} \frac{1}{n}}{\lim_{n \rightarrow \infty} 8 - \lim_{n \rightarrow \infty} \frac{1}{n}} = \frac{3+0}{8-0} = \frac{3}{8}$$

#### 4.2.5. $e$ soni

$\{x_n\} = \left\{ \left(1 + \frac{1}{n}\right)^n \right\}$  ketma-ketlik berilgan bo'lsin. Bu ketma-ketlik

uchun Nyuton binomi formulasini  $a=1$ ,  $b=\frac{1}{n}$  da qo'llaymiz:

$$x_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \dots + \frac{n(n-1)(n-2)\dots(n-(n-1))}{n!} \cdot \frac{1}{n^n}$$

yoki

$$\left(1 + \frac{1}{n}\right)^n = 2 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \dots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right) \quad (2.1)$$

(2.1) tenglikdan ko'rinadiki,  $n$  ning o'sishi bilan uning o'ng tomonida musbat qo'shiluvchilar soni ortib boradi. Bundan tashqari,

$n$  ning o'sishi bilan  $\frac{1}{n}$  kamayadi va  $\left(1 - \frac{1}{n}\right), \left(1 - \frac{2}{n}\right), \dots$  kattaliklar ortadi.

Shu sababli  $\{x_n\} = \left\{ \left(1 + \frac{1}{n}\right)^n \right\}$  ketma-ketlik monoton o'suvchi bo'ladi.

(2.1) tenglikka ko'ra,

$$\left(1 + \frac{1}{n}\right)^n > 2. \quad (2.2)$$

(2.1) tenglikda qavs ichidagi har bir ifoda birdan kichik va shu bilan birga,  $n > 2$  da  $\frac{1}{n!} < \frac{1}{2^{n-1}}$  bo'ladi.

Bundan

$$x_n < 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} = 1 + \frac{1}{1 - \frac{1}{2}} = 3 \quad (2.3)$$



kelib chiqadi. Demak, (2.2) va (2.3) tengsizliklarga ko'ra,  $2 < x_n < 3$ , ya'ni  $\{x_n\}$  - chegaralangan.

Shunday qilib,  $\{x_n\}$  ketma-ketlik monoton o'suvchi va chegaralangan. U holda Veershtrass teoremagasiga ko'ra u yaqinlashadi, ya'ni chekli limitga ega bo'ladi. Bu limitni  $e$  harfi bilan belgilaymiz.

Demak,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e. \quad (2.4)$$

$e$  soniga *Neper soni* deyiladi.  $e$  soni irratsional son.  $e$  soni matematikaning bir qancha masalalarida muhim rol o'ynaydi.  $e$  soni, masalan, natural logarifmning asosi bo'ladi:  $x > 0$  sonining natural logarifmi  $\ln x$  bilan belgilanadi. Bu paragrafda  $e$  soniga faqat ta'rif berildi va u  $2 < e < 3$  tengsizlikni qanoatlantirishi ko'rsatildi. Keyinchlik  $e$  sonining qiymatini istalgan aniqlikda topish usullari ko'rsatiladi.

#### 4.2.6. Mashqlar

1. Ketma-ketlikning dastlabki to'rtta hadi berilgan. Uning umumiy hadini toping:

1)  $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \dots$

2)  $5, \frac{25}{2}, \frac{125}{6}, \frac{625}{24}, \dots$

3)  $-1, 1, -1, 1, \dots$

4)  $1, 5, 1, 5, \dots$

2. Chegaralangan ketma-ketliklarni ko'rsating:

1)  $x_n = \frac{n}{2+n^2}$ ;

2)  $x_n = \cos n\pi + 2\lg n\pi$ ;

3)  $x_n = \frac{1-n}{\sqrt{n}}$ ;

4)  $x_n = \sqrt{n^2 + 1} - n$ ;

5)  $x_n = (-1)^n \cdot n$ ;

6)  $x_n = \ln(n+1) - \ln n$ .

3. Ketma-ketliklardan qaysilari monoton va qaysilari qat'iy monoton?

1)  $x_n = \frac{n}{3n-2}$ ;

2)  $x_1 = 1, x_n = \frac{2}{x_{n-1} + 1}$ ;

3)  $x_n = \frac{3^n}{n}$ ;

4)  $x_n = \frac{n}{5^n}$ .

5)  $x_n = \lfloor \sqrt{n} \rfloor$

6)  $x_n = \frac{3^n}{n!}$ .

4.  $1, \frac{1}{7}, \frac{1}{17}, \dots, \frac{1}{2n^2-1}$  ketma-ketlik cheksiz kichik ekanligini isbotlang.

5.  $\frac{17}{14}, \frac{37}{29}, \frac{65}{50}, \dots, \frac{4n^2+1}{3n^2+2}$  ketma-ketlik  $\frac{4}{3}$  ga teng limitga ega ekanligini ketma-

ketlikning limiti ta'rifidan foydalanib isbotlang.

6. Ketma-ketliklarning limitini toping:

$$1) x_n = \frac{5-n^2}{3+2n^7};$$

$$2) x_n = \frac{3n^2+2}{4-n^3};$$

$$3) x_n = \frac{3n+n^3}{2n^2+3n+7};$$

$$4) x_n = \left( \frac{2n^2+3n-1}{n^2-2n+1} \right)^3;$$

$$5) x_n = \frac{(n+2)^2 - (2-n)^2}{2n+7};$$

$$6) x_n = \frac{(n+1)^3 - (n-1)^3}{3n^2+2};$$

$$7) x_n = \frac{3n^3}{1+3n^2} + \frac{1-5n^2}{5n+1};$$

$$8) x_n = \frac{3}{n+2} - \frac{5n}{2n+1};$$

$$9) x_n = \sqrt{n+2} - \sqrt{n-2};$$

$$10) x_n = \sqrt{n^2+n} - \sqrt{n^2-n};$$

$$11) x_n = \sqrt{n(n-5)} - n;$$

$$12) x_n = \sqrt[3]{n^3-4n^2} - n;$$

$$13) x_n = \frac{2n+1}{\sqrt[3]{n^2+n+5}};$$

$$14) x_n = \frac{\sqrt[3]{n^4-1}}{\sqrt{n+1}};$$

$$15) x_n = \frac{n!+(n+1)!}{(n+1)!-2n!};$$

$$16) x_n = \frac{(2n+1)!+(2n+2)!}{(2n+3)!-(2n+2)!};$$

$$17) x_n = \frac{2-5+4-7+\dots+2n-(2n+3)}{n+5};$$

$$18) x_n = \frac{1+2+3+\dots+n}{n^2-2n+1};$$

$$19) x_n = \frac{1}{1 \cdot 7} + \frac{1}{7 \cdot 13} + \dots + \frac{1}{(6n-5)(6n+1)};$$

$$20) x_n = \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{2n(2n+2)};$$

$$21) x_n = \frac{\frac{1}{3^n}-1}{\frac{1}{3^n}+1};$$

$$22) x_n = \frac{6 \cdot 6^n + 5}{2 \cdot 3^n + 1} - 3^{n+1};$$

$$23) x_n = \frac{3}{4} + \frac{5}{16} + \frac{9}{64} + \dots + \frac{1+2^n}{4^n};$$

$$24) x_n = \frac{1+3+9+\dots+3^{n-1}}{2 \cdot 3^{n-2} + 5};$$

$$25) x_n = \frac{1}{n} \cos n^2 - \frac{3n}{6n+1};$$

$$26) x_n = \frac{1}{n} \sin n^3 + \frac{2n^2}{n^3-1};$$

$$27) x_n = \left( 1 - \frac{1}{n} \right)^n;$$

$$28) x_n = \left( \frac{n-1}{1+n} \right)^{2n-5};$$

$$29) x_n = \left( \frac{2n+1}{2n-1} \right)^{3n-4};$$

$$30) x_n = \left( \frac{n^2-1}{n^2+1} \right)^{3n-n^2};$$

### 4.3. BIR O'ZGARUVCHINING FUNKSIYASI

#### 4.3.1. Funksiya

Ikki to'plam elementlari orasidagi bog'lanishni o'rnatishga asoslangan funksiya tushunchasi matematik analiz kursida o'rganilsada, nafaqat bu kursning, balki butun matematikaning asosiy tushunchalaridan biri hisoblanadi.

#### *Funksiya tushunchasi*

**1-ta'rif.** Agar  $X$  to'plamning har bir  $x$  soniga biror  $f$  qoidaga ko'ra,  $Y$  to'plamning bitta  $y$  soni mos qo'yilgan bo'lsa,  $X$  to'plamda *funksiya* berilgan deyiladi va  $f: x \rightarrow y$  yoki  $y = f(x)$  kabi belgilanadi.

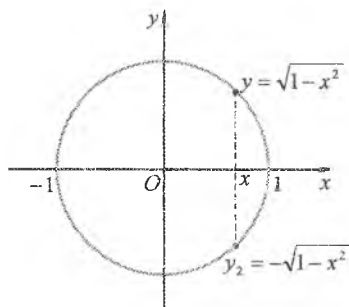
Bunda  $f$  funksiya  $X$  to'plamni  $Y$  to'plamga akslantiradi deb aytiladi.  $X$  to'plam  $f$  *funksiyaning aniqlanish sohasi* deb ataladi va  $D(f)$  bilan belgilanadi,  $y \in Y$  to'plam  $f$  *funksiyaning qiymatlar sohasi* deb ataladi va  $E(f)$  bilan belgilanadi.

Bunda  $x$  *funksiyaning argumenti* yoki *erkli o'zgaruvchi*,  $y$  *funksiya* yoki  $x$  ga *bog'liq o'zgaruvchi* deb ataladi.

$y = f(x)$  *funksiyaning*  $x = x_0 (x_0 \in X)$  *nuqtadagi xususiy qiymati*  $f(x_0) = y_0$  yoki  $y|_{x=x_0} = y_0$  kabi belgilanadi. Masalan,  $f(x) = 3x^2 - 2$  bo'lsa,  $f(0) = -2$ ,  $f(1) = 1$ .

$y = f(x)$  *funksiyaning grafigi* deb  $Oxy$  koordinatalar tekisligining absissasi  $x$  argumentning qiymatlaridan va ordinatasi  $y$  funksiyaning mos qiymatlaridan tashkil topgan barcha  $(x; f(x))$ ,  $x \in D(f)$  nuqtalari to'plamiga aytiladi. Funksiyaning grafigi tutash chiziqdan (egri chiziqdan yoki to'g'ri chiziqdan) iborat bo'lishi yoki ayrim nuqtalardan tashkil topishi mumkin, masalan,  $y = n!$ ,  $n \in \mathbb{N}$  funksiyaning grafigi 1,2,6,... nuqtalardan iborat bo'ladi

Har qanday chiziq ham biror funksiyaning grafigi bo'lavermaydi, masalan,  $x^2 + y^2 = 1$  aylana



4-shakl.

funksiyaning grafigi bo'lmaydi, chunki har bir  $x \in (-1;1)$  uchun  $y$  ning bitta emas balki ikkita qiymati mos keladi:  $y_1 = \sqrt{1-x^2}$  va  $y_2 = -\sqrt{1-x^2}$  (4-shakl). Bunda funksiya ta'rifining bir qiymatlilik sharti buziladi.

Ammo aylananing quyi yarim tekislikdagi bo'lagi  $y = -\sqrt{1-x^2}$  funk-siyaning, yuqori yarim tekislikdagi bo'lagi esa  $y = \sqrt{1-x^2}$  funksiyaning grafigi bo'ladi.

### *Funksiyaning berilish usullari*

Funksiyaning berilishi, ya'ni  $x$  ning har bir qiymatiga  $y$  ning yagona qiymatini topish turli usulda berilgan bo'lishi mumkin. Amalda funksiya berilishining analitik, jadval va grafik usullari ko'p qo'llaniladi.

*Analitik usulda*  $x$  va  $y$  o'zgaruvchilar orasidagi bog'lanish bir yoki bir nechta formula orqali beriladi. Masalan,

$$y = x^3, \quad y = \sin 2x, \quad y = \begin{cases} x-5, & \text{agar } x < 3, \\ x^2 + 2, & \text{agar } x \geq 3. \end{cases}$$

Funksiya  $y = f(x)$  ko'rinishda yozilganda, ayrim hollarda funksiyaning aniqlanish sohasi  $D(f)$  ko'rsatilmaydi. Bunda, funksiyaning aniqlanish sohasi  $x$  ning  $f(x)$  funksiya ma'noga ega bo'ladigan barcha qiymatlari to'plamidan iborat deb qaraladi.

*Jadval usulida*  $x$  va  $y$  o'zgaruvchilar orasidagi bog'lanish jadval orqali beriladi. Masalan, logorifmik fuksiyalarning, trigonometrik fuksiyalarning jadvallari.

Amalda jadval orqali funksiyaning kuzatish natijalari yoki uning tajribada olingan qiymatlari beriladi.

*Grafik usulda* fuksiya-ning grafigi beriladi. Bunda funksiyaning argumentning u yoki bu qiymatlariga mos qiymatlari bevosita shu grafikdan topiladi.

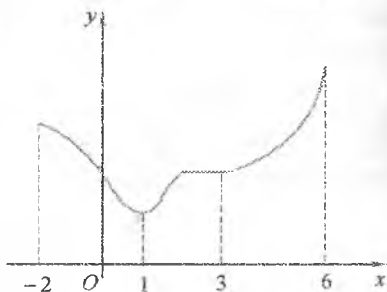
$x$  va  $y$  o'zgaruvchilar orasidagi bog'lanish yuqorida keltirilgan uch usul bilan chegaralanib qolmasdan, boshqa shakllarda berilishi ham mumkin. Masalan, EHMning hisoblash programmasi shaklida, tavsiflardangina iborat holda.

### *Funksiyaning monotonligi*

$y = f(x)$  funksiya  $X$  to'plamda aniqlangan va  $I = (a;b) \subset X$  bo'lsin.

**2-ta'rif.** Agar  $\forall x_1, x_2 \in I$  uchun  $x_1 < x_2$  bo'lganda  $f(x_1) < f(x_2)$  ( $f(x_1) > f(x_2)$ ) tengsizlik bajarilsa,  $y = f(x)$  funksiyaga  $I$  intervalda *o'suvchi* (*kamayuvchi*) deyiladi.

**3-ta'rif.** Agar  $\forall x_1, x_2 \in I$  uchun  $x_1 < x_2$  bo'lganda  $f(x_1) \leq f(x_2)$  ( $f(x_1) \geq f(x_2)$ ) tengsizlik bajarilsa,  $y = f(x)$  funksiyaga  $I$  intervalda *kamaymaydigan* (*o'smaydigan*) deyiladi.



5-shakl.

Masalan, grafigi 5-shaklda berilgan funksiya  $(-2;1)$  intervalda kamayuvchi,  $(1;6)$  intervalda kamaymaydigan,  $(3;6)$  intervalda o'suvchi.

Barcha bunday funksiyalar  $I$  intervalda *monoton funksiya* nomi bilan umumlashtiriladi. Bunda o'suvchi va kamayuvchi funksiyalarga  $I$  intervalda *qat'iy monoton* funksiyalar deyiladi. Funksiya monoton bo'lgan intervallar *monotonlik intervallari* deb ataladi.

### Funksiyaning juft va toqligi

$y = f(x)$  funksiya  $X$  to'plamda aniqlangan bo'lsin.

Agar  $\forall x \in X$  uchun  $-x \in X$  va  $f(-x) = f(x)$  bo'lsa,  $f(x)$  funksiyaga *juft funksiya* deyiladi. Masalan,  $y = x^2$ ,  $y = \cos x$ ,  $y = \sqrt{1+x^2}$  - juft funksiyalar.

Juft funksiyaning grafigi ordinata o'qiga nisbatan simmetrik bo'ladi.

Agar  $\forall x \in X$  uchun  $-x \in X$  va  $f(-x) = -f(x)$  bo'lsa  $f(x)$  funksiyaga *toq funksiya* deyiladi. Masalan,  $y = x^3$ ,  $y = \sin x$  - toq funksiyalar.

Toq funksiyaning grafigi koordinata boshiga nisbatan simmetrik bo'ladi.

Juft ham, toq ham bo'lmagan funksiya umumiy ko'rinishdagi funksiya deb ataladi. Masalan,  $y = x - 2$ ,  $y = \sqrt{x}$  - umumiy ko'rinishdagi funksiyalar.

$f(x)$  va  $g(x)$  funksiyalar juft funksiyalar bo'lsa,

$$f(x) + g(x), f(x) - g(x), f(x) \cdot g(x), \frac{f(x)}{g(x)}, (g(x) \neq 0)$$

funksiyalar ham juft bo'ladi.

$f(x)$  va  $g(x)$  funksiyalar toq funksiyalar bo'lsa,

$$f(x) + g(x), f(x) - g(x)$$

funksiyalar toq bo'ladi,

$$f(x) \cdot g(x), \frac{f(x)}{g(x)}, (g(x) \neq 0)$$

funksiyalar esa juft bo'ladi.

*1-misol.*  $f(x) = \ln(2x + \sqrt{1 + 4x^2})$  funksiyalarning toq ekanini ko'rsating.

*Yechish.* Toq funksiya uchun

$$f(-x) = -f(x) \text{ yoki } f(x) + f(-x) = 0$$

bo'ladi.

Tekshirib ko'ramiz:

$$\begin{aligned} f(x) + f(-x) &= \ln(2x + \sqrt{1 + 4x^2}) + \ln(-2x + \sqrt{1 + 4x^2}) = \\ &= \ln(1 + 4x^2 - 4x^2) = \ln 1 = 0. \end{aligned}$$

Bu munosabatdan  $x \in D(f)$  bo'lsa,  $-x \in D(f)$  bo'lishligi kelib chiqadi.

Demak, funksiya toq.

### *Funksiyaning chegaralanganligi*

$y = f(x)$  funksiya  $X$  to'plamda aniqlangan bo'lsin.

**4-ta'rif.** Agar shunday o'zgarmas  $M$  soni topilsa va  $\forall x \in X$  uchun  $f(x) \leq M$  tengsizlik bajarilsa,  $f(x)$  funksiya  $X$  to'plamda *yuqoridan chegaralangan* deyiladi.

**5-ta'rif.** Agar shunday o'zgarmas  $m$  soni topilsa va  $\forall x \in X$  uchun  $f(x) \geq m$  tengsizlik bajarilsa  $f(x)$  funksiya  $X$  to'plamda *quyidan chegaralangan* deyiladi.

**6-ta'rif.** Agar  $f(x)$  funksiya ham quyidan, ham yuqoridan chegaralangan bo'lsa, y'ani shunday o'zgarmas  $m$  va  $M$  sonlari topilsa va  $\forall x \in X$  uchun  $m \leq f(x) \leq M$  tengsizlik bajarilsa,  $f(x)$  funksiya  $X$  to'plamda *chegaralangan* deyiladi.

Masalan,  $y = 1 - x^4$  funksiya yuqoridan  $M = 1$  soni bilan chegaralangan,  $y = 2 + x^2$  funksiya quyidan  $m = 2$  soni bilan

chegaralangan,  $y = \sin x$  funksiya quyidan  $m = -1$  soni bilan va yuqoridan  $M = 1$  soni bilan chegaralangan.

### Funksiyaning davriyligi

$y = f(x)$  funksiya  $X$  to'plamda aniqlangan bo'lsin.

**7-ta'rif.** Agar shunday o'zgarmas  $T (T \neq 0)$  son topilsa va  $\forall x \in X$  uchun  $x + T \in X$ ,  $x - T \in X$ ,  $f(x \pm T) = f(x)$  bo'lsa,  $f(x)$  funksiyaga *davriy funksiya* deyiladi. Bunda  $T$  larning eng kichik musbat qiymati  $T_0$  ga  $f(x)$  *funksiyaning davri* deyiladi.

Masalan,  $y = \sin x$  funksiyaning davri  $2\pi$ ,  $\operatorname{tg} x$  funksiyaning davri  $\pi$ .

**2-misol.**  $f(x) = 4 \sin 3x + 3 \cos 3x$  funksiyaning eng katta qiymatini va davrini toping.

*Yechish.*  $a \cos x + b \sin x = \sqrt{a^2 + b^2} \cos(x - \varphi)$  ( $\varphi = \operatorname{arctg} \frac{b}{a}$ ) formulaga ko'ra,

$$f(x) = \sqrt{3^2 + 4^2} \cos(3x - \varphi) = 5 \cos(3x - \varphi), \quad \varphi = \operatorname{arctg} \frac{4}{3}.$$

Bu funksiyaning eng katta qiymati  $f\left(\frac{2k\pi + \varphi}{3}\right) = 5$ .

$A \cos(ax \pm \varphi)$  funksiyaning davri  $T_0 = \frac{2\pi}{a}$  bo'ladi. Bundan  $T_0 = \frac{2\pi}{3}$ .

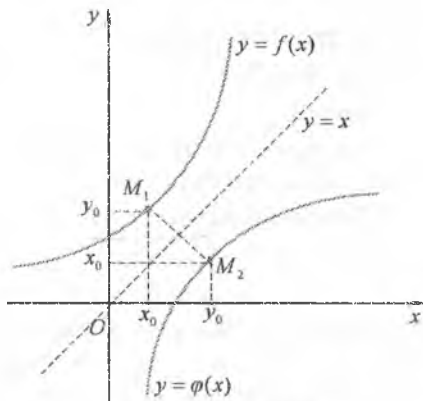
### 4.3.2. Teskari funksiya

Aniqlanish sohasi  $X$  va qiymatlar sohasi  $Y$  bo'lgan  $y = f(x)$  funksiya berilgan bo'lsin. Agar bunda har bir  $y \in Y$  qiymat yagona  $x \in X$  qiymatga mos qo'yilgan bo'lsa, u holda aniqlanish sohasi  $Y$  va qiymatlar sohasi  $X$  bo'lgan  $x = \varphi(y)$  funksiya aniqlangan bo'ladi. Bu funksiya  $y = f(x)$  ga *teskari funksiya* deb ataladi va  $x = \varphi(y) = f^{-1}(y)$  kabi belgilanadi.

$y = f(x)$  va  $x = \varphi(y)$  funksiyalar o'zaro *teskari funksiyalar* deyiladi. Bunda  $y = f(x)$  funksiyaga teskari  $x = \varphi(y)$  funksiyani topish uchun  $f(x) = y$  tenglamani  $x$  ga nisbatan yechish (agar mumkin bo'lsa) yetarli. Masalan,  $y = a^x$  funksiya teskari funksiya  $x = \log_a y$  funksiya bo'ladi.  $y = x^2$  funksiya  $x \in [0; 1]$  da  $x = \sqrt{y}$  teskari funksiya mavjud,  $x \in [-1; 1]$  da

esa mavjud emas, chunki bunda  $y$  ning har bir qiymatiga  $x$  ning ikkita qiymati, masalan,  $y=1$  ga  $x_1=-1$ ,  $x_2=1$  mos keladi.

Teskari funksiya ta'rifiga ko'ra,  $y=f(x)$  funksiya  $X$  va  $Y$  to'plamlar o'rtasida bir qiymatli moslik o'rnatishiga  $y=f(x)$  funksiya teskari funksiyaga ega bo'ladi. Bunda *har qanday qat'iy monoton funksiya teskari funksiyaga ega bo'ladi* deyish mumkin bo'ladi. Bunda agar funksiya o'ssa (kamaysa), u holda unga teskari funksiya ham o'sadi (kamayadi).



6-shakl.

$y=f(x)$  va unga teskari  $x=\varphi(y)$  funksiyalar bitta egri chiziq bilan ifodalanadi, ya'ni ularning grafigi ustma-ust tushadi.

Odatdagidek, argument (erkli o'zgaruvchi)

$x$  bilan va funksiya (bog'liq o'zgaruvchi)  $y$  bilan belgilansa,  $y=f(x)$  funksiyaga teskari funksiya  $y=\varphi(x)$  deb yoziladi. Bu  $y=f(x)$  egri chiziqning  $M_1(x_0; y_0)$  nuqtasi  $y=\varphi(x)$  egri chiziqning  $M_2(y_0; x_0)$  nuqtasi bo'lishini bildiradi. Bu nuqtalar  $y=x$  to'g'ri chiziqqa nisbatan simmetrik bo'ladi (6-shakl). Shu sababli *o'zaro teskari*  $y=f(x)$  va  $y=\varphi(x)$  funksiyalarning grafiglari I va III choraklar koordinata burchaklarining bissektrisalariga nisbatan simmetrik bo'ladi.

### 4.3.3. Murakkab funksiya

$X$  to'plamda qiymatlar sohasi  $Z$  bo'lgan  $z=\varphi(x)$  funksiya aniqlangan bo'lsin. Agar  $Z$  to'plamda  $y=f(z)$  funksiya aniqlangan bo'lsa, u holda  $X$  to'plamda  $y=f(\varphi(x))$  murakkab funksiya (yoki  $z=\varphi(x)$  va  $y=f(z)$  funksiyalarning superpozitsiyasi) aniqlangan deyiladi.

$z=\varphi(x)$  o'zgaruvchi murakkab funksiyaning oraliq argumenti deb ataladi. Murakkab funksiyaning oraliq argumentlari bir nechta bo'lishi ham mumkin.



Masalan,  $y = \cos 5x$  murakkab funksiya, chunki u  $y = f(z) = \cos z$  va  $z = \varphi(x) = 5x$  funksiyalarning superpozitsiyasidan iborat.

#### 4.3.4. Elementar funksiyalar sinfi

Quyida keltirilgan funksiyalarga *asosiy elementar funksiyalar* deyiladi.

1. **O'zgarmas funksiya**  $y = C$  ( $C \in R$ ).

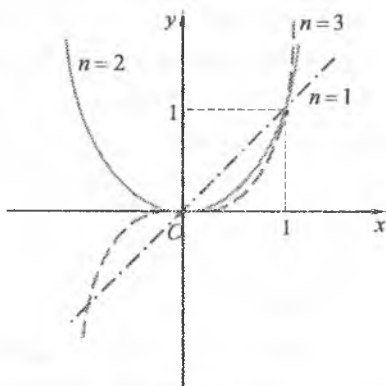
O'zgarmas funksiya:  $D(f) = (-\infty; +\infty)$ ,  $E(f) = \{C\}$  chegaralangan, juft, davri ixtiyoriy  $T$ .

O'zgarmas funksiyaning grafigi absissalar o'qiga parallel to'g'ri chiziqdan iborat bo'ladi.

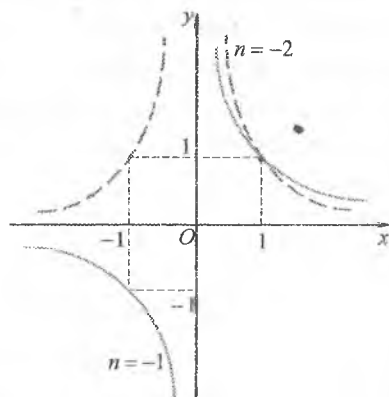
2. **Darajali funksiya**  $y = x^\alpha$ ,  $\alpha \in R, \alpha \neq 0$ .

Hamma darajali funksiyaning grafiklari (1:1) nuqtadan o'tadi.

1)  $\alpha = n$ ,  $n$  - butun musbat son. Bunda funksiyaning grafigi koordinatalar boshida absissalar o'qiga urunadi ( $n \geq 2$  da);  $n$  juft son bo'lganda ordinatalar o'qiga nisbatan simmetrik,  $n$  toq son bo'lganda esa koordinatalar boshiga nisbatan simmetrik bo'ladi (7-shakl).  $n=1$  da I va III choraklar koordinata burchaklari bissektrisalarining grafigini ifodalaydi (7-shakl).



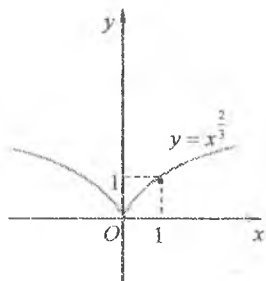
7-shakl.



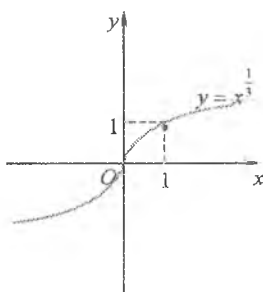
8-shakl.

2)  $\alpha = -n$ ,  $n$  - butun musbat son. Bunda funksiyaning grafigi  $n$  juft son bo'lganda ordinatalar o'qiga nisbatan simmetrik,  $n$  toq son bo'lganda esa koordinatalar boshiga nisbatan simmetrik bo'ladi (8-shakl).  $n=1$  da teskari proporsional bog'lanish grafigini ifodalaydi (8-shakl).

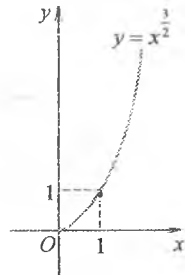
3)  $\alpha = r, r = \frac{m}{n}$ ,  $m$  va  $n$  - o'zaro tub butun sonlar. Bunda  $n$  juft son bo'lganda  $D(f) = [0; +\infty)$ ,  $n$  toq son bo'lganda  $D(f) = (-\infty; +\infty)$ . Funksiyaning grafigi  $n$  toq va  $m$  juft son bo'lganda ordinatalar o'qiga nisbatan simmetrik (9-shakl),  $n$  va  $m$  toq sonlar bo'lganda koordinatalar boshiga nisbatan simmetrik bo'ladi (10-shakl).  $r < 1$  da grafik koordinatalar boshida ordinatalar o'qiga urinadi (9, 10-shakl),  $r > 1$  da grafik koordinatalar boshida abssissalar o'qiga urinadi (11-shakl).



9-shakl.

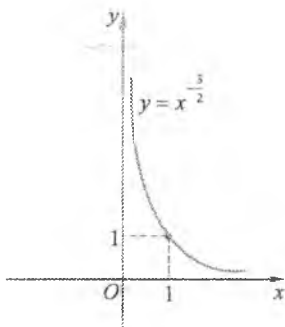


10-shakl.

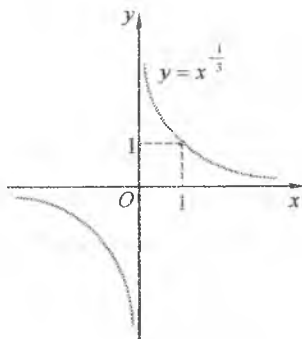


11-shakl.

4)  $\alpha = q, q = \frac{m}{n} < 0$ ,  $m$  va  $n$  - o'zaro tub butun sonlar,  $n \neq -1$ . Bunda  $n$  juft son bo'lganda  $D(f) = (0; +\infty)$  (12-shakl),  $n$  toq son bo'lganda  $D(f) = (-\infty; 0) \cup (0; +\infty)$ . Funksiyaning grafigi  $n$  toq va  $m$  juft son bo'lganda ordinatalar o'qiga nisbatan simmetrik,  $n$  va  $m$  toq sonlar bo'lganda esa koordinatalar boshiga nisbatan simmetrik bo'ladi (13-shakl).



12-shakl.



13-shakl.

•  $y = \operatorname{arctg} x$ :  $D(f) = (-\infty; +\infty)$ ,  $E(f) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ , toq, monoton o'suvchi (20-shakl);

•  $y = \operatorname{arcctg} x$ :  $D(f) = (-\infty; +\infty)$ ,  $E(f) = (0; \pi)$ , monoton kamayuvchi (21-shakl).

Asosiy elementar funksiyalardan chekli sondagi arifmetik amallar (qo'shish, ayirish, ko'paytirish, bo'lish) va superpozitsiyalash yordamida hosil qilingan bitta formula bilan berilgan funksiyaga *elementar funksiya* deyiladi.

Masalan,  $y = P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$ ,  $y = \lg^2(\sin 2x) + e^{2x}$ ,

$y = \arccos \frac{1}{x} + \sqrt[3]{x^2}$  — elementar funksiyalar.

*Elementar bo'lmagan funksiyalarga* quyidagi funksiyalar misol bo'ladi:

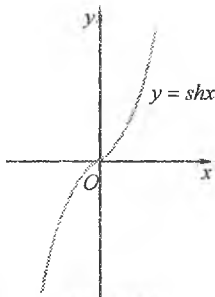
$$y = \operatorname{sign} x = \begin{cases} 1, & \text{agar } x > 0, \\ 0, & \text{agar } x = 0, \\ -1, & \text{agar } x < 0, \end{cases} \quad y = \begin{cases} \frac{1}{x^3}, & \text{agar } x > 0, \\ x^3, & \text{agar } x \leq 0, \end{cases}$$

$$y = 1 - \frac{x^3}{3! \cdot 3} + \frac{x^5}{5! \cdot 5} - \frac{x^7}{7! \cdot 7} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)! (2n+1)} + \dots$$

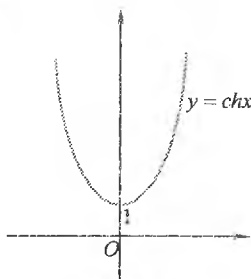
### 4.3.5. Giperbolik funksiyalar

Ko'rsatkichli funksiyalardan hosil qilinadigan quyidagi elementar funksiyalarga *giperbolik funksiyalar* deyiladi:

• *giperbolik sinus*:  $y = \operatorname{sh} x$ ,  $\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$  (22-shakl);

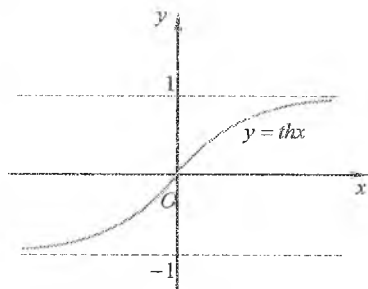


22-shakl.

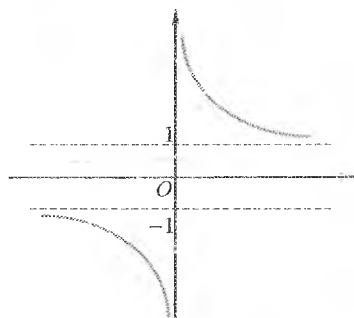


23-shakl.

- *giperbolik kosinus*:  $y = chx$ ,  $chx = \frac{e^x + e^{-x}}{2}$  (23-shakl);
- *giperbolik tangens*:  $y = thx$ ,  $thx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$  (24-shakl);
- *giperbolik kotangens*:  $y = cthx$ ,  $cthx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$  (25-shakl).



24-shakl.



25-shakl.

#### 4.3.6. Oshkormas va parametrik ko'rinishda berilgan funksiyalar

Agar  $x$  va  $y$  o'zgaruvchilar orasidagi bog'lanish  $y = f(x)$  ko'rinishda ifodalansa, bu funksiyaning oshkor ko'rinishdagi berilishi hisoblanadi. Shuningdek, ayrim hollarda funksiyaning oshkormas ko'rinishidan foydalanishga to'g'ri keladi.

Funksiya  $X$  to'plamda aniqlangan bo'lsin. Agar har bir  $x \in X$  songa mos qo'yilgan yagona  $y$  son  $F(x, y) = 0$  tenglamani qanoatlantirsa,  $y = f(x)$  funksiyaga  $F(x, y) = 0$  tenglama bilan oshkormas ko'rinishda berilgan funksiya deyiladi.

Agar  $x$  va  $y$  o'zgaruvchilar orasidagi bog'lanish ikkita  $x = x(t)$  va  $y = y(t)$ ,  $t \in X$  funksiyalar berilgan bo'lsin. U holda  $Oxy$  koordinatalar tekisligining koordinatalari  $(x(t); y(t))$  bo'lgan barcha nuqtalari to'plamiga parametrik ko'rinishda berilgan chiziq (egri chiziq yoki to'g'ri chiziq) deyiladi. Bunda  $t$  parametr deb ataladi.

Agar parametrik ko'rinishda berilgan chiziq  $y = f(x)$  funksiyaning grafigini ifodalasa, bu funksiyaga *parametrik ko'rinishda berilgan funksiya* deyiladi.

### 4.3.7. Mashqlar

#### 1. Funksiyaning aniqlanish sohasini toping:

- 1)  $f(x) = \frac{1+x^2}{x^3+8}$ ;
- 2)  $f(x) = \frac{1+x}{x^2+5x+6}$ ;
- 3)  $f(x) = \sqrt{4-x^2}$ ;
- 4)  $f(x) = \frac{5}{(x-1)\sqrt{x+2}}$ ;
- 5)  $f(x) = \sqrt{\frac{10-x}{x^2-11x+18}}$ ;
- 6)  $f(x) = \frac{\sqrt{4-3x^2-x^4}}{\cos \pi x}$ ;
- 7)  $f(x) = \sqrt{x-7} + \sqrt{10-x}$ ;
- 8)  $f(x) = \sqrt{2x+1} - \sqrt{x+1}$ ;
- 9)  $f(x) = \sqrt{x-2} + \sqrt{2-x} + \sqrt{x^2+4}$ ;
- 10)  $f(x) = \sqrt{x^3-8} + \frac{3}{\sqrt[3]{2-x}}$ ;
- 11)  $f(x) = \arcsin x - \arccos(4-x)$ ;
- 12)  $f(x) = \arcsin(x-2) + 3 \ln(x-2)$ ;
- 13)  $f(x) = \log_3 \ln \lg x$ ;
- 14)  $f(x) = \ln \sin x$ ;
- 15)  $f(x) = e^{\sqrt{x}} \log_2(2-3x)$ ;
- 16)  $f(x) = \ln \left( \frac{\sqrt{x-3} + \sqrt{7-x}}{\sqrt[3]{(x-6)^2}} \right)$ ;
- 17)  $f(x) = \sqrt{3-4x} + \arccos x \frac{3-4x}{6}$ ;
- 18)  $f(x) = \arccos \frac{x+2}{3} + 2^x$ ;
- 19)  $f(x) = \frac{3}{\sqrt[3]{x^2-3x+2}} - 5 \sin 2x$ ;
- 20)  $f(x) = \frac{x - \ln(x+3)}{\sqrt{8-x^3}}$ ;

#### 2. Funksiyaning qiymatlar sohasini toping:

- 1)  $f(x) = x^2 + 4x + 2$ ;
- 2)  $f(x) = \sqrt{7-x} + 2$ ;
- 3)  $f(x) = 2 \sin x - 5$ ;
- 4)  $f(x) = \sin x + \cos x$ ;
- 5)  $f(x) = 2^{x^2} - 1$ ;
- 6)  $f(x) = 2e^{-x^2} + 1$ ;
- 7)  $f(x) = \sqrt{9-x^2}$ ;
- 8)  $f(x) = \frac{1}{\pi} \arctg x$ ;
- 9)  $f(x) = 3|x| - \frac{1}{5}$ ;
- 10)  $f(x) = \frac{2x-3}{|2x-3|}$ ;
- 11)  $f(x) = \frac{9}{2x^2+4x+5}$ ;
- 12)  $f(x) = \frac{2}{\sqrt{2x^2-4x+3}}$ ;

3.  $f(x) = x^3 3^x$  funksiya berilgan. Quyidagilarni toping:

1)  $f(1)$ ;      2)  $f(-\sqrt[3]{4})$ ;      3)  $f(-x)$ ;      4)  $f\left(\frac{1}{x}\right)$

4. Funksiyaning monotonlik oraliqlarini toping:

1)  $f(x) = x^2 - 5x + 6$ ;      2)  $f(x) = x^3 + \arcsin x$ ;  
3)  $f(x) = \frac{1}{x^3}$ ;      4)  $f(x) = \arctg x - x$ .

5. Funksiyaning juft, toq yoki umumiy ko'rishda ekanini aniqlang:

1)  $f(x) = x^3 - 3x - x^5$ ;      2)  $f(x) = x^4 + 5x^2 + 1$ ;  
3)  $f(x) = \frac{\operatorname{tg} 2x}{x}$ ;      4)  $f(x) = \operatorname{ctg} 3x + \cos 2x$ ;  
5)  $f(x) = \ln\left(\frac{3+x}{3-x}\right)$ ;      6)  $f(x) = \ln(x + \sqrt{x^2 + 1})$ ;  
7)  $f(x) = 2|x| - 3$ ;      8)  $f(x) = x|x|$ ;  
9)  $f(x) = 3^{x^2}(x + \sin x)$ ;      10)  $f(x) = \left(\frac{2^x - 2^{-x}}{2}\right)x$ .

6. Funksiyaning eng katta va eng kichik qiymatlarini toping:

1)  $f(x) = (k-n)\cos^2 x + n$  ( $0 < k < n$ );      2)  $f(x) = 4\sin x^5$ ;  
3)  $f(x) = \sin 2x + \cos 2x$ ;      4)  $f(x) = 3\sin x + 4\cos x$ ;  
5)  $f(x) = \sin^4 x + \cos^4 x$ ;      6)  $f(x) = |\cos 4x|$ .

7. Funksiyaning monoton, qat'iy monoton yoki chegaralangan ekanini aniqlang:

1)  $f(x) = \sin^2 x$ ;      2)  $f(x) = \frac{x+2}{x+7}$ ;  
3)  $f(x) = \sqrt{3x-4}$ ;      4)  $f(x) = \begin{cases} x, & \text{agar } x < 0, \\ -3, & \text{agar } x \geq 0. \end{cases}$

8. Funksiyaning davrini toping:

1)  $f(x) = -2\cos\frac{x}{3}$ ;      2)  $f(x) = \operatorname{ctg}(2x-3)$ ;  
3)  $f(x) = \operatorname{tg} x - \cos\frac{x}{2}$ ;      4)  $f(x) = \sin\frac{x}{2}\cos\frac{x}{2}\cos x\cos 2x$ ;  
5)  $f(x) = \sin^4 x - \cos^4 x$ ;      6)  $f(x) = \sin 2x + \cos 3x$ ;  
7)  $f(x) = |\sin 2x|$ ;      8)  $f(x) = |\cos 3x|$ ;

$$9) f(x) = \sin \frac{3x}{2} + \cos \frac{2x}{3};$$

$$10) f(x) = \operatorname{tg} \frac{2x}{3} - \operatorname{ctg} \frac{3x}{2} + \sin \frac{x}{3}.$$

9. Funksiyaga teskari funksiyani toping:

$$1) y = 3x + 5;$$

$$2) y = \frac{x}{1+x};$$

$$3) y = 4 + \log_3 x;$$

$$4) y = 2 \sin 3x.$$

10.  $f(g(x))$  va  $g(f(x))$  murakkab funksiyalarni toping:

$$1) f(x) = 3x + 1, g(x) = x^3;$$

$$2) f(x) = \sin x, g(x) = |x|;$$

$$3) f(x) = \frac{x+1}{x}, g(x) = \frac{1}{4-x};$$

$$4) f(x) = 2^{3^x}, g(x) = \log_2 x.$$

11. Funksiyaning grafigini chizing:

$$1) y = x^2 + 4x + 3;$$

$$2) y = -2 \sin 3x;$$

$$3) y = \frac{2x-1}{2x+1};$$

$$4) y = -x^2 |x|;$$

$$5) y = x \sin x;$$

$$6) y = x + \sin x.$$

$$7) y = \arccos |x|;$$

$$8) y = 3^{\frac{1}{x}}.$$

12. Ayniyatni isbotlang:

$$1) 1 - \operatorname{th}^2 x = \frac{1}{\operatorname{ch}^2 x};$$

$$2) \operatorname{cth}^2 x - 1 = \frac{1}{\operatorname{sh}^2 x};$$

$$3) \operatorname{ch}^2 x = \frac{\operatorname{ch} 2x + 1}{2};$$

$$4) \operatorname{sh}^2 x = \frac{\operatorname{ch} 2x - 1}{2};$$

$$5) \operatorname{sh}(\ln x) = \frac{x^2 - 1}{2x};$$

$$6) \operatorname{ch}(\ln x) = \frac{x^2 + 1}{2x}.$$

13. Qaysi nuqta  $y + \cos y - x = 0$  funksiya grafigiga tegishli ekanini aniqlang:

$$A(1;0); B(0;0); C\left(\frac{\pi}{2}; \frac{\pi}{2}\right); D(\pi-1;\pi).$$

14. Qaysi nuqta  $\begin{cases} x = t - 1, \\ y = t^2 + 1 \end{cases}$  parametrik tenglamalar bilan berilgan egri chiziqqa

tegishli ekanini aniqlang:  $A(1;5); B\left(\frac{1}{2}; \frac{13}{4}\right); C(2;8); D(0;1).$

15. Berilgan funksiyani  $y = y(x)$  ko'rinishga keltiring:

$$1) \begin{cases} x = t + 2, \\ y = t^2 + 4t + 5; \end{cases}$$

$$2) \begin{cases} x = 3 \sin t, \\ y = 2 \cos t. \end{cases}$$

## 4.4. FUNKSIYANING LIMITI

### 4.4.1. Funksiyaning limiti

#### *Funksiya limitining ta'riflari*

Biror  $X$  sonli to'plam berilgan bo'lsin.

**1-ta'rif.** Agar  $x_0 \in X$  nuqtaning ixtiyoriy  $\varepsilon$  ( $\varepsilon > 0$ ) atrofida  $X$  to'plamning cheksiz ko'p elementlari yotsa,  $x_0$  nuqtaga  $X$  toplamning *limit nuqtasi* deyiladi.

Masalan,  $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$  to'plam uchun  $x_0 = 0$  limit nuqta bo'ladi.

$f(x)$  funksiya  $X$  toplamda aniqlangan va  $x_0$  nuqta  $X$  toplamning limit nuqtasi bo'lsin.

**2- ta'rif.** (*funksiya limitining «ketma-ketlik tilidagi» yoki Geyne ta'rif*). Agar  $X$  toplamning nuqtalaridan tuzilgan  $x_0$  nuqtaga yaqinlashuvchi har qanday  $\{x_n\}$  ketma-ketlik ( $x_n \neq x_0$ ) olinganda ham, bu ketma-ketlikka mos  $\{f(x_n)\}$  ketma-ketlik hamma vaqt yagona  $A$  limitga intilsa,  $A$  soniga  $f(x)$  funksiyaning  $x_0$  nuqtadagi yoki  $x \rightarrow x_0$  dagi limiti deyiladi va bu  $\lim_{x \rightarrow x_0} f(x) = A$  deb yoziladi.

**3- ta'rif.** (*funksiya limitining « $\varepsilon - \delta$  tilidagi» yoki Koshi ta'rif*). Agar  $\forall \varepsilon > 0$  son uchun shunday  $\delta > 0$  son topilsa va  $x$  ning  $0 < |x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha  $x \in X$ ,  $x \neq x_0$  qiymatlarida  $|f(x) - A| < \varepsilon$  tengsizlik bajarilsa,  $A$  soniga  $f(x)$  funksiyaning  $x_0$  nuqtadagi yoki  $x \rightarrow x_0$  dagi limiti deyiladi va bu  $\lim_{x \rightarrow x_0} f(x) = A$  deb yoziladi.

Funksiya limiti uchun berilgan Geyne va Koshi ta'riflari o'zaro ekvivalent ekanligi isbotlangan. Shu sababli funksiyaning nuqtadagi limitini topishda bu ta'riflarning istalgan biridan foydalanish mumkin.

$x_0$  ga intiluvchi  $\{x_n\}$  ketma-ketlikni yetarlicha ko'p usul bilan tanlash mumkin bo'lganligi uchun Geyni ta'rifidan funksiyaning limitini topishdan ko'ra, funksiyaning nuqtada limitga ega bo'lmashligini ko'rsatishda foydalanish qulaylikka ega bo'ladi. Buning uchun  $x_0$  ga nuqtada limitga ega bo'lmagan birorta  $\{f(x_n)\}$  ketma-ketlikni topish yetarli yoki har xil limitlarga ega bo'lgan  $\{f(x'_n)\}$  va  $\{f(x''_n)\}$  ketma-ketliklarni ko'rsatish kifoya.



1-misol.  $f(x) = \sin \frac{1}{x}$  funksiya  $x=0$  nuqtada limitga ega bo'lmashligini ko'rsating.

Yechish.  $x=0$  nuqtaga intiluvchi ikkita  $\left\{ \frac{1}{n\pi} \right\}$  va  $\left\{ \frac{1}{\frac{\pi}{2} + 2n\pi} \right\}$  ketma-ketliklarni qaraymiz. Mos  $\{f(x)\}$  ketma-ketliklar har xil limitlarga intiladi:  $\{\sin n\pi\}$  ketma-ketlik nolga intiladi,  $\left\{ \sin \left( \frac{\pi}{2} + 2n\pi \right) \right\}$  ketma-ketlik esa birga intiladi.

Demak,  $f(x) = \sin \frac{1}{x}$  funksiya  $x=0$  nuqtada limitga ega bo'lmaydi.

2-misol.  $\lim_{x \rightarrow 1} (3x - 2) = 1$  ekanini ko'rsating.

Yechish.  $\forall \varepsilon > 0$  son olamiz. Shunday  $\delta > 0$  sonni ko'rsatishimiz kerakki,  $|x-1| < \delta$  bo'lganida  $|f(x)-1| < \varepsilon$  bo'lsin.

Bunda  $f(x) = 3x - 2$ :  $|f(x) - 1| = |(3x - 2) - 1| = |3x - 3| = 3|x - 1|$  bo'lgani uchun  $\delta = \frac{\varepsilon}{3}$  deb olsak,  $|x-1| < \delta$  bo'lganda  $|f(x)-1| < \varepsilon$  bo'ladi.

Demak,  $\lim_{x \rightarrow 1} (3x - 2) = 1$ .

Xususan,  $\varepsilon = 1$  da  $\delta = \frac{1}{3}$ ,  $\varepsilon = \frac{1}{2}$  da  $\delta = \frac{1}{6}$ .

Shunday qilib,  $\delta$  son  $\varepsilon$  songa bog'liq bo'ladi. Shu sababli keyingi ta'riflarda  $\delta = \delta(\varepsilon)$  deb olamiz.

**Izoh.** Funksiyaning  $x_0$  nuqtadagi limiti ta'rifida  $x_0$  nuqtaning o'zi qaralmaydi. Shunday qilib, funksiyaning  $x_0$  nuqtadagi qiymati funksiyaning bu nuqtadagi limitiga ta'sir qilmaydi. Bundan tashqari, funksiya  $x_0$  nuqtada aniqlanmagan bo'lishi ham mumkin. Shu sababli  $x_0$  nuqtaning atrofida ( $x \neq x_0$  bo'lganda) teng bo'lgan ( $x = x_0$  da har xil qiymatga ega bo'lgan yoki ulardan bittasi yoki har ikkalasi aniqlanmagan) ikkita funksiya  $x \rightarrow x_0$  da bitta limitga ega bo'lishi yoki ularning har ikkalasi limitga ega bo'lmashligi mumkin.

3-misol.  $f(x) = \begin{cases} x^2, & x \neq 0, \\ 1, & x = 0 \end{cases}$  bo'lsa,  $\lim_{x \rightarrow 0} f(x)$  limitni toping.

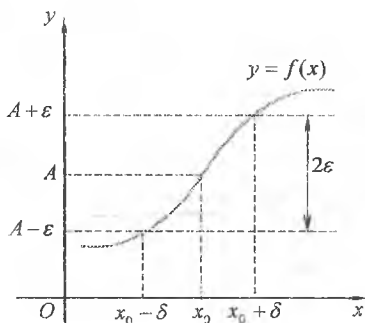
Yechish.  $g(x) = x^2$ ,  $-\infty < x < +\infty$  funksiya  $x=0$  dan tashqari, barcha

nuqtalarda  $f(x)$  funksiya bilan ustma-ust tushadi va  $\lim_{x \rightarrow 0} g(x) = 0$  bo'ladi. Shu sababli  $\lim_{x \rightarrow 0} f(x) = 0$ .

Funksiyaning nuqtadagi limiti ta'rifini *geometrik nuqtayi nazardan* shunday talqin qilish mumkin: agar  $A$  soni  $f(x)$  funksiyaning  $x_0$  nuqtadagi limiti bo'lsa,  $A$  nuqtaning istalgan  $\varepsilon$  atrofi uchun

$x_0$  nuqtaning shunday  $\delta$  atrofi topiladi va  $\delta$  atrofda barcha  $x$  ( $x \neq x_0$ ) nuqtalarda  $f(x)$  funksiyaning mos qiymatlari  $A$  nuqtaning  $\varepsilon$  atrofida yotadi. Boshqacha aytganda  $f(x)$  funksiyaning  $\delta$  atrofda grafigi

$y = A - \varepsilon$  va  $y = A + \varepsilon$  to'g'ri chiziqlar bilan chegaralangan, kengligi  $2\varepsilon$  ga teng bo'lgan tasmda joylashadi (26-shakl).



26-shakl.

**4-ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $\delta = \delta(\varepsilon) > 0$  son topilsa va  $x$  ning  $x_0 < x < x_0 + \delta$  ( $x_0 - \delta < x < x_0$ ) tengsizlikni qanoatlantiruvchi barcha qiymatlarida  $|f(x) - A| < \varepsilon$  tengsizlik bajarilsa,  $A$  soniga  $f(x)$  funksiyaning  $x_0$  nuqtadagi *o'ng (chap) limiti* deyiladi va  $\lim_{x \rightarrow x_0+0} f(x) = A$  yoki  $f(x+0) = A$  ( $\lim_{x \rightarrow x_0-0} f(x) = A$  yoki  $f(x-0) = A$ ) kabi belgilanadi.

$f(x)$  funksiyaning  $x_0$  nuqtadagi o'ng va chap limitlari *bir tomonlama limitlar* deb ataladi. Agar  $f(x)$  funksiyaning  $x_0$  nuqtadagi o'ng va chap limitlari mavjud va bir-biriga teng, ya'ni  $f(x_0+0) = f(x_0-0) = A$  bo'lsa,  $x_0$  nuqtada  $f(x)$  funksiyaning limiti mavjud va  $\lim_{x \rightarrow x_0} f(x) = A$  bo'ladi.

$y = f(x)$  funksiya  $(-\infty; +\infty)$  intervalda aniqlangan bo'lsin.

**5-ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $\delta = \delta(\varepsilon) > 0$  son topilsa va  $x$  ning  $x > \delta$  ( $x < -\delta$ ) tengsizlikni qanoatlantiruvchi barcha qiymatlarida  $|f(x) - A| < \varepsilon$  tengsizlik bajarilsa,  $A$  soniga  $f(x)$  funksiyaning  $x \rightarrow +\infty$  ( $x \rightarrow -\infty$ ) dagi limiti deyiladi va  $\lim_{x \rightarrow +\infty} f(x) = A$  ( $\lim_{x \rightarrow -\infty} f(x) = A$ ) kabi belgilanadi.

Funksiyaning cheksizlikdagi limiti ta'rifini *geometrik nuqtayi*

nazardan bunday talqin qilish mumkin: agar  $\lim_{x \rightarrow +\infty} f(x) = A$  ( $\lim_{x \rightarrow -\infty} f(x) = A$ ) bo'lsa,  $\forall \varepsilon > 0$  son uchun shunday  $\delta = \delta(\varepsilon) > 0$  son topiladiki,  $x \in (\delta; +\infty)$  ( $x \in (-\infty; -\delta)$ ) larda  $f(x)$  funksiyaning qiymatlari  $A$  nuqtaning  $\varepsilon$  atrofida yotadi.

### *Limitlar haqidagi teoremlar*

Funksiya limitining «ketma-ketlik tilidagi» ta'rifi yaqinlashuvchi (limitga ega) ketma-ketliklarning xossalari funksiyaning limiti uchun o'tkazish imkonini beradi. Bu xossalarni ifodalovchi teoremlar bilan tanishamiz va ularning ayrimlarini isbotlaymiz. Bu teoremlarda qaralayotgan funksiyalar  $x \rightarrow x_0$  da limitga ega deb hisoblaymiz.

**1-teorema.** Funksiya  $x \rightarrow x_0$  da yagona limitga ega bo'ladi.

**2-teorema.** Ikkita funksiya algebraik yig'indisining limiti bu funksiyalar limitlarining algebraik yig'indisiga teng, ya'ni

$$\lim_{x \rightarrow x_0} (f(x) \pm g(x)) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x).$$

*Isboti.* Ixtiyoriy  $\{x_n\}$  kema-ketlik olamiz.

Bu ketma-ketlik uchun  $x_n \rightarrow x_0$ ,  $x_n \neq x_0$ ,  $x_n \in D(f) \cap D(g)$  bo'lsin.

U holda

$$\begin{aligned} \lim_{x \rightarrow x_0} (f(x) \pm g(x)) &= \lim_{n \rightarrow \infty} ((f(x_n) \pm g(x_n))) = \\ &= \lim_{n \rightarrow \infty} f(x_n) \pm \lim_{n \rightarrow \infty} g(x_n) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x). \end{aligned}$$

Demak,

$$\lim_{x \rightarrow x_0} (f(x) \pm g(x)) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x).$$

**3-teorema.** Ikkita funksiya ko'paytmasining limiti bu funksiyalar limitlarining ko'paytmasiga teng, ya'ni

$$\lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x).$$

**1-natija.** O'zgarmas funksiyaning limiti uning o'ziga teng, ya'ni

$$\lim_{x \rightarrow x_0} C = C.$$

**2-natija.** O'zgarmas ko'paytuvchini limit belgisidan tashqariga chiqazish mumkin, ya'ni

$$\lim_{x \rightarrow x_0} (k \cdot f(x)) = k \cdot \lim_{x \rightarrow x_0} f(x), \quad k \in \mathbb{R}.$$

**3-natija.** Funktsiyaning musbat ko'rsatkichli darajasining limiti bu funktsiya limitining shu tartibli darajasiga teng, ya'ni

$$\lim_{x \rightarrow x_0} (f(x))^p = (\lim_{x \rightarrow x_0} f(x))^p, \quad p > 0.$$

**4-teorema.** Ikki funktsiya bo'linmasining limiti bu funktsiyalar limitlarining nisbatiga teng, ya'ni

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)}, \quad \lim_{x \rightarrow x_0} g(x) \neq 0.$$

**5-teorema.** Agar  $x_0$  nuqtaning biror atrofidagi barcha  $x$  uchun  $f(x) \leq g(x)$  tengsizlik bajarilsa, u holda  $\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$  bo'ladi.

**6-teorema.** Agar  $x_0$  nuqtaning biror atrofidagi barcha  $x$  uchun  $f(x) \leq \varphi(x) \leq g(x)$  tengsizlik bajarilsa va  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = A$  bo'lsa, u holda  $\lim_{x \rightarrow x_0} \varphi(x) = A$  bo'ladi.

**7-teorema.**  $\lim_{x \rightarrow x_0} g(x) = 0$ ,  $\lim_{x \rightarrow x_0} f(x) = C \neq 0$  bo'lsin.

U holda:

1) agar  $|x - x_0| < \delta$  ( $\delta > 0$ ) tengsizlikni qanoatlantiruvchi barcha  $x$  uchun  $\frac{f(x)}{g(x)} > 0$  bo'lsa,  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = +\infty$  bo'ladi;

2) agar  $|x - x_0| < \delta$  ( $\delta > 0$ ) tengsizlikni qanoatlantiruvchi barcha  $x$  uchun  $\frac{f(x)}{g(x)} < 0$  bo'lsa,  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = -\infty$  bo'ladi.

**8-teorema.** Agar  $\lim_{x \rightarrow x_0} f(x) = A \neq \infty$  bo'lsa, u holda  $x_0$  nuqtaning yetarlicha kichik atrofidan olingan  $x$  ( $x \neq x_0$ ) qiymatlarda  $f(x)$  funktsiya chegaralangan bo'ladi.

*Isboti.* Shartga ko'ra,  $\lim_{x \rightarrow x_0} f(x) = A \neq \infty$ . Funktsiya limitining Koshi ta'rifiga ko'ra,  $\forall \varepsilon > 0$  son uchun shunday  $\delta > 0$  son topiladi va  $x$  ning  $0 < |x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha qiymatlarida  $|f(x) - A| < \varepsilon$ , ya'ni  $A - \varepsilon < f(x) < A + \varepsilon$  tengsizlik bajariladi. Demak,  $x$  ning  $0 < |x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$f(x)$  funksiyaning qiymatlari  $(A-\varepsilon; A+\varepsilon)$  oraliqda bo'ladi. Bu funksiyaning  $(x_0-\delta; x_0+\delta)$ ,  $x \neq x_0$  oraliqda chegaralanganligini bildiradi.

**9-teorema.** Agar: 1)  $\lim_{x \rightarrow x_0} \varphi(x) = t_0$  va  $x_0$  nuqtaning shunday  $(x_0-\delta; x_0+\delta)$ ,  $\delta > 0$  atrofi mavjud va bu atrofdan olingan barcha  $x$  lar uchun  $\varphi(x) \neq t_0$  bo'lsa, 2)  $\lim_{t \rightarrow t_0} f(t) = B$  bo'lsa, u holda  $x \rightarrow x_0$  da murakkab  $f(\varphi(x))$  funksiya limitga ega va  $\lim_{x \rightarrow x_0} f(\varphi(x)) = B$  bo'ladi.

Yuqorida keltirilgan teoremlar  $x \rightarrow \pm\infty$  da ham o'rinli bo'ladi.

4- misol.  $\lim_{x \rightarrow 5} \frac{x^2 - 8x + 15}{x^2 - 25}$  limitni toping.

*Yechish.* Bu limit uchun ikki funksiya bo'linmasining limiti haqidagi teoremani qo'llab bo'lmaydi, chunki  $x \rightarrow 5$  da kasrning maxrajini nolga teng bo'ladi. Bundan tashqari, suratning limiti ham nolga teng.

Bunday hollarda  $\frac{0}{0}$  ko'rinishdagi aniqmaslik berilgan deyiladi.

Bu aniqmaslikni ochish uchun kasrning surati va maxrajini ko'paytuvchilarga ajratamiz va kasrni  $x-5 \neq 0$  ( $x \rightarrow 5$ , lekin  $x \neq 5$ ) ga bo'lib, topamiz:

$$\lim_{x \rightarrow 5} \frac{(x-5)(x-3)}{(x-5)(x+5)} = \lim_{x \rightarrow 5} \frac{x-3}{x+5} = \frac{2}{10} = \frac{1}{5}.$$

5- misol.  $\lim_{x \rightarrow \infty} \frac{2x^3 + 3x^2 + 1}{x^3 + 4x^2 - x}$  limitni toping.

*Yechish.* Bu misolda  $x \rightarrow \infty$  da  $\frac{\infty}{\infty}$  ko'rinishdagi aniqmaslik hosil bo'ladi. Kasrning surat va maxrajini  $x$  ning yuqori darajasiga, ya'ni  $x^3$  ga bo'lib, topamiz:

$$\lim_{x \rightarrow \infty} \frac{2x^3 + 3x^2 + 1}{x^3 + 4x^2 - x} = \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x} + \frac{1}{x^3}}{1 + \frac{4}{x} - \frac{1}{x^2}} = \frac{2+0+0}{1+0-0} = 2.$$

### Ajoyib limitlar

**Birinchi ajoyib limit:**  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$

*Isboti.*  $0 < x < \frac{\pi}{2}$  bo'lsin.

Radiusi  $R=1$  ga teng bo'lgan aylananing radian o'lchovi  $x$  ga teng bo'lgan markaziy burchagiga mos yoyini qaraymiz (27-shakl).

Shakldan quyidagilarga ega bo'lamiz:

$\Delta MOA$  yuzi  $<$   $MOA$  sektor yuzi  $<$   $\Delta LOA$  yuzi;

$$\Delta MOA \text{ yuzi: } S_1 = \frac{1}{2} OA \cdot MK = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x;$$

$$MOA \text{ sektor yuzi: } S_2 = \frac{1}{2} OA \cdot \overset{\frown}{MA} = \frac{1}{2} \cdot 1 \cdot x = \frac{1}{2} x;$$

$$\Delta LOA \text{ yuzi: } S_3 = \frac{1}{2} OA \cdot LA = \frac{1}{2} \cdot 1 \cdot \operatorname{tg} x = \frac{1}{2} \operatorname{tg} x.$$

Bundan  $\sin x < x < \operatorname{tg} x$  kelib chiqadi. Tengsizlikni  $\sin x > 0$  ga bo'lamiz:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \text{yoki} \quad \cos x < \frac{\sin x}{x} < 1.$$

Endi  $x < 0$  bo'lsin.

$$\frac{\sin(-x)}{-x} = \frac{\sin x}{x}, \quad \cos(-x) = \cos x \quad \text{ekanidan}$$

$x < 0$  da ham

$$\cos x < \frac{\sin x}{x} < 1.$$

$\lim_{x \rightarrow 0} \cos x = 1$ ,  $\lim_{x \rightarrow 0} 1 = 1$  bo'lgani uchun oxirgi

tenglikdan 6-teoremaga ko'ra,

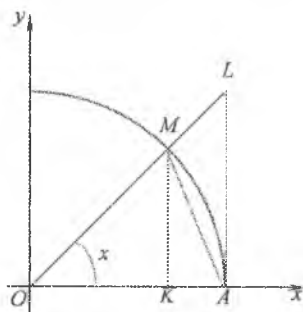
$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

6-misol.  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$  limitni toping.

*Yechish.*  $x \rightarrow 0$  da  $\frac{0}{0}$  ko'rinishdagi aniqmaslik berilgan.

Almashtirishlar bajaramiz:

$$\frac{1 - \cos x}{x^2} = \frac{2 \sin^2\left(\frac{x}{2}\right)}{x^2} = \frac{1}{2} \left( \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \right)^2.$$



27-shakl.

$x \rightarrow 0$  da  $\frac{x}{2} \rightarrow 0$  va 1-ajoyib limitga ko'ra,  $\lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 1$ .

Demak,

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{2} \left( \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \right)^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}.$$

**Ikkinchi ajoyib limit:**  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$

*Isboti.* Ma'lumki,  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ ,  $n \in \mathbb{N}$ .

$x > 1$  bo'lsin.  $n = [x]$  deb olamiz. U holda  $x = n + \alpha$ , bu yerda  $0 \leq \alpha \leq 1$ .

$n < x < n+1$  tengsizlikdan topamiz:

$$\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n}$$

yoki

$$\left(1 + \frac{1}{n+1}\right)^n < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{n}\right)^{n+1}. \quad (4.1)$$

$x \rightarrow \infty$  da  $n \rightarrow \infty$ .

U holda

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 1 \cdot e = e;$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)^n = \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)^{n+1}}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)} = \frac{e}{1} = e.$$

Shuning uchun (4.1) tengsizlikdan 6-teoremaga ko'ra,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

kelib chiqadi.

Endi  $x < -1$  bo'lsin.  $x = -y$  deb olamiz.

U holda

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{y \rightarrow \infty} \left(1 - \frac{1}{y}\right)^{-y} = \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y-1}\right)^y = \\ &= \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y-1}\right)^{y-1} \cdot \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y-1}\right) = e \cdot 1 = e. \end{aligned}$$

Demak,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

7- misol.  $\lim_{x \rightarrow \infty} \left(\frac{2x}{2x-1}\right)^{3x-1}$  limitni toping.

*Yechish.*  $x \rightarrow \infty$  da  $1^\infty$  ko'rinishdagi aniqmaslik berilgan.

Qavs ichidagi kasrning butun qismini ajratib, almashtirishlar bajaramiz:

$$\left(1 + \frac{1}{2x-1}\right)^{3x-1} = \left[\left(1 + \frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{3x-1}{2x-1}} = \left[\left(1 + \frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{3-\frac{1}{x}}{2-\frac{1}{x}}}.$$

$x \rightarrow \infty$  da  $2x-1 \rightarrow \infty$  bo'lgani sababli 2-ajoyib limitni qo'llab, topamiz:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{2x-1}\right)^{2x-1} = e.$$

$$\lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x}}{2 - \frac{1}{x}} = \frac{3}{2} \text{ ekanidan } \lim_{x \rightarrow \infty} \left(\frac{2x}{2x-1}\right)^{3x-1} = \sqrt{e^3}.$$

#### 4.4.2. Cheksiz kichik funksiyalar

##### *Ta'riflar va asosiy teoremlar*

**6-ta'rif.** Agar  $\lim_{x \rightarrow x_0} f(x) = 0$  bo'lsa,  $f(x)$  funksiyaga  $x \rightarrow x_0$  da cheksiz kichik funksiya deyiladi.

Funksiyaning limiti ta'rifiga ko'ra,  $\lim_{x \rightarrow x_0} f(x) = 0$  tenglik quyidagicha



talqin qilinadi:  $\forall \varepsilon > 0$  son uchun shunday  $\delta = \delta(\varepsilon) > 0$  son topiladi va  $x$  ning  $0 < |x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha qiymatlarda  $|f(x)| < \varepsilon$  tengsizlik bajariladi.

$x \rightarrow x_0 + 0$ ,  $x \rightarrow x_0 - 0$ ,  $x \rightarrow +\infty$ ,  $x \rightarrow -\infty$  da cheksiz kichik funksiya shu kabi ta'riflanadi.

Cheksiz kichik funksiyalar ko'pincha cheksiz kichik kattaliklar yoki cheksiz kichik deb ataladi va odatda, grek alifbosining  $\alpha, \beta$  kabi harflari bilan belgilanadi.

Cheksiz kichik funksiyalarga  $x \rightarrow 0$  da  $\alpha = x^3$ ,  $x \rightarrow 3$  da  $\beta = x - 3$ ,  $x \rightarrow k\pi, k \in \mathbb{Z}$  da  $\gamma = \sin x$  funksiyalar misol bo'ladi.

**7-ta'rif.** Agar  $\lim_{x \rightarrow x_0} f(x) = \infty$  bo'lsa,  $f(x)$  funksiyaga  $x \rightarrow x_0$  da cheksiz katta funksiya deyiladi.

Bunda  $f(x)$  funksiya faqat musbat qiymatlar qabul qilsa,  $\lim_{x \rightarrow x_0} f(x) = +\infty$  deb, faqat manfiy qiymatlar qabul qilsa,  $\lim_{x \rightarrow x_0} f(x) = -\infty$  deb

yoziyadi. Masalan,  $x \rightarrow 1$  da  $f(x) = \frac{1}{x-1}$  cheksiz katta funksiya bo'ladi.

Cheksiz kichik funksiyalar uchun o'rinli bo'ladigan teoremlar bilan tanishamiz.

**10-teorema.**  $\lim_{x \rightarrow x_0} f(x) = A$  bo'lishi uchun  $x \rightarrow x_0$  da  $\alpha(x) = f(x) - A$  funksiya cheksiz kichik bo'lishi zarur va yetarli.

*Isboti.* Zarurligi.  $\lim_{x \rightarrow x_0} f(x) = A$  bo'lsin.  $f(x) - A = \alpha(x)$  funksiyani olamiz.

U holda

$$\lim_{x \rightarrow x_0} \alpha(x) = \lim_{x \rightarrow x_0} |f(x) - A| = \lim_{x \rightarrow x_0} f(x) - \lim_{x \rightarrow x_0} A = A - A = 0.$$

Demak,  $\alpha(x) = f(x) - A$  funksiya cheksiz kichik.

*Yetarliligi.*  $f(x) - A = \alpha(x)$ , bu yerda  $\alpha(x)$  cheksiz kichik funksiya bo'lsin. Bundan  $f(x) = A + \alpha(x)$ .

U holda

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} (A + \alpha(x)) = \lim_{x \rightarrow x_0} A + \lim_{x \rightarrow x_0} \alpha(x) = A + 0 = A.$$

Quyidagi teoremlar  $x \rightarrow x_0$  da deb qaraladi.

**11-teorema.** Chekli sondagi cheksiz kichik funksiyalarning algebraik yig'indisi cheksiz kichik funksiya bo'ladi.

**12-teorema.** Cheksiz kichik funksiyaning chegaralangan funksiyaga ko'paytmasi cheksiz kichik funksiya bo'ladi.

**4-natija.** Chekli sondagi cheksiz kichik funksiyalarning ko'paytmasi cheksiz kichik funksiya bo'ladi.

**5-natija.** Cheksiz kichik funksiyaning chekli o'zgarmas songa ko'paytmasi cheksiz kichik funksiya bo'ladi.

**13-teorema.** Cheksiz kichik funksiyaning nolga teng bo'lmagan limitga ega funksiyaga bo'linmasi cheksiz kichik funksiya bo'ladi.

Yuqorida keltirilgan teoremlar  $x \rightarrow \infty$ ,  $x \rightarrow x_0 - 0$ ,  $x \rightarrow x_0 + 0$  da ham o'rinli bo'ladi.

**14-teorema.** Agar  $x \rightarrow x_0$  da  $\alpha(x)$  funksiya cheksiz kichik bo'lsa, u holda  $x \rightarrow x_0$  da  $\frac{1}{\alpha(x)}$  funksiya cheksiz katta bo'ladi va aksincha agar  $x \rightarrow x_0$  da  $f(x)$  funksiya cheksiz katta bo'lsa, u holda  $x \rightarrow x_0$  da  $\frac{1}{f(x)}$  funksiya cheksiz kichik bo'ladi.

**8-misol.**  $\alpha(x) = (x-2)^3 \sin^2 \frac{1}{x-2}$  funksiya  $x \rightarrow 2$  da cheksiz kichik bo'lishini ko'rsating.

*Yechish.*  $\lim_{x \rightarrow 2} (x-2)^3 = 0$  ekanidan  $\beta(x) = (x-2)^3$  funksiya cheksiz kichik.

$$g(x) = \sin^2 \frac{1}{x-2}, \quad x \neq 2 \text{ funksiya chegaralangan, chunki } \left| \sin^2 \frac{1}{x-2} \right| \leq 1.$$

$\alpha(x)$  funksiya cheksiz kichik  $\beta(x)$  funksiyaning chegaralangan  $g(x)$  funksiyaga ko'paytmasidan iborat. Demak, 12-teoremaga ko'ra,  $\alpha(x)$  funksiya  $x \rightarrow 2$  da cheksiz kichik.

### *Cheksiz kichik funksiyalarni taqqoslash*

Ma'lumki, cheksiz kichik funksiyalarning yig'indisi, ayirmasi va ko'paytmasi cheksiz kichik funksiyalar bo'ladi. Bu tasdiqni cheksiz kichik funksiyalarning bo'linmasi uchun ta'kidlab bo'lmaydi, chunki bitta cheksiz kichik funksiyaning boshqa cheksiz kichik funksiyaga nisbati har xil natijaga olib kelishi, ya'ni chekli son bo'lishi, cheksiz katta bo'lishi, cheksiz kichik bo'lishi yoki limitga ega bo'lmasligi mumkin.

Cheksiz kichik funksiyalar bir-biri bilan nisbati yordamida taqqoslanadi.

$\alpha(x)$  va  $\beta(x)$  funksiyaar  $x \rightarrow x_0$  da cheksiz kichik funksiyalar

bo'lsin.

1. Agar  $\lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)} = A \neq 0$  ( $A$  - chekli son) bo'lsa,  $\alpha(x)$  va  $\beta(x)$  bir xil tartibli cheksiz kichik funksiyalar deyiladi.

2. Agar  $\lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)} = 0$  bo'lsa,  $\alpha(x)$  funksiya  $\beta(x)$  funksiyaga nisbatan yuqori tartibli cheksiz kichik funksiya deyiladi va bu  $\alpha = o(\beta)$  kabi belgilanadi.

3. Agar  $\lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)} = \infty$  bo'lsa,  $\alpha(x)$  funksiya  $\beta(x)$  funksiyaga nisbatan quyi tartibli cheksiz kichik funksiya deyiladi.

4. Agar  $\lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)}$  mavjud bo'lmasa,  $\alpha(x)$  va  $\beta(x)$  funksiyalarga taqqoslanmaydigan cheksiz kichik funksiyalar deyiladi.

Misollar keltiramiz:

1)  $x \rightarrow 0$  da  $\sin 3x$  va  $\sin x$  funksiyalar bir xil tartibli cheksiz kichik funksiyalar, chunki 
$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin x} = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} = \frac{3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}}{\lim_{x \rightarrow 0} \frac{3x}{x}} = \frac{3}{1} = 3;$$

2)  $x \rightarrow 0$  da  $2x^3$  funksiya  $5x$  funksiyaga nisbatan yuqori tartibli cheksiz kichik funksiya, chunki 
$$\lim_{x \rightarrow 0} \frac{2x^3}{5x} = \lim_{x \rightarrow 0} \frac{2x^2}{5} = 0;$$

3)  $x \rightarrow 0$  da  $\sin x$  funksiya  $x^2$  funksiyaga nisbatan quyi tartibli cheksiz kichik funksiyalar, chunki 
$$\lim_{x \rightarrow 0} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{x} = 1 \cdot \frac{1}{0} = \infty;$$

4)  $x \sin \frac{1}{x}$  va  $x$  funksiyalar  $x \rightarrow 0$  da taqqoslanmaydigan cheksiz kichik funksiyalar, chunki 
$$\lim_{x \rightarrow 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0} \sin \frac{1}{x}$$
 limit mavjud emas.

## Ekvivalent cheksiz kichik funksiyalar

Bir xil tartibli cheksiz kichik funksiyalar orasida ekvivalent cheksiz kichik funksiyalar muhim ahamiyatga ega.

$\alpha(x)$  va  $\beta(x)$  funksiyalar  $x \rightarrow x_0$  da cheksiz kichik funksiyalar bo'lsin.

**8-ta'rif.** Agar  $\lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)} = 1$  bo'lsa, u holda  $x \rightarrow x_0$  da  $\alpha(x)$  va  $\beta(x)$  ekvivalent cheksiz kichik funksiyalar deyiladi va  $\alpha(x) \sim \beta(x)$  kabi belgilanadi.

Masalan,  $x \rightarrow 0$  da  $\sin x$  va  $x$  funksiyalar ekvivalent cheksiz kichik funksiyalar, chunki  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

Cheksiz kichik funksiyalar uchun o'rinli bo'ladigan teoremlar bilan tanishamiz.

**15-teorema.** Agar ikkita cheksiz kichik funksiya nisbatida cheksiz kichik funksiyalarning har ikkalasini yoki ulardan bittasini ekvivalent cheksiz kichik funksiya bilan almashtirilsa, u holda bu nisbatning limiti o'zgarmaydi.

*Isboti.*  $x \rightarrow x_0$  da  $\alpha \sim \alpha'$  va  $\beta \sim \beta'$  bo'lsin.

U holda

$$\lim_{x \rightarrow x_0} \frac{\alpha}{\beta} = \lim_{x \rightarrow x_0} \frac{\alpha}{\alpha'} \cdot \frac{\alpha'}{\beta'} \cdot \frac{\beta'}{\beta} = \lim_{x \rightarrow x_0} \frac{\alpha}{\alpha'} \cdot \lim_{x \rightarrow x_0} \frac{\beta'}{\beta} \cdot \lim_{x \rightarrow x_0} \frac{\alpha'}{\beta'} = 1 \cdot 1 \cdot \lim_{x \rightarrow x_0} \frac{\alpha'}{\beta'} = \lim_{x \rightarrow x_0} \frac{\alpha'}{\beta'}$$

ya'ni

$$\lim_{x \rightarrow x_0} \frac{\alpha}{\beta} = \lim_{x \rightarrow x_0} \frac{\alpha'}{\beta'}$$

**16-teorema.** Ikkita ekvivalent cheksiz kichik funksiyaning ayirmasi ularning har biriga nisbatan yuqori tartibli cheksiz kichik funksiya bo'ladi.

**17-teorema.** Chekli sondagi har xil tartibli cheksiz kichik funksiyalarning yig'indisi eng quyi tartibli qo'shiluvchiga ekvivalent bo'ladi.

Cheksiz kichik funksiyalarning yig'indisiga ekvivalent bo'lgan cheksiz kichik funksiya *bu yig'indining bosh qismi* deyiladi. Cheksiz kichik funksiyalarning yig'indisini uning bosh qismi bilan almashtirish *yuqori tartibli cheksiz kichik funksiyalarni tashlab yuborish* deb yuritiladi.

9-misol.  $\lim_{x \rightarrow 0} \frac{3x + 7x^2 + 4x^5}{2 \sin x}$  limitni toping.

Yechish.  $\lim_{x \rightarrow 0} \frac{3x + 7x^2 + 4x^5}{2 \sin x} = \lim_{x \rightarrow 0} \frac{3x}{2x} = \lim_{x \rightarrow 0} \frac{3}{2} = \frac{3}{2}$ , chunki  $x \rightarrow 0$  da  $\sin x \sim x$  va 17-teoremaga ko'ra,  $x \rightarrow 0$  da  $3x + 7x^2 + 4x^5 \sim 3x$ .

$\frac{0}{0}$  ko'rinishdagi aniqlanmasliklarni ochishda ekvivalent cheksiz kichik funksiyalarni almashtirish prinsipidan va ekvivalent cheksiz kichik funksiyalarning xossaligidan foydalanish mumkin.

Limitlarni hisoblashda quyidagi ekvivalentliklar qo'llaniladi.

- |   |   |
|---|---|
| 1. $x \rightarrow 0$ da $\sin x \sim x$ ;                     | 2. $x \rightarrow 0$ da $\operatorname{tg} x \sim x$ ;    |
| 3. $x \rightarrow 0$ da $\arcsin x \sim x$ ;                  | 4. $x \rightarrow 0$ da $\operatorname{arctg} x \sim x$ ; |
| 5. $x \rightarrow 0$ da $1 - \cos x \sim \frac{x^2}{2}$ ;     | 6. $x \rightarrow 0$ da $e^x - 1 \sim x$ ;                |
| 7. $x \rightarrow 0$ da $a^x - 1 \sim x \ln a$ ;              | 8. $x \rightarrow 0$ da $\ln(1+x) \sim x$ ;               |
| 9. $x \rightarrow 0$ da $\log_a(1+x) \sim x \cdot \log_a e$ ; |   |
| 10. $x \rightarrow 0$ da $(1+x)^n - 1 \sim nx$ .              |   |

10-misol.  $\lim_{x \rightarrow 0} \frac{2^{3x} - 7^x}{\sin 4x - \operatorname{arctg} 3x}$  limitni toping.

Yechish.  $\lim_{x \rightarrow 0} \frac{2^{3x} - 7^x}{\sin 4x - \operatorname{arctg} 3x} = \lim_{x \rightarrow 0} \frac{(2^{3x} - 1) - (7^x - 1)}{\sin 4x - \operatorname{arctg} 3x}$ .

$x \rightarrow 0$  da  $2^{3x} - 1 \sim 3x \ln 2$ ,  $7^x - 1 \sim x \ln 7$ ,  $\sin 4x \sim 4x$  va  $\operatorname{arctg} 3x \sim 3x$  ekvivalentliklardan foydalanamiz:

$$\lim_{x \rightarrow 0} \frac{2^{3x} - 7^x}{\sin 4x - \operatorname{arctg} 3x} = \lim_{x \rightarrow 0} \frac{3x \ln 2 - x \ln 7}{4x - 3x} = \frac{3 \ln 2 - \ln 7}{1} = \ln \frac{8}{7}.$$

### 4.4.3. Mashqlar

1. Funksiya limitining ta'rifi yordamida isbotlang:

1)  $\lim_{x \rightarrow 2} (2x - 3) = 1$ ;

2)  $\lim_{x \rightarrow -1} (1 - 3x) = 4$ ;

3)  $\lim_{x \rightarrow 1} x^2 = 1$ ;

4)  $\lim_{x \rightarrow 1} \left( \frac{2}{4-x} \right) = 2$ .

2.  $f(x)$  funksiyaning  $x = x_0$  nuqtalardagi chap va o'ng limitlarini toping:

1)  $f(x) = [x]$ ,  $x_0 = 3$ ;

2)  $f(x) = 2^{\frac{1}{x}}$ ,  $x_0 = 0$ ;

3)  $f(x) = \begin{cases} x, & x < 2, \\ x^2 - 4, & x \geq 2, \end{cases}$   $x_0 = 2$ ;

4)  $f(x) = \frac{2(1-x) - |1-x|}{4(1-x) + |1-x|}$ ,  $x_0 = 1$ .

3.  $f(x) = \operatorname{sign} x$  funksiyaning  $x_0 = 0$  nuqtada limitga ega emasligini ko'rsating.

4.  $f(x) = x - [x]$  funksiyaning  $x_0 = 2$  nuqtada limitga ega emasligini ko'rsating.

5. Limitlarni toping:

1)  $\lim_{x \rightarrow -3} (2x^2 + 3x - 1)$ ;

2)  $\lim_{x \rightarrow 2} \frac{3^x - 9}{3^x + 9}$ ;

3)  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 2x - 3}$ ;

4)  $\lim_{x \rightarrow 5} \frac{x^2 - 7x + 10}{2x^2 - 11x + 5}$ ;

5)  $\lim_{x \rightarrow 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2}$ ;

6)  $\lim_{x \rightarrow 1} \frac{\sqrt{2-x} - 1}{\sqrt{5-x} - 2}$ ;

7)  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{8-x} - 2}{x}$ ;

8)  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1}{x}$ ;

9)  $\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 + 6x + 3}{2x^2 + 3x + 1}$ ;

10)  $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^3 - x^2 - x + 1}$ ;

11)  $\lim_{x \rightarrow 2} \left( \frac{2x+1}{x-2} - \frac{x-7}{x^2-5x+6} \right)$ ;

12)  $\lim_{x \rightarrow 1} \left( \frac{3}{x^3-1} + \frac{1}{1-x} \right)$ ;

13)  $\lim_{x \rightarrow \infty} \frac{4x^4 - 3x + 2}{x^2 - 3x^4}$ ;

14)  $\lim_{x \rightarrow \infty} \frac{3x^5 - 4}{x^3 + 3x - x^5}$ ;

15)  $\lim_{x \rightarrow \infty} \frac{x^3 + 2x}{x^4 - 2x^2 + 3}$ ;

16)  $\lim_{x \rightarrow \infty} \frac{x^5 - 2x^2}{2x^3 + x - 4}$ ;

17)  $\lim_{x \rightarrow \infty} x(\sqrt{4x^2 - 1} - 2x)$ ;

18)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 - 4} + x)$ ;

19)  $\lim_{x \rightarrow \infty} \left( \frac{x^3}{x^2 - 2} - x \right)$ ;

20)  $\lim_{x \rightarrow \infty} \left( \frac{x^3}{5x^2 + 1} - \frac{x^2}{5x + 2} \right)$ ;

21)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} 2x}{x}$ ;

22)  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x + \sin x}$ ;

23)  $\lim_{x \rightarrow \frac{\pi}{2}} \left( \frac{\pi}{2} - x \right) \operatorname{tg} x$ ;

24)  $\lim_{x \rightarrow \pi} \frac{\sin 3x}{\sin 2x}$ ;

25)  $\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x \sin 2x}$ ;

26)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x - \sin x}{x^3}$ ;

$$27) \lim_{x \rightarrow 0} \frac{\sin 3x}{\sqrt{x+2} - \sqrt{2}};$$

$$29) \lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \operatorname{ctg} x \right);$$

$$31) \lim_{x \rightarrow 1} (x-1) \operatorname{ctg} \pi x;$$

$$33) \lim_{x \rightarrow -1} \frac{\arcsin(x+1)}{x^2 + x};$$

$$35) \lim_{x \rightarrow 0} \left( \frac{2x-1}{2x+1} \right)^{\frac{1}{3x-2}};$$

$$37) \lim_{x \rightarrow 0} \left( \frac{3x-2}{x+3} \right)^{\frac{1}{x-4}};$$

$$39) \lim_{x \rightarrow 2} \frac{e^x - e^2}{x-2};$$

$$41) \lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}};$$

$$43) \lim_{x \rightarrow 1} (3-2x)^{\frac{x}{2(1-x)}};$$

$$45) \lim_{x \rightarrow 0} \frac{e^{2x} - e^{3x}}{\operatorname{tg} x - 2 \sin x};$$

$$47) \lim_{x \rightarrow +\infty} (4x+1)(\ln(3x+2) - \ln(3x-1));$$

$$28) \lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x};$$

$$30) \lim_{x \rightarrow \frac{\pi}{2}} \left( \operatorname{tg} x - \frac{1}{\cos x} \right);$$

$$32) \lim_{x \rightarrow \frac{1}{2}} \left( \frac{1}{2} - x \right) \operatorname{tg} \pi x;$$

$$34) \lim_{x \rightarrow 2} \frac{\operatorname{arctg}(x-2)}{x^2 - 2x};$$

$$36) \lim_{x \rightarrow \infty} \left( \frac{3x-4}{3x+2} \right)^{\frac{4-x}{1}};$$

$$38) \lim_{x \rightarrow -\infty} \left( \frac{2x+3}{x+2} \right)^{\frac{4x}{1}};$$

$$40) \lim_{x \rightarrow e} \frac{\ln x - 1}{x - e};$$

$$42) \lim_{x \rightarrow 0} (\cos 2x)^{\frac{1}{1+\operatorname{ctg}^2 x}};$$

$$44) \lim_{x \rightarrow 2} (3-x)^{\frac{2x-3}{2-x}};$$

$$46) \lim_{x \rightarrow 0} \frac{e^{3x} - e^x}{\arcsin x + 3x};$$

$$48) \lim_{x \rightarrow +\infty} x(\ln(x+1) - \ln x).$$

### 6. Quyidagilarni isbotlang:

1)  $x \rightarrow 0$  da  $\alpha(x) = \operatorname{tg} 2x$  va  $\beta(x) = 3x + x^3$  funksiyalar bir xil tartibli;

2)  $x \rightarrow 1$  da  $\alpha(x) = \frac{x-1}{x+1}$  va  $\beta(x) = \sqrt{x} - 1$  funksiyalar ekvivalent;

3)  $x \rightarrow +\infty$  da  $\alpha(x) = \frac{1}{1+x^2}$  va  $\beta(x) = \frac{1}{x\sqrt{x+2}}$  funksiyalar uchun  $\alpha = o(\beta)$ ;

4)  $x \rightarrow 0$  da  $\alpha(x) = \arcsin 2x + x^2$  va  $\beta(x) = 1 - \cos x$  funksiyalar uchun  $\beta = o(\alpha)$ .

### 7. Limitlarni ekvivalent cheksiz kichik funksiyalardan foydalanib toping:

$$1) \lim_{x \rightarrow 0} \frac{\operatorname{tg} 2x}{\ln(1+3x)};$$

$$2) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 + 2x^3 + 3x^4};$$

$$3) \lim_{x \rightarrow 0} \frac{\operatorname{arctg} 3x}{\sin x - \sin 4x};$$

$$4) \lim_{x \rightarrow 0} \frac{3^{2x} - 1}{\arcsin 2x};$$

- 5)  $\lim_{x \rightarrow 2} \frac{\operatorname{tg} 5(x-2)}{x^2 + x - 6}$ ;      6)  $\lim_{x \rightarrow 1} \frac{x^2 + 3x - 4}{\operatorname{arctg}(x-1)}$ ;
- 7)  $\lim_{x \rightarrow 0} \frac{3^{\sin x} - 1}{\operatorname{tg} 2x}$ ;      8)  $\lim_{x \rightarrow 0} \frac{e^{\sin 2x} - 1}{\arcsin x + 2x^2}$ ;
- 9)  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1 + \sin^2 x} - 1}{1 - \cos x}$ ;      10)  $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \operatorname{tg} x} - 1}{x \arcsin 3x}$ ;
- 11)  $\lim_{x \rightarrow 0} \frac{\sin \sqrt{x}}{e^{\sqrt{x}} - e^{x\sqrt{x}}}$ ;      12)  $\lim_{x \rightarrow 0} \frac{e^{\sin x} - e^{3x}}{\operatorname{arctg} 2x - \arcsin 3x}$ ;
- 13)  $\lim_{x \rightarrow 0} \frac{e^{\operatorname{tg} x} - 1}{\ln(1 + \arcsin 2x)}$ ;      14)  $\lim_{x \rightarrow 0} \frac{3^{2x} - 5^x}{\arcsin 2x - x^3}$ ;
- 15)  $\lim_{x \rightarrow \pi} \frac{\sin 3x}{3 \operatorname{tg} 4x}$ ;      16)  $\lim_{x \rightarrow \pi} \frac{\ln(2 + \cos x)}{\sin x (e^{\operatorname{tg} x} - 1)}$ ;
- 17)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x - \sin x}{x^3 + 3x^4}$ ;      18)  $\lim_{x \rightarrow 0} \frac{x \ln(\cos 3x)}{\operatorname{tg} x - \sin x}$ ;
- 19)  $\lim_{x \rightarrow 0} \frac{e^{x^2} - \cos 2x}{x \sin x}$ ;      20)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{e^{\cos x} - 1}{x \cos x}$ ;
- 21)  $\lim_{x \rightarrow \infty} x \cdot (e^{1/x^3} - 1)$ ;      22)  $\lim_{x \rightarrow \infty} x \cdot (2^{1/x} - 3^{-1/x})$ ;
- 23)  $\lim_{x \rightarrow 0} \frac{(e^{2x} - 1) \cdot \operatorname{tg} 3x}{\ln(1 - 3x^2)(1 - \cos 2x)}$ ;      24)  $\lim_{x \rightarrow 0} \frac{(\sqrt{1 + \operatorname{tg} x} - 1) \cdot \sin 3x}{x(e^{\arcsin x} - 1)}$ ;

## 4.5. FUNKSIYANING UZLUKSIZLIGI

### 4.5.1. Funksiya uzluksizligining ta'riflari

$f(x)$  funksiya  $x_0$  nuqtada va uning biror atrofida aniqlangan bo'lsin.

**1-ta'rif.** Agar  $f(x)$  funksiya  $x_0$  nuqtada chekli limitga ega bo'lib, bu limit funksiyaning shu nuqtadagi qiymatiga teng, ya'ni

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \quad (5.1)$$

bo'lsa,  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz deyiladi.

$\lim_{x \rightarrow x_0} f(x) = f(x_0)$  tenglik uchta shartning bajarilishini anglatadi:

1)  $f(x)$  funksiya  $x_0$  nuqtada va uning atrofida aniqlangan;



2)  $f(x)$  funksiya  $x \rightarrow x_0$  da limitga ega;

3) funksiyaning  $x_0$  nuqtadagi limiti uning shu nuqtadagi qiymatiga teng.

$x_0 = \lim_{x \rightarrow x_0} x$  ekanidan (5.1) tenglikni

$$\lim_{x \rightarrow x_0} f(x) = f(\lim_{x \rightarrow x_0} x) \quad (5.2)$$

ko'rinishda yozish mumkin. Demak, uzluksiz funksiya uchun limitga o'tish va funksiya belgilarining o'rnini almashtirish mumkin.

Funksiya limitning ta'rifi asosida funksiya uzluksizligining ta'rifini « $\varepsilon - \delta$  tilida» quyidagicha ifodalash mumkin.

**2- ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $\delta > 0$  son topilsaki,  $x$  ning  $|x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha qiymatlarida  $|f(x) - f(x_0)| < \varepsilon$  tengsizlik bajarilsa,  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz deyiladi.

(5.1) tenglikni

$$\lim_{x \rightarrow x_0} (f(x) - f(x_0)) = 0 \quad (5.3)$$

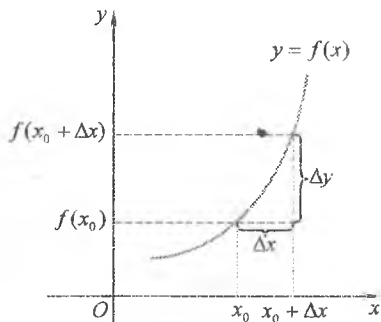
ko'rinishda yozamiz.

$x - x_0$  ayirmaga  $x$  argumentning  $x_0$  nuqtadagi orttirmasi deyiladi va  $\Delta x$  bilan belgilanadi,  $f(x) - f(x_0)$  ayirmaga esa  $f(x)$  funksiyaning  $x_0$  nuqtadagi orttirmasi deyiladi va  $\Delta y$  bilan belgilanadi.

Shunday qilib,  $\Delta x = x - x_0$ ,  
 $\Delta y = f(x_0 + \Delta x) - f(x_0)$ .

Demak,  $f(x)$  funksiyaning  $x_0$  nuqtadagi orttirmasi  $x$  ning fiksirlangan  $x_0$  qiymatida argument orttirmasining funksiyasi bo'ladi (28-shakl).

(5.3) tenglik yangi belgilashlarda



28-shakl.

$$\lim_{\Delta x \rightarrow 0} \Delta y = 0 \quad (5.4)$$

ko'rinishni oladi.

(5.4) tenglikni uzluksizlikning argument orttirmasi va funksiya orttirmasi tushunchalariga asoslangan ta'rifi sifatida quyidagicha ifodalash mumkin.

**3-ta'rif.** Agar  $x$  argumentning  $x_0$  nuqtadagi cheksiz kichik ortirmasiga  $f(x)$  funksiyaning shu nuqtadagi cheksiz kichik ortirmasi mos kelsa,  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz deyiladi.

Funksiyaning nuqtadagi uzluksizligini tekshirishda keltirilgan ta'riflarning istalgan biridan foydalanish mumkin.

*1-misol.*  $y = \cos x$  funksiyani uzluksizlikka tekshiring.

*Yechish.*  $y = \cos x$  funksiya  $x \in R$  da aniqlangan. Istalgan  $x$  nuqtani olamiz va bu nuqtada  $\Delta y$  ni topamiz:

$$\Delta y = \cos(x + \Delta x) - \cos x = -2 \sin\left(x + \frac{\Delta x}{2}\right) \cdot \sin \frac{\Delta x}{2}.$$

Bundan  $\lim_{\Delta x \rightarrow 0} \Delta y = \lim_{\Delta x \rightarrow 0} \left(-2 \sin\left(x + \frac{\Delta x}{2}\right) \cdot \sin \frac{\Delta x}{2}\right) = 0$  kelib chiqadi, chunki

chegaralangan  $\sin\left(x + \frac{\Delta x}{2}\right)$  funksiyaning cheksiz kichik  $\sin \frac{\Delta x}{2}$

funksiyaga ko'paytmasi cheksiz kichik bo'ladi.

Demak, 3-ta'rifga ko'ra,  $y = \cos x$  funksiya  $x$  nuqtada uzluksiz.

**4-ta'rif.** Agar  $\lim_{x \rightarrow x_0 + 0} f(x) = f(x_0) \left( \lim_{x \rightarrow x_0 - 0} f(x) = f(x_0) \right)$  bo'lsa,  $f(x)$  funksiya  $x_0$  nuqtada o'ngdan (chapdan) uzluksiz deyiladi.

1-ta'rif va 4-ta'riflardan quyidagi xulosa kelib chiqadi:  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz bo'lishi uchun u shu nuqtada ham chapdan, ham o'ngdan uzluksiz bo'lishi zarur va yetarli.

#### 4.5.2. Uzluksiz funksiyalarning xossalari

##### *Nuqtada uzluksiz funksiyalarning xossalari*

**1-teorema** (uzluksiz funksiyalar ustida arifmetik amallar).  $f(x)$  va  $g(x)$  funksiyalar  $x_0$  nuqtada uzluksiz bo'lsa, u holda  $f(x) \pm g(x)$ ,  $f(x) \cdot g(x)$  va  $\frac{f(x)}{g(x)}$  ( $g(x_0) \neq 0$ ) funksiyalar ham  $x_0$  nuqtada uzluksiz bo'ladi.

*Isboti.*  $f(x)$  va  $g(x)$  funksiyalar  $x_0$  nuqtada uzluksiz bo'lgani uchun ular bu nuqtada  $f(x_0)$  va  $g(x_0)$  limitlarga ega. U holda

funksiyaning limiti haqidagi teoremlarga ko'ra,  $f(x) \pm g(x)$ ,  $f(x) \cdot g(x)$  va  $\frac{f(x)}{g(x)}$  ( $g(x_0) \neq 0$ ) funksiyalarning  $x_0$  nuqtadagi limitlari mavjud va ular mos ravishda  $f(x_0) \pm g(x_0)$ ,  $f(x_0) \cdot g(x_0)$  va  $\frac{f(x_0)}{g(x_0)}$  ( $g(x_0) \neq 0$ ) ga teng bo'ladi. U holda 1-ta'rifga ko'ra,  $f(x) \pm g(x)$ ,  $f(x) \cdot g(x)$  va  $\frac{f(x)}{g(x)}$  ( $g(x_0) \neq 0$ ) funksiyalar  $x_0$  nuqtada uzluksiz.

Bu teorema chekli sondagi funksiyalarning algebraik yig'indisi va ko'paytmasi uchun ham o'rinli bo'ladi.

**2-teorema.** (*murakkab funksiyaning uzluksizligi*).  $z = \varphi(x)$  funksiya  $x_0$  nuqtada uzluksiz,  $y = f(z)$  funksiya esa  $z_0 = \varphi(x_0)$  nuqtada uzluksiz bo'lsin. U holda  $y = f(\varphi(x))$  murakkab funksiya  $x_0$  nuqtada uzluksiz bo'ladi.

*Isboti.*  $z = \varphi(x)$  funksiya  $x_0$  nuqtada uzluksizligidan  $z = \varphi(x)$ ,  $\lim_{x \rightarrow x_0} \varphi(x) = \varphi(x_0)$ , ya'ni  $x \rightarrow x_0$  da  $z \rightarrow z_0$  bo'ladi.

Shu sababli  $z = \varphi(x)$  funksiyaning uzluksiligidan

$$\lim_{x \rightarrow x_0} f(\varphi(x)) = \lim_{z \rightarrow z_0} f(z) = f(z_0) = f(\varphi(x_0))$$

kelib chiqadi. Bu  $f(\varphi(x))$  murakkab funksiyaning  $x_0$  nuqtada uzluksizligini bildiradi.

2-teorema yordamida (5.2) tenglikni quyidagicha umumilashtirish mumkin.

Agar  $z = \varphi(x)$  funksiya  $x_0$  nuqtada  $A$  limitga ega bo'lib,  $y = f(z)$  funksiya  $z = A$  nuqtada uzluksiz bo'lsa, u holda  $y = f(\varphi(x))$  murakkab funksiya uchun

$$\lim_{x \rightarrow x_0} f(\varphi(x)) = f(\lim_{x \rightarrow x_0} \varphi(x)) \quad (5.5)$$

bo'ladi.

Bu tenglik *uzluksiz funksiya belgisi ostida limitga o'tish qoidasini* ifodalaydi va funksiyaning limitini topishda foydalaniladi.

2- misol.  $\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x}$  ( $a > 0, a \neq 1$ ) limitni toping.

$$Yechish. \lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \log_a(1+x) = \lim_{x \rightarrow 0} \log_a(1+x)^{\frac{1}{x}}$$

$\log_a(1+x)^{\frac{1}{x}}$  funksiya  $y = \log_a z$  va  $z = (1+x)^{\frac{1}{x}}$  funksiyalarning murakkab funksiyasi.  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$  va  $y = \log_a z$  funksiya  $z = e$  nuqtada uzluksiz. U holda (5.5) tenglikka ko'ra,

$$\lim_{x \rightarrow 0} \log_a (1+x)^{\frac{1}{x}} = \log_a \left( \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right) = \log_a e.$$

Xususan,  $a=1$  da  $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$ .

Sonlar o'qida aniqlangan  $f(x) = C$  funksiyani qaraymiz.  $\forall x_0 \in R$  da  $\lim_{x \rightarrow x_0} f(x) = C = f(x_0)$  bo'ladi. Demak,  $f(x) = C$  o'zgarmas funksiya istalgan  $x_0$  nuqtada uzluksiz.

$f(x) = x$  funksiya ham  $x_0$  nuqtada uzluksiz, chunki  $\lim_{x \rightarrow x_0} x = x_0$ .

Bundan 1-teoreмага ko'ra,  $f(x) = x$  funksiya ko'paytmalaridan iborat  $y = x^n$  ( $n \in N$ ) darajali funksiya hamda o'zgarmas va darajali funksiyalardan arifmetik amallar orqali hosil qilingan  $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  ko'phad (butun-ratsional funksiya) istalgan  $x_0 \in R$  nuqtada uzluksiz bo'ladi.

Shu kabi yuqorida keltirilgan teoremlar va limitlar haqidagi teoremlar yordamida asosiy elementar funksiyalar o'zining aniqlanish sohasida uzluksiz bo'lishini ko'rsatish va ushbu teoremani isbotlash mumkin.

**3-teorema.** Elementar funksiyalar o'zining aniqlanish sohasidagi barcha nuqtalarda uzluksiz bo'ladi.

**4-teorema.** Agar  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz va  $f(x_0) > A$  ( $f(x_0) < A$ ) bo'lsa, u holda shunday  $\delta > 0$  son topiladi va  $\forall x \in (x_0 - \delta; x_0 + \delta)$  uchun  $f(x) > A$  ( $f(x) < A$ ) bo'ladi.

*Isboti.*  $f(x_0) > A$  bo'lsin. Aniqlik uchun  $f(x_0) = A + h$  deymiz, bu yerda  $h > 0$ .  $\varepsilon = \frac{h}{2}$  son olamiz.  $f(x)$  funksiyaning  $x_0$  nuqtada uzluksizligidan shunday  $\delta > 0$  son topiladi va  $x$  ning  $|x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha

qiymatlarida  $|f(x) - f(x_0)| < \frac{h}{2}$  tengsizlik bajariladi. Bundan  $\forall x \in (x_0 - \delta; x_0 + \delta)$  uchun

$$f(x) > f(x_0) - \frac{h}{2} = A + h - \frac{h}{2} = A + \frac{h}{2} > A$$

bo'ldi.

$f(x_0) < A$  bo'lsin.  $-f(x)$  funksiyani qaraymiz.  $-f(x_0) > -A$  bo'lgani sababli yuqoridagi isbotga asosan,  $x_0$  nuqtaning  $\delta > 0$  atrofi topiladi va bu atrofda  $-f(x) > -A$  yoki  $f(x) < A$  bo'ldi.

**5-teorema** (*uzluksiz funksiya ishorasining turg'unligi*). Agar  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz va  $f(x_0) \neq 0$  bo'lsa, u holda shunday  $\delta > 0$  son topiladi va  $(x_0 - \delta; x_0 + \delta)$  intervalda  $f(x)$  funksiya ishorasini saqlaydi, ya'ni  $f(x_0)$  funksiya bilan bir ishorali bo'ldi.

Teoremaning isboti 4-teoremadan  $A=0$  bo'lganda kelib chiqadi.

### *Kesmada uzluksiz funksiyalarning xossalari*

Agar  $f(x)$  funksiya  $(a; b)$  intervalning har bir nuqtasida uzluksiz bo'lsa, u holda u  $(a; b)$  intervalda uzluksiz deyiladi.

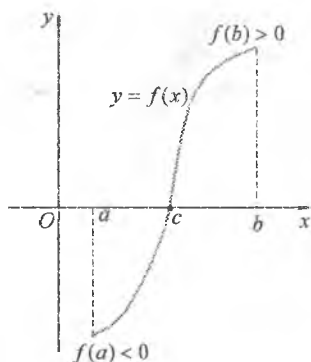
Agar  $f(x)$  funksiya  $(a; b)$  intervalda uzluksiz bo'lib,  $a$  nuqtada o'ngdan uzluksiz va  $b$  nuqtada chapdan uzluksiz bo'lsa, u holda  $f(x)$  funksiyaga  $[a; b]$  kesmada uzluksiz deyiladi.

Kesmada uzluksiz funksiyalar bir qancha muhim xossalarga ega. Bu xossalarni teoremlar orqali ifodalaymiz. Bunda teoremlarning isbotini keltirmasdan, faqat geometrik talqinini ko'rsatish bilan kifoyalanamiz.

**6-teorema** (*Bolsano-Koshining birinchi teoremasi*).  $f(x)$  funksiya  $[a; b]$  kesmada uzluksiz va kesmaning oxirlarida turli ishorali qiymatlar qabul qilinsin. U holda shunday  $c \in (a; b)$  nuqta topiladiki, bu nuqtada  $f(c) = 0$  bo'ldi.

Teoremaning geometrik talqini: uzluksiz funksiyaning grafigi  $Ox$  o'qning bir tomonidan ikkinchi tomoniga o'tganida  $Ox$  o'qni kesadi (29-shakl).

**7-teorema** (*Bolsano-Koshining ikkinchi teoremasi*).  $f(x)$  funksiya  $[a; b]$  kesmada uzluksiz va  $f(a) = A$ ,  $f(b) = B$ ,  $C - A$  va  $B$  orasidagi ixtiyoriy son bo'lsin. U holda shunday  $c \in [a; b]$  nuqta topiladiki,  $f(c) = C$  bo'ldi.



29-shakl.

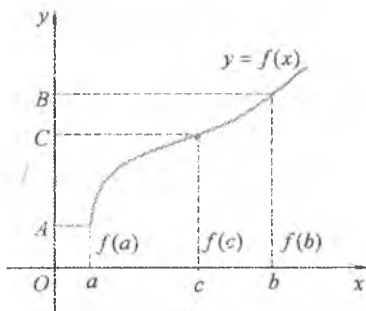
Teoremaning geometrik talqini: uzluksiz funksiya bir qiymatdan ikkinchi qiymatga o'tganida barcha oraliq qiymatlarni qabul qiladi (30-shakl).

**8-teorema** (*Veyershtrassning birinchi teoremasi*): Agar  $f(x)$  funksiya  $[a;b]$  kesmada uzluksiz bo'lsa, u holda u bu kesmada chegaralangan bo'ladi.

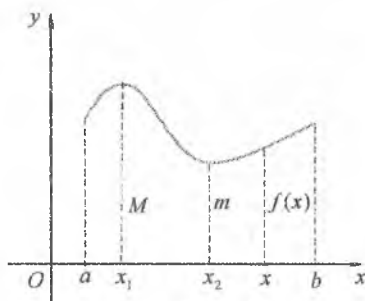
31-shaklda keltirilgan  $y=f(x)$  funksiya  $[a;b]$  kesmada uzluksiz. Bunda  $\forall x \in [a;b]$  uchun  $m \leq f(x) \leq M$ .

**1-izoh.** Teorema  $[a;b]$  kesma  $(a;b)$  interval bilan almashtirilganida o'rinli bo'lmisligi mumkin.

Masalan,  $f(x) = \frac{1}{x}$  funksiya  $(0;1)$  intervalda uzluksiz, lekin chegaralanmagan, chunki  $\lim_{x \rightarrow 0} \frac{1}{x} = +\infty$ .



30-shakl.



31-shakl.

**9-teorema** (*Veyershtrassning ikkinchi teoremasi*): Agar  $f(x)$  funksiya  $[a;b]$  kesmada uzluksiz bo'lsa, u holda u shu kesmada o'zining eng kichik va eng katta qiymatlariga erishadi.

31-shaklda keltirilgan  $y=f(x)$  funksiya  $[a;b]$  kesmada uzluksiz. Bunda u  $x_1$  nuqtada o'zining eng katta  $M$  qiymatini va  $x_2$  nuqtada o'zining eng kichik  $m$  qiymatini qabul qiladi.

**2-izoh.** Bu teorema  $(a;b)$  interval uchun o'rinli bo'lmisligi mumkin. Masalan,  $f(x)=x$  funksiya  $(0;1)$  intervalda uzluksiz, lekin

o'zining eng kichik va eng katta qiymatlariga erishmaydi.

**10-teorema** (teskari funksiyaning uzluksizligi haqidagi). Agar  $y = f(x)$  funksiya  $[a; b]$  kesmada uzluksiz va qat'iy monoton bo'lib,  $[c; d]$  uning qiymatlar sohasi bo'lsa, u holda berilgan funksiyaga teskari  $y = \varphi(x)$  funksiya  $[c; d]$  kesmada uzluksiz va qat'iy monoton bo'ladi.

### 4.5.3. Funksiyaning uzulish nuqtalari

Agar  $f(x)$  funksiya uchun  $x_0$  nuqtada funksiya uzluksizligi 1-ta'rifining hech bo'lmaganda bitta sharti bajarilmasa, *funksiya  $x_0$  nuqtada uzilishga ega* deyiladi. Bunda  $x_0$  nuqta *f(x) funksiyaning uzilish nuqtasi* deb ataladi.

32-shaklda g'rafiklar bilan berilgan funksiyalarga qaraymiz.

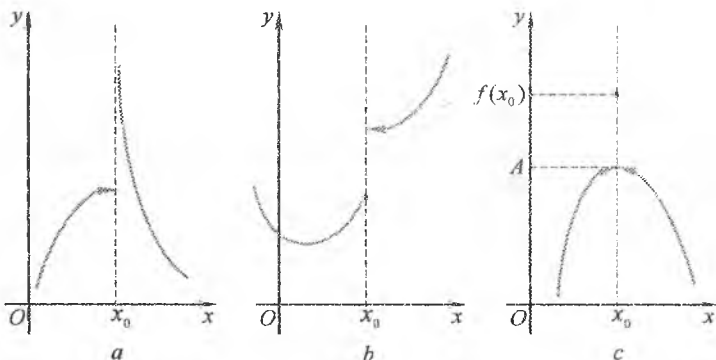
Bu funksiyalarning har biri uchun  $x_0$  – uzilish nuqtasi.

Birinchi holda (32,a-shakl) ta'rifning 1-sharti bajarilmaydi, chunki funksiya  $x_0$  nuqtada aniqlanmagan.

Ikkinchi holda (32,b-shakl) ta'rifning 2-sharti buzilgan, chunki  $\lim_{x \rightarrow x_0} f(x)$  limit mavjud emas.

Uchinchi holda (32,c-shakl) ta'rifning 3-sharti bajarilmaydi, chunki  $\lim_{x \rightarrow x_0} f(x) = A \neq f(x_0)$ .

Funksiyaning barcha uzilish nuqtalari birinchi va ikkinchi tur uzilish nuqtalariga bo'linadi.



32-shakl.

**5-ta'rif.** Agar  $x_0$  uzilish nuqtasida  $f(x)$  funksiya chekli bir tomonlama limitlarga ega, ya'ni  $\lim_{x \rightarrow x_0, -0} f(x) = A_1$  va  $\lim_{x \rightarrow x_0, +0} f(x) = A_2$  bo'lsa,  $x_0$  nuqtaga  $f(x)$  funksiyaning *birinchi tur uzilish nuqtasi* deyiladi. Bunda:

a)  $A_1 = A_2$  bo'lsa,  $x_0$  *bartaraf qilinadigan uzilish nuqtasi* deb ataladi;

b)  $A_1 \neq A_2$  bo'lsa,  $x_0$  *sakrash nuqtasi*,  $|A_1 - A_2|$  kattalik *funksiyaning sakrashi* deb ataladi.

Masalan:  $g(x) = \begin{cases} 2x-1, & -1 \leq x < 1, \\ 4-2x, & 1 \leq x \leq 3 \end{cases}$  funksiya uchun  $x_0 = 1$  - sakrash

nuqtasi, bunda funksiyaning sakrashi  $|1-2|=1$  ga teng;

$\varphi(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 2, & x = 0 \end{cases}$  funksiya uchun  $x_0 = 0$  - bartaraf qilinadigan

uzilish nuqtasi, bunda  $\varphi(x) = 2$  o'rniga  $\varphi(x) = 1$  deb olinsa uzilish

bartaraf qilinadi, ya'ni  $\varphi(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$  uzluksiz funksiya hosil

bo'ladi.

**6-ta'rif.** Agar  $x_0$  uzilish nuqtasida  $f(x)$  funksiyaning bir tomonlama limitlaridan kamida bittasi mavjud bo'lnasa yoki cheksizlikka teng bo'lsa,  $x_0$  nuqtaga  $f(x)$  funksiyaning *ikkinchi tur uzilishi nuqtasi* deyiladi.

Masalan,  $f(x) = \frac{1}{x}$  funksiya uchun  $x_0 = 0$  - ikkinchi tur uzilish nuqtasi.

*3-misol.*  $f(x) = \frac{|2x-3|}{2x-3}$  funksiyaning uzilish nuqtalarini toping va

har bir uzilish nuqtasining turini aniqlang.

*Yechish.* Funksiya sonlar o'qining  $x = \frac{3}{2}$  nuqtasidan boshqa nuqtalarida aniqlangan va uzluksiz.



Bunda

$$f(x) = \begin{cases} -1, & x < \frac{3}{2}, \\ 1, & x > \frac{3}{2}. \end{cases}$$

U holda

$$f\left(\frac{3}{2}-0\right) = -1, \quad f\left(\frac{3}{2}+0\right) = 1.$$

Demak,  $x = \frac{3}{2}$  sakrash nuqtasi va funksiyaning sakrashi  $\mu = |1 - (-1)| = 2$ .

#### 4.5.4. Tekis uzluksizlik

$f(x)$  funksiya  $(a; b)$  intervalda uzluksiz bo'lsin. U holda istalgan  $x_0 \in (a; b)$  nuqtada  $\forall \varepsilon > 0$  son uchun shunday  $\delta > 0$  son topiladi va  $|x - x_0| < \delta$  tengsizlikni qanoatlantiruvchi barcha  $x_0 \in (a; b)$  uchun  $|f(x) - f(x_0)| < \varepsilon$  tengsizlik bajariladi. Bunda  $\delta$  ham  $\varepsilon$  ga, ham  $x_0$  ga bog'liq bo'ladi:  $\delta = \delta(\varepsilon; x_0)$ . Bitta  $\varepsilon > 0$  son uchun har xil  $x \in (a; b)$  nuqtalarda  $\delta$  son turlicha bo'lishi mumkin va bunda barcha  $x \in (a; b)$  da yagona  $\delta$  sonning mavjud bo'lishi kelib chiqmaydi. Bunday  $\delta = \delta(\varepsilon) > 0$  son mavjud bo'lishining talabi  $f(x)$  funksiyaning  $(a; b)$  intervalda uzluksiz bo'lishi talabiga nisbatan kuchli talab hisoblanadi. •

**7-ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $\delta = \delta(\varepsilon) > 0$  son topilsa va  $(a; b)$  intervalning  $|x' - x''| < \delta$  tengsizlikni qanoatlantiruvchi ixtiyoriy  $x'$  va  $x''$  sonlari uchun  $|f(x') - f(x'')| < \varepsilon$  tengsizlik bajarilsa,  $f(x)$  funksiya  $(a; b)$  intervalda tekis uzluksiz deyiladi.

Masalan,  $f(x) = x$  funksiya butun sonlar o'qida tekis uzluksiz. Bunda  $\delta = \varepsilon$  deb olish yetarli.

Agar  $f(x)$  funksiya  $(a; b)$  intervalda tekis uzluksiz bo'lsa, u holda u har bir  $x \in (a; b)$  nuqtada uzluksiz bo'ladi. Teskari tasdiq o'rinli bo'lmaydi. Agar bunda  $(a; b)$  interval  $[a; b]$  kesma bilan almashtirilsa, teskari tasdiq ham o'rinli bo'ladi.

**11-teorema (Kantor teoremasi).** Agar  $f(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lsa, u holda u  $[a; b]$  kesmada tekis uzluksiz bo'ladi.

### 4.5.5. Mashqlar

1. Funksiyaning uzluksizligi ta'rifidan foydalanib, berilgan funksiyalarning  $\forall x_0 \in R$  da uzluksiz ekanini isbotlang:

1)  $f(x) = 3x^2 - 7$ ;

2)  $f(x) = x^3 + 7x - 6$ .

2. Uzluksiz funksiyalarning xossalaridan foydalanib, berilgan funksiyalarning  $(-\infty; +\infty)$  intervalda uzluksiz ekanini isbotlang:

1)  $f(x) = \cos 3x - e^{2x-1}$ ;

2)  $f(x) = \sqrt[3]{x-3} + \sin^2 x + \frac{3}{x^2+2}$ .

3. Berilgan funksiyalarni uzluksizlikka tekshiring va grafagini chizing:

1)  $f(x) = \frac{x}{|x|}$ ;

2)  $f(x) = x^2 + \frac{|x+1|}{x+1}$ ;

3)  $f(x) = \begin{cases} x^2, & x \neq 2, \\ 3, & x = 2; \end{cases}$

4)  $f(x) = \begin{cases} 3x-1, & x < 0, \\ \frac{1}{x-1}, & x \geq 0; \end{cases}$

5)  $f(x) = 2^{\frac{x}{x^2-1}}$ ;

6)  $f(x) = \frac{3}{1+2^{1/x}}$ ;

7)  $f(x) = \begin{cases} 1, & x < -3, \\ \sqrt{9-x^2}, & -3 \leq x \leq 3, \\ x-3, & x > 3; \end{cases}$

8)  $f(x) = \begin{cases} x^2, & x \leq 2, \\ 4, & 2 < x < 5, \\ -x+7, & x \geq 5; \end{cases}$

9)  $f(x) = \frac{|x-3|}{x^2-2x-3}$ ;

10)  $f(x) = \frac{|\sin x|}{(x-1)\sin x}$ .

4.  $a$  ning qanday qiymatlarida berilgan funksiyalar uzluksiz bo'ladi?

1)  $f(x) = \begin{cases} x^2+3x-10, & x < 2, \\ x-2, & x = 2, \\ a^2-x, & x \geq 2; \end{cases}$

2)  $f(x) = \begin{cases} 3^x, & x \geq 0, \\ a \cos x + 2, & x < 0. \end{cases}$

5.  $f(x)$  funksiyaning  $x_0$  nuqtadagi uzulish turini aniqlang:

1)  $f(x) = \frac{3x+4}{x-3}$ ,  $x_0 = 3$ ;

2)  $f(x) = \frac{x^2-9}{x+3}$ ,  $x_0 = -3$ ;

3)  $f(x) = \operatorname{arctg} \frac{5}{2x-1}$ ,  $x_0 = \frac{1}{2}$ ;

4)  $f(x) = \frac{3}{4^{x-3}-1}$ ,  $x_0 = 3$ .

6. Murakkab funksiyani uzluksizlikka tekshiring:

1)  $f(z) = \frac{2}{z^2+1}$ ,  $z = \begin{cases} x+2, & x < 0, \\ x-2, & x \geq 0; \end{cases}$

2)  $f(z) = 2z^2 - 3$ ,  $z = \operatorname{tg} x$ .

7.  $f(x) = \frac{1}{(x+3)(x-4)}$  funksiyani  $[a, b]$  kesmada uzluksizlikka tekshiring:

1)  $[a, b] = [-4; 1]$ ;

2)  $[a, b] = [-2; 3]$ .

8.  $f(x)$  funksiyani  $[0; 2], [-3; 1], [4; 5]$  kesmalarda uzluksizlikka tekshiring:

1)  $f(x) = \frac{1}{x^2 + 2x - 3}$ ;

2)  $f(x) = \ln \frac{x-4}{x+5}$ .

9. Tenglamalar berilgan kesmada kamida bitta ildizga ega bo'lishini ko'rsating:

1)  $x^3 - 5x^2 + 3x + 2 = 0$ ,  $[-1; 1]$ ;

2)  $\sin x - x + 1 = 0$ ,  $[1; 2]$ .

10. Limitlarni toping:

1)  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x}$  ( $a > 0, a \neq 0$ ),

2)  $\lim_{x \rightarrow 0} \frac{(1+x)^m - 1}{x}$ .

- Funksiyaning hosilasi va differensial
- Differensiallash qoidalari va formulalari
- Differensial hisobning asosiy teoremlari
- Funksiyalarni hosilalar yordamida tekshirish



**Gotfrid Vilgelm Leybnis**  
(1646–1716) –  
nemis matematigi,  
faylasufi, fizigi,  
xuzuqshunosi  
va tilshunosi.

Leybnis cheksiz kichiklarga asostangan differensial va integral hisobni yaratgan, kombinatorikani fan sifatida asostagan, matematik mantiqqa asos solgan, 0 va 1 sonlari bilan ikkilik sanoq sistemasi ni tasvirlagan.

Mexanikada energiya-ning saqlanish qonuni-ni asostlab bergan.

# BIR O'ZGARUVCHI FUNKSIYASINING DIFFERENSIAL HISOBI

*Differensial hisob* – bu matematik analizning hosila va differensial tushunchalari hamda ularning funksiyalarni tekshirishga tatbiqi masalalari o'rganiladigan bo'limidir. Differensial hisobning rivojlanishi integral hisobning rivojlanishi bilan uzviy bog'liq. Ular birgalikda tabiatshunoslik va texnika uchun muhim ahamiyatga ega bo'lgan matematik analizning asosini tashkil qiladi.

Differensial hisobning matematik fan sifatida yuzaga chiqishini, odatda, *I.Nyuton* va *G.Leybnis* (XVII asrning ikkinchi yarmida) nomlari bilan bog'lashadi. Ular differensial hisobning asosiy qoidalarini mustaqil ravishda ishlab chiqishgan.

## 5.1. FUNKSIYANING HOSILASI VA DIFFERENSIALI

### 5.1.1. Hosila tushunchasiga olib keluvchi masalalar

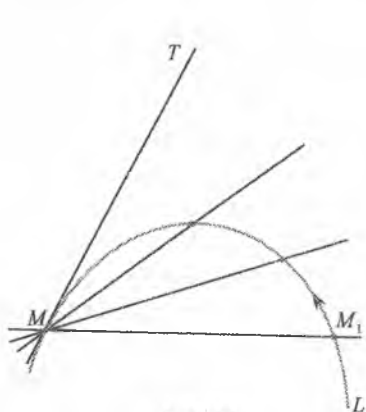
#### *Egri chiziqqa o'tkazilgan urinma*

Avval egri chiziqqa o'tkazilgan urinmaning umumiy ta'rifini beramiz. Uzuksiz  $L$  egri chiziqda  $M$  va  $M_1$  nuqtalarni olamiz (1-shakl).

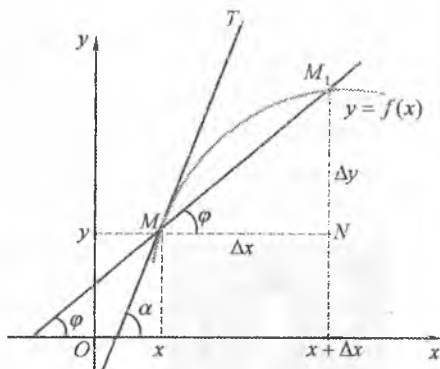
$M$  va  $M_1$  nuqtalar orqali o'tuvchi  $MM_1$  to'g'ri chiziqqa *kesuvchi* deyiladi.

$M$ , nuqta  $L$  egri chiziq bo‘ylab siljib,  $M$  nuqtaga yaqinlashsin. U holda  $MM_1$  kesuvchi  $M$  nuqta atrofida buriladi va qandaydir  $MT$  limit holatiga intiladi.

Berilgan  $L$  egri chiziqqa berilgan  $M$  nuqtada o‘tkazilgan urinma deb,  $MM_1$  kesuvchining  $M_1$  nuqta  $L$  egri chiziq bo‘ylab siljib,  $M$  nuqtaga yaqinlashgandagi  $MT$  limit holatiga (agar mavjud bo‘lsa) aytiladi.



1-shakl.



2-shakl.

Endi  $M(x; y)$  nuqtada vertikal bo‘lmagan urinmaga ega bo‘lgan  $y = f(x)$  uzluksiz funksiya grafigini qaraymiz va uning  $k = \operatorname{tg} \alpha$  burchak koeffitsiyentini topamiz, bu yerda  $\alpha$  – urinmaning  $Ox$  o‘q bilan tashkil qilgan burchagi. Buning uchun  $M$  nuqta va grafikning  $x + \Delta x$  absissali  $M_1$  nuqtasi orqali kesuvchi o‘tkazamiz (2-shakl). Kesuvchining  $Ox$  o‘q bilan tashkil qilgan burchagini  $\varphi$  bilan belgilaymiz.

2-shakldan topamiz:

$$\operatorname{tg} \varphi = \frac{M_1 N}{MN} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$\Delta x \rightarrow 0$  da funksiyaning uzluksizligiga asosan,  $\Delta y$  ham nolga

intiladi. Shu sababli  $\Delta x \rightarrow 0$  da  $M_1$  nuqta egri chiziq bo‘ylab siljib,  $M$  nuqtaga yaqinlashadi. Bunda  $MM_1$  kesuvchi  $M$  nuqta atrofida buriladi va  $MT$  urinmaga yaqinlashib boradi, ya‘ni  $\varphi \rightarrow \alpha$ . Bundan  $\lim_{\Delta x \rightarrow 0} \varphi = \alpha$  yoki  $\lim_{\Delta x \rightarrow 0} \operatorname{tg} \varphi = \operatorname{tg} \alpha$ .

Shuning uchun urinmaning burchak koeffitsiyenti

$$k = \operatorname{tg} \alpha = \lim_{\Delta x \rightarrow 0} \operatorname{tg} \varphi = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (1.1)$$

### To'g'ri chiziqli harakat tezligi

$M$  moddiy nuqta (biror jism) qandaydir to'g'ri chiziq bo'ylab harakat qilayotgan bo'lsin. Vaqtning har bir  $t$  qiymatiga boshlang'ich  $M_0$  holatdan  $M$  holatgacha bo'lgan muayyan  $s = M_0M$  masofa mos keladi. Bu masofa  $t$  vaqtga bog'liq, ya'ni  $s$  masofa vaqtning funksiyasi bo'ladi:  $s = s(t)$ .

$s(t)$  funksiyaga *nuqtaning harakat qonuni* deyiladi.

Nuqtaning  $t$  vaqtdagi harakat tezligini aniqlash masalasini qaraymiz.

Agar biror  $t$  vaqtda nuqta  $M$  holatda bo'lsa, u holda  $t + \Delta t$  ( $\Delta t$  – vaqtning orttirmasi) vaqtda nuqta  $M_1$  holatga o'tadi, bu yerda  $M_0M_1 = s + \Delta s$  ( $\Delta s$  – masofaning orttirmasi). Demak,  $M$  nuqtaning  $\Delta t$  vaqt oralig'idagi ko'chishi  $\Delta s = s(t + \Delta t) - s(t)$  ga teng bo'ladi (3-shakl).

$\frac{\Delta s}{\Delta t}$  nisbat *nuqtaning  $\Delta t$  vaqt oralig'idagi o'rtacha tezligini*

ifodalaydi:  $v_{or} = \frac{\Delta s}{\Delta t}$ . Bunda

o'rtacha tezlik  $\Delta t$  qiymatga bog'liq bo'ladi:  $\Delta t$  qancha kichik bo'lsa, o'rtacha tezlik nuqtaning

berilgan  $t$  vaqtdagi tezligini shuncha aniq ifodalaydi.

Harakat o'rtacha tezligining  $\Delta t$  vaqt oralig'i nolga intilgandagi limitiga *nuqtaning berilgan vaqtdagi harakat tezligi* (yoki *oniq tezligi*) deyiladi. Bu  $t$  ondagi tezlikni  $v(t)$  bilan belgilaymiz.

U holda

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} \quad \text{yoki} \quad v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}. \quad (1.2)$$

Shunday qilib, nuqtaning berilgan  $t$  vaqtdagi harakat tezligini aniqlash uchun (1.2) limitni hisoblash kerak bo'ladi.

Tabiatning turli sohalariga tegishli ko'plab masalalar (1.1) va (1.2) ko'rinishdagi limitlarni topishga olib keladi.

Bunday masalalardan yana ayrimlarini keltiramiz:

1) agar  $Q=Q(t)$  o'tkazgichning ko'ndalang kesimi orqali vaqtning  $t$  onida o'tuvchi elektr toki bo'lsa, u holda *elektr tokining  $t$  ondagi momenti*

$$I(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{Q(t + \Delta t) - Q(t)}{\Delta t}; \quad (1.3)$$

2) agar  $N=N(t)$  vaqtning  $t$  onida reaksiyaga kirishuvchi kimyoviy modda miqdori bo'lsa, u holda *kimyoviy moddaning  $t$  ondagi reaksiyaga kirishish tezligi*

$$V(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta N}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{N(t + \Delta t) - N(t)}{\Delta t}; \quad (1.4)$$

3) agar  $m=m(x)$  bir jinsli bo'lmagan sterjenning  $O(0;0)$  va  $M(x;0)$  nuqtalar orasidagi massasi bo'lsa, u holda *sterjenning  $x$  nuqtadagi zichligi*

$$\gamma(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{m(x + \Delta x) - m(x)}{\Delta x}. \quad (1.5)$$

Ko'rilgan masalalar fizik mazmunining turlicligiga qaramasdan, (1.1)-(1.5) limitlar bir xil ko'rinishga ega: ularda funksiya orttirmasining argument orttirmasiga nisbatining argument orttirmasi nolga intilgandagi limitini topish talab qilinadi.

### 5.1.2. Hosilaning ta'rifi, geometrik va mexanik ma'nolari

#### *Hosilaning ta'riflari*

$y=f(x)$  funksiya  $(a;b)$  intervalda aniqlangan bo'lsin. Ixtiyoriy  $x_0 \in (a;b)$  nuqtani olamiz va bu nuqtada  $x$  argumentga  $\Delta x$  orttirma ( $x_0 + \Delta x \in (a;b)$ ) beramiz. Bunda funksiya  $\Delta y = f(x_0 + \Delta x) - f(x_0)$  orttirma oladi.

**1-ta'rif.** Agar  $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$  limit mavjud va chekli bo'lsa, bu limitga  $f(x)$  funksiyaning  $x_0$  nuqtadagi hosilasi deyiladi va u  $f'(x_0)$  (yoki  $y'(x_0)$  yoki  $y'|_{x=x_0}$ ) kabi belgilanadi.

Shunday qilib,

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \quad (1.6)$$

Agar  $x_0$  ning biror qiymatida  $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = +\infty$  ( $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = -\infty$ ) bo'lsa, u holda funksiya  $x_0$  nuqtada musbat ishorali (manfiy ishorali) cheksiz hosilaga ega deyiladi. Shu sababli 1-ta'rif bilan aniqlanadigan hosila chekli hosila deb yuritiladi.

*1-misol.*  $f(x) = x^3$  funksiyaning  $x = x_0$  nuqtadagi hosilasini toping.

*Yechish.*  $x_0$  nuqtada  $x$  argumentga  $\Delta x$  ortirma beramiz va funksiyaning mos ortirmasini topamiz:

$$\begin{aligned} \Delta y &= f(x_0 + \Delta x) - f(x_0) = (x_0 + \Delta x)^3 - x_0^3 = \\ &= (x_0 + \Delta x - x_0)(x_0^2 + 2x_0\Delta x + \Delta x^2 + x_0^2 + x_0\Delta x + x_0^2) = \Delta x(3x_0^2 + 3x_0\Delta x + \Delta x^2). \end{aligned}$$

Ortirmalar nisbatini tuzamiz:

$$\frac{\Delta y}{\Delta x} = 3x_0^2 + 3x_0\Delta x + \Delta x^2.$$

Bu nisbatning  $\Delta x \rightarrow 0$  dagi limitini topamiz:

$$y'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (3x_0^2 + 3x_0\Delta x + \Delta x^2) = 3x_0^2.$$

**2-ta'rif.**  $y = f(x)$  funksiyaning  $x_0$  nuqtadagi o'ng (chap) hosilasi

deb  $f'_+(x_0) = \lim_{\Delta x \rightarrow 0^+} \frac{\Delta y}{\Delta x}$  ( $f'_-(x_0) = \lim_{\Delta x \rightarrow 0^-} \frac{\Delta y}{\Delta x}$ ) limitga aytiladi.

*2-misol.*  $f(x) = |x - 3|$  funksiyaning  $x_0 = 3$  nuqtadagi o'ng va chap hosilalarini toping.

*Yechish.* Berilgan funksiyaning  $x_0 = 3$  nuqtadagi ortirmasini topamiz:

$$\Delta y = f(3 + \Delta x) - f(3) = |3 + \Delta x - 3| - |3 - 3| = |\Delta x|.$$

U holda

$$\lim_{\Delta x \rightarrow 0^+} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{\Delta x}{\Delta x} = 1,$$

$$\lim_{\Delta x \rightarrow 0^-} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = -1.$$



Bu misolda  $f'_+(0) \neq f'_-(0)$ . Shu sababli  $f(x) = |x-3|$  funksiya uchun  $\Delta x \rightarrow 0$  da  $\frac{\Delta y}{\Delta x} = \frac{|\Delta x|}{\Delta x}$  nisbatning limiti mavjud emas va  $f(x) = |x-3|$  funksiya  $x_0 = 3$  nuqtada hosilaga ega bo'lmaydi.

Funksiya hosilasining yuqorida keltirilgan ta'riflaridan ushbu tasdiqlar kelib chiqadi: agar funksiya  $x_0$  nuqtada hosilaga ega bo'lsa, funksiya shu nuqtada bir-biriga teng bo'lgan o'ng va chap hosilalarga ega bo'lib,  $f'_+(x_0) = f'_-(x_0) = f'(x_0)$  bo'ladi; agar funksiya  $x_0$  nuqtada o'ng va chap hosilalarga ega bo'lib,  $f'_+(x_0) = f'_-(x_0)$  bo'lsa, funksiya shu nuqtada hosilaga ega va  $f'_+(x_0) = f'_-(x_0) = f'(x_0)$  bo'ladi.

Funksiyaning hosilasini topish amaliga *funksiyani differensiallash* deyiladi.

Agar  $y = f(x)$  funksiya biror oraliqda aniqlangan va bu oraliqning har bir nuqtasida  $f'(x)$  hosila mavjud bo'lsa, u holda

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

formula  $f'(x)$  hosilani  $x$  ning funksiyasi sifatida aniqlaydi. Bundan keyin, agar  $y = f(x)$  funksiyani differensiallashda differensiallash nuqtasi ko'rsatilmagan bo'lsa, hosilani  $x$  ning mumkin bo'lgan barcha qiymatlarida topamiz va  $y'(x)$  deb yozamiz.

### *Hosilaning geometrik va mexanik ma'nolari*

Egri chiziqqa o'tkazilgan urinma haqidagi masalada urinmaning burchak koeffitsiyenti uchun ushbu

$$k = \operatorname{tg} \alpha = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

tenglik hosil qilingan edi.

Bu tenglikni  $k = f'(x)$  ko'inishda yozamiz, ya'ni  $f'(x)$  hosila  $y = f(x)$  funksiya grafigiga  $M(x, f(x))$  nuqtada o'tkazilgan urinmaning burchak koeffitsiyentiga teng. Bu jumla hosilaning *geometrik ma'nosini* ifodalaydi.

To'g'ri chizikli harakat haqidagi masalada ushbu

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

limit hosil qilingan edi.

Bu limitni  $v(t) = s'(t)$  ko'rishda yozamiz, ya'ni moddiy nuqta harakat qonunidan  $t$  vaqt bo'yicha olingan hosila nuqtaning  $t$  ondagi to'g'ri chiziqli harakati tezligiga teng. Bu jumla *hosilaning mexanik ma'nosini* ifodalaydi.

Umumlashtirgan holda, agar  $y = f(x)$  funksiya biror fizik jarayonni ifodalasa, u holda  $y'$  hosila bu jarayon tezligini ifodalaydi deyish mumkin. Bu jumla *hosilaning fizik ma'nosini* anglatadi.

### **Funksiya grafigiga o'tkazilgan urinma va normal tenglamalari**

$y = f(x)$  funksiya grafigiga  $M(x_0; y_0)$  (bu yerda  $y_0 = f(x_0)$ ) nuqtada o'tkazilgan urinma tenglamasini hosilaning geometrik ma'nosidan keltirib chiqaramiz.

Urinma  $M_0(x_0; y_0)$  nuqtadan o'tadi. Shu sababli uning tenglamasini

$$y - y_0 = k(x - x_0)$$

ko'rishda izlaymiz.

Hosilaning geometrik ma'nosiga ko'ra,

$$k_w = f'(x_0).$$

Bundan

$$y - y_0 = f'(x_0)(x - x_0) \tag{1.7}$$

*urinma tenglamasi* kelib chiqadi.

*Egri chiziqqa o'tkazilgan normal* deb, urinish nuqtasida urinmaga perpendikular bo'lgan to'g'ri chiziqqa aytiladi.

Egri chiziqqa  $M_0(x_0; y_0)$  nuqtada o'tkazilgan normal shu nuqtada o'tkazilgan urunmaga perpendikulyar bo'lgani sababli

$$k_{norm} = -\frac{1}{k_w} = -\frac{1}{f'(x_0)}.$$

Bundan, agar  $f'(x_0) \neq 0$  bo'lsa

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \tag{1.8}$$

*normal tenglamasi* kelib chiqadi.

### 5.1.3. Funksiyaning differensiallanuvchiligi

**3-ta'rif.** Agar  $y = f(x)$  funksiyaning  $x_0$  nuqtadagi  $\Delta x$  orttirmaga mos orttirmasini

$$\Delta y = A\Delta x + \alpha(\Delta x)\Delta x \quad (1.9)$$

ko'rinishda ifodalash mumkin bo'lsa,  $f(x)$  funksiya  $x_0$  nuqtada differensiallanuvchi deyiladi, bu yerda  $A = f'(x_0)$  ga bog'liq bo'lmagan son,  $\alpha(\Delta x) = o(\Delta x) \rightarrow 0$  da cheksiz kichik funksiya, ya'ni  $\lim_{\Delta x \rightarrow 0} \alpha(\Delta x) = 0$ .

Funksiyaning nuqtada differensiallanuvchanligi bilan shu nuqtadagi hosilasi orasidagi bog'lanishni aniqlaymiz.

**1-teorema.**  $y = f(x)$  funksiya  $x_0$  nuqtada differensiallanuvchi bo'lishi uchun u shu nuqtada hosilaga ega bo'lishi zarur va yetarli.

*Isboti. Zarurligi.*  $y = f(x)$  funksiya  $x_0$  nuqtada differensiallanuvchi bo'lsin.

U holda ta'rifga ko'ra,

$$\Delta y = A\Delta x + \alpha(\Delta x)\Delta x$$

yoki

$$\frac{\Delta y}{\Delta x} = A + \alpha(\Delta x).$$

Bundan

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = A = f'(x_0),$$

ya'ni  $y = f(x)$  funksiya  $x_0$  nuqtada hosilaga ega.

*Yetariligi.*  $y = f(x)$  funksiya  $x_0$  nuqtada hosilaga ega bo'lsin.

U holda  $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = A = f'(x_0)$  belgilash kiritamiz, bunda

$\alpha(\Delta x) = \frac{\Delta y}{\Delta x} - A$  funksiya  $\Delta x \rightarrow 0$  da cheksiz kichik bo'ladi.

Bundan

$$\Delta y = A\Delta x + \alpha(\Delta x)\Delta x,$$

ya'ni funksiyaning  $x_0$  nuqtada differensiallanuvchi bo'lishi kelib chiqadi.

**2-teorema.** Agar  $y = f(x)$  funksiya  $x_0$  nuqtada differensiallanuvchi bo'lsa, u holda u shu nuqtada uzluksiz bo'ladi.

*Isboti.*  $y = f(x)$  funksiya  $x_0$  nuqtada differensiallanuvchi bo'lsin. Ta'rifga ko'ra,  $\Delta y = A\Delta x + \alpha(\Delta x)\Delta x$ .

Bundan  $\lim_{\Delta x \rightarrow 0} \Delta y = \lim_{\Delta x \rightarrow 0} (A\Delta x + \alpha(\Delta x)\Delta x) = 0$ , ya'ni funksiya  $x_0$  nuqtada uzluksiz.

Teoremaning teskarisi hamma vaqt ham o'rinli bo'lmaydi, ya'ni funksiyaning biror nuqtada uzluksiz bo'lishidan uning shu nuqtada differensiallanuvchi bo'lishi hamma vaqt ham kelib chiqmaydi. Masalan,  $y = |x - 3|$  funksiya  $x = 3$  nuqtada uzluksiz bo'lsa ham, u shu nuqtada hosilaga ega emas (2-misol), ya'ni differensiallanuvchi emas.

Agar  $y = f(x)$  funksiya  $(a; b)$  intervalning  $[a; b]$  kesmaning) har bir nuqtasida hosilaga ega bo'lsa, u shu *intervalda (kesmada) differensiallanuvchi* deyiladi.

### 5.1.4. Funksiyaning differensiali

#### *Differensial tushunchasi*

$y = f(x)$  funksiya  $(a; b)$  intervalda aniqlangan bo'lib,  $x_0 \in (a; b)$  nuqtada differensiallanuvchi bo'lsin. U holda  $\Delta y = f'(x_0)\Delta x + \alpha(\Delta x)\Delta x$  bo'ladi.

**3-ta'rif.**  $y = f(x)$  funksiya orttirmasining  $\Delta x$  ga nisbatan chiziqli bo'lgan bosh qismi  $f'(x_0)\Delta x$  ga  $y = f(x)$  funksiyaning  $x_0$  nuqtadagi differensial deyiladi va  $dy$  (yoki  $df(x)$ ) bilan belgilanadi:

$$dy = f'(x_0)\Delta x. \quad (1.10)$$

Erkli o'zgaruvchi  $x$  ning, ya'ni  $y = x$  funksiyaning differensialini topamiz.

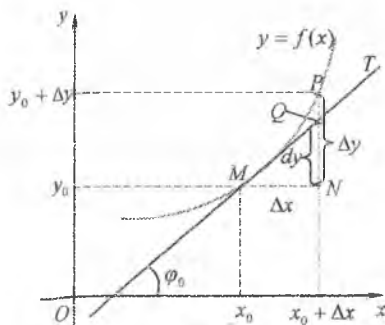
$$y' = x' = 1 \text{ bo'lgani uchun} \quad (1.10)$$

formuladan  $dy = dx = \Delta x$  kelib chiqadi, ya'ni erkli o'zgaruvchining differensialini uning orttirmasiga teng:  $dx = \Delta x$ .

Shu sababli (1.10) tenglikni quyidagicha yozish mumkin:

$$dy = f'(x_0)dx. \quad (1.11)$$

boshqacha aytganda, *funksiyaning differensialini funksiya hosilasi bilan*



4- shakl.

erkli o'zgaruvchi differensialning ko'paytmasiga teng.

(1.11) tenglikdan  $f'(x_0) = \frac{dy}{dx}$  bo'ladi, ya'ni funksiyaning  $x_0$  nuqtadagi hosilasi funksiyaning shu nuqtadagi differensialning argument differensial nisbatiga teng.

### **Differensialning geometrik ma'nosi**

Differensialning geometrik ma'nosini aniqlaymiz. Bunig uchun  $y = f(x)$  funksiya grafigiga  $M_0(x_0; f(x_0))$  nuqtada  $MT$  urinma o'tkazamiz va bu urinmaning  $x_0 + \Delta x$  nuqtadagi ordinatasini qaraymiz (4-shakl).

$MNQ$  uchburchakdan  $NQ = tg\varphi_0 \Delta x = dy$  ekanligi kelib chiqadi.

Urinmaning geometrik ma'nosiga ko'ra,  $tg\varphi_0 = f'(x_0)$ .

Bundan  $NQ = f'(x_0)\Delta x = dy$ .

Demak,  $y = f(x)$  funksiyaning  $x_0$  nuqtadagi differensial funksiya grafigiga  $M_0(x_0; f(x_0))$  nuqtada o'tkazilgan urinmaning orttirmasiga teng. Bu jumla *differensialning geometrik ma'nosini* ifodalaydi.

### **Differensialning taqribiy hisoblashlarga tatbiqi**

Ko'pchilik masalalarni yechishda  $y = f(x)$  funksiyaning  $x_0$  nuqtadagi orttirmasi funksiyaning shu nuqtadagi differensialiga taqriban almashtiriladi, ya'ni  $\Delta y \approx dy$  deb olinadi. Buni  $f(x) \approx f(x_0) + f'(x_0)\Delta x$  ko'rinishda yozish mumkin.

Bunday almashtirish yordamida biror  $A$  miqdorning taqribiy qiymati quyidagi tartibda hisoblanadi:

1°.  $A$  ni  $x$  nuqtada biror  $f(x)$  funksiya qiymatiga tenglashtiriladi:  
 $A = f(x)$ ;

2°.  $x_0$  nuqta  $x$  ga yaqin va  $f(x_0)$  ni hisoblash qulay qilib tanlanadi;

3°.  $f(x_0)$  hisoblanadi;

4°.  $f'(x_0)$  hisoblanadi;

5°.  $x_0$ ,  $f(x_0)$ ,  $f'(x_0)$  qiymatlar  $f(x) \approx f(x_0) + f'(x_0)\Delta x$  formulaga qo'yiladi.

**3-misol.**  $2,008^3$  ning taqribiy qiymatini toping .

**Yechish.**

1°.  $A = 2,008^3$ ,  $f(x) = x^3$  deyimiz, u holda  $f(x) = A$  va  $x = 2,008$ ;

$$2^\circ. x_0 = 2 \text{ deb olamiz;}$$

$$3^\circ. f(x_0) = 2^3 = 8;$$

$$4^\circ. f'(x) = 3x^2, \quad f'(x_0) = 12;$$

$$5^\circ. f(x) \approx f(x_0) + f'(x_0)\Delta x = 8 + 12 \cdot 0,08 = 8,096.$$

### 5.1.5. Mashqlar

1. Hosila ta'rifidan foydalanib, funksiyalarning hosilasini toping:

$$1) f(x) = \sqrt{3x-1};$$

$$2) f(x) = \frac{1}{2-5x};$$

$$3) f(x) = \operatorname{ctg} 2x;$$

$$4) f(x) = ch 2x.$$

2.  $f'(x_0)$  ni hosila ta'rifidan foydalanib hisoblang:

$$1) f(x) = e^{-3x}, \quad x_0 = 0;$$

$$2) f(x) = \ln(1-4x), \quad x_0 = 0;$$

$$3) f(x) = \operatorname{tg}\left(2x + \frac{\pi}{4}\right), \quad x_0 = \pi;$$

$$4) f(x) = \frac{1-x}{1+x}, \quad x_0 = 1.$$

3. Berilgan funksiyalarning  $f'(x_0)$  va  $f''(x_0)$  hosilalarini toping:

$$1) f(x) = |3x-2|, \quad x_0 = \frac{2}{3};$$

$$2) f(x) = |x-2| + |x+2|, \quad x_0 = 2;$$

$$3) y = \begin{cases} x & \text{agar } x \leq 2 \text{ bo'lsa,} \\ -x^2 + 3x & \text{agar } x > 2 \text{ bo'lsa, } x_0 = 2; \end{cases}$$

$$4) f(x) = \sqrt{e^{x^2} - 1}, \quad x_0 = 0.$$

4. Moddiy nuqta  $Ox$  o'qi bo'ylab  $x = \frac{t^3}{3} - 2t^2 + 3t$  qonun bilan harakatlanmoqda. Qaysi nuqtalarda moddiy nuqtaning harakat yo'nalishi o'zgaradi?

5. Moddiy nuqta  $s = s(t)$  qonun bilan to'g'ri chiziqli harakat qilmoqda. Qaysi vaqtda material nuqtaning tezlanishi  $a(m/c^2)$  ga teng bo'ladi?

$$1) s(t) = 2t^3 - \frac{5}{2}t^2 + 3t + 1(m), \quad a = 19;$$

$$2) s(t) = t^3 + \frac{3}{2}t^2 - 4t + 3(m), \quad a = 9.$$

6. O'tkazgich orqali o'tuvchi tok miqdori  $t=0$  vaqtdan boshlab  $q = 3t^2 - 1$  qonun bilan aniqlanadi. Ikkinchi sekund oxiridagi tok kuchini aniqlang.

## 5.2. DIFFERENSIALLASH QOIDALARI VA FORMULALARI

### 5.2.1. Yig'indi, ayirma, ko'paytma va bo'linmani differensiallash

Funksiyaning hosilasi ta'rifidan foydalanib, ikki funksiya yig'indisi, ayirmasi, ko'paytmasi va bo'linmasini differensiallash qoidalarini keltirib chiqaramiz.

**1-teorema.** Agar  $u = u(x)$  va  $v = v(x)$  funksiyalar  $x$  nuqtada differensiallanuvchi bo'lsa,  $u$  holda bu funksiyalarning yig'indisi, ayirmasi, ko'paytmasi va bo'linmasi (bo'linmasi  $v(x) \neq 0$  shart bajarilganda) ham  $x$  nuqtada differensiallanuvchi va quyidagi formulalar o'rinli bo'ladi:

$$1. (u \pm v)' = u' \pm v'; \quad 2. (u \cdot v)' = u'v + v'u; \quad 3. \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}, \quad (v \neq 0).$$

*Isboti.* 1. Funksiyaning hosilasi va limitlar haqidagi teoremlardan foydalanib topamiz:

$$\begin{aligned} (u \pm v)' &= \lim_{\Delta x \rightarrow 0} \frac{(u(x + \Delta x) \pm v(x + \Delta x)) - (u(x) \pm v(x))}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \left( \frac{u(x + \Delta x) - u(x)}{\Delta x} \pm \frac{v(x + \Delta x) - v(x)}{\Delta x} \right) = \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \pm \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} = u' \pm v'. \end{aligned}$$

2. Formulani isbotlashda 5.1.3. banddagi 2-teoremadan foydalanamiz:  $x$  nuqtada differensiallanuvchi  $u = u(x)$  va  $v = v(x)$  funksiyalar shu nuqtada uzluksiz bo'ladi. Shu sababli  $\Delta x \rightarrow 0$  da  $\Delta u \rightarrow 0$  va  $\Delta v \rightarrow 0$ .

$$\begin{aligned} (u \cdot v)' &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) \cdot v(x + \Delta x) - u(x) \cdot v(x)}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{(u(x) + \Delta u) \cdot (v(x) + \Delta v) - u(x) \cdot v(x)}{\Delta x} = \end{aligned}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{u(x) \cdot v(x) + u(x) \cdot \Delta v + v(x) \cdot \Delta u + \Delta u \cdot \Delta v - u(x) \cdot v(x)}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0} \left( v(x) \cdot \frac{\Delta u}{\Delta x} + u(x) \cdot \frac{\Delta v}{\Delta x} + \Delta v \cdot \frac{\Delta u}{\Delta x} \right) = v(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + u(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} +$$

$$+ \lim_{\Delta x \rightarrow 0} \Delta v \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = u' \cdot v + u \cdot v' + 0 \cdot u' = u'v + v'u.$$

$$3. \left( \frac{u}{v} \right)' = \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{v(x) + \Delta v}{\Delta x} \frac{u(x) + \Delta u - u(x)}{v(x)} =$$

$$= \lim_{\Delta x \rightarrow 0} \frac{u(x) \cdot v(x) + v(x) \cdot \Delta u - u(x) \cdot v(x) - u(x) \cdot \Delta v}{\Delta x \cdot (v(x) + \Delta v) \cdot v(x)} = \lim_{\Delta x \rightarrow 0} \frac{v \cdot \Delta u - u \cdot \Delta v}{\Delta x \cdot (v^2 + v \cdot \Delta v)} =$$

$$= \lim_{\Delta x \rightarrow 0} \frac{v \cdot \frac{\Delta u}{\Delta x} - u \cdot \frac{\Delta v}{\Delta x}}{v^2 + v \cdot \Delta v} = \frac{v \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} - u \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}}{v^2 + v \cdot \lim_{\Delta x \rightarrow 0} \Delta v} = \frac{u'v - v'u}{v^2}.$$

### 5.2.2. Teskari funksiyani differensiallash

$y = f(x)$  funksiya teskari funksiya mavjudligi haqidagi teoremaning shartlarini qanoatlantirsin va  $x = \varphi(y)$  teskari funksiyaga ega bo'lsin.

**2-teorema.** Agar  $y = f(x)$  funksiya  $x$  nuqtada  $f'(x) \neq 0$  hosilaga ega bo'lsa, u holda  $x = \varphi(y)$  funksiya mos  $y = f(x)$  nuqtada differensiallanuvchi bo'ladi va

$$\varphi'(y) = \frac{1}{f'(x)}, \text{ ya'ni } x'_y = \frac{1}{y'_x}.$$

*Isboti.*  $x = \varphi(y)$  funksiyaning argumenti  $y$  ga  $\Delta y \neq 0$  orttirma beramiz. U holda  $y = f(x)$  funksiyaning qat'iy monotonlikdan  $x = \varphi(y)$  funksiya  $\Delta x \neq 0$  orttirma oladi. Shu sababli  $\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$  kabi yozish

mumkin.

$x = \varphi(y)$  teskari funksiya  $y$  nuqtada uzluksiz bo'lgani uchun  $\Delta y \rightarrow 0$  da  $\Delta x \rightarrow 0$  bo'ladi:  $\Delta x \rightarrow 0$  da oxirgi tenglikning o'ng tomoni  $\frac{1}{f'(x)}$  ( $f'(x) \neq 0$ ) ga, chap tomoni  $\varphi'(y)$  hosilaga teng bo'ladi.



Demak,

$$\varphi'(y) = \frac{1}{f'(x)}.$$

Shunday qilib, *teskari funksiyaning hosilasi berilgan funksiya hosilasining teskari qiymatiga teng.*

### 5.2.3. Murakkab funksiyani differensiallash

$y = f(u)$  va  $u = \varphi(x)$  bo'lsin. U holda  $y = f(\varphi(x))$  funksiya erkli argumenti  $x$  dan va oraliq argumenti  $u$  dan iborat murakkab funksiya bo'ladi.

**3-teorema.** Agar  $u = \varphi(x)$  funksiya  $x$  nuqtada  $\varphi'(x)$  hosilaga va  $y = f(u)$  funksiya mos  $u = \varphi(x)$  nuqtada  $f'(u)$  hosilaga ega bo'lsa, u holda  $f(\varphi(x))$  murakkab funksiya  $x$  nuqtada differensiallanuvchi bo'ladi va

$$y'(x) = f'(u)\varphi'(x).$$

*Isboti.*  $y = f(u)$  funksiya  $u$  nuqtada differensiallanuvchi bo'lgani uchun  $\Delta y = f'(u)\Delta u + \alpha(\Delta u)\Delta u$  bo'ladi.

Bundan

$$\frac{\Delta y}{\Delta x} = f'(u)\frac{\Delta u}{\Delta x} + \alpha(\Delta u)\frac{\Delta u}{\Delta x}.$$

$u = \varphi(x)$  funksiya  $x$  nuqtada hosilaga ega. Shu sababli  $u = \varphi(x)$  funksiya  $x$  nuqtada uzluksiz va  $\Delta x \rightarrow 0$  da  $\Delta u \rightarrow 0$ .

U holda

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(u)\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \rightarrow 0} \alpha(\Delta u) \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}.$$

Bundan

$$y'(x) = f'(u) \cdot \varphi'(x) + 0 \cdot \varphi'(x)$$

yoki

$$y'(x) = f'(u) \cdot \varphi'(x).$$

Shunday qilib, *murakkab funksiyaning hosilasi berilgan funksiyaning oraliq argument bo'yicha hosilasi bilan oraliq argumentning erkli argument bo'yicha hosilasining ko'paytmasiga teng.*

Bu qoida oraliq argumentlar bir nechta bo'lganda ham o'z kuchida qoladi. Masalan,  $y = f(u)$ ,  $u = g(v)$ ,  $v = h(x)$  bo'lsa,  $y'_x = y'_u \cdot u'_v \cdot v'_x$  bo'ladi.

$y = f(u)$  va  $u = \varphi(x)$  differensiallanuvchi va  $y = f(\varphi(x))$  murakkab funksiyani hosil qiluvchi funksiyalar bo'lsin.

Murakkab funksiyaning hosilasi haqidagi teorema ko'ra,  $y'_x = y'_u u'_x$  bo'ladi.

Bu tenglikning har ikkala tomonini  $dx$  ga ko'paytiramiz:

$$y'_x dx = y'_u u'_x dx.$$

Endi  $y'_x dx = dy$  va  $u'_x dx = du$  ekanini hisobga olsak,

$$dy = y'_u du.$$

$dy = y'_x dx$  va  $dy = y'_u du$  formulalarni solishtirish ko'rsatadiki,  $y = f(x)$  funksiyaning differensial argument erkli o'zgaruvchi yoki biror argumentning funksiyasi bo'lishidan qat'iy nazar bir xil formula bilan topiladi.

Differensialning bu xossasiga *differensialning invariantlik xossasi* deyiladi.

$dy = y'_x dx$  formula tashqi ko'rinishidan  $dy = y'_u du$  formulaga o'xshashda, aslini olganda ular orasida farq mavjud: birinchi formulada  $x$  erkli o'zgaruvchi va shu sababli  $dx = \Delta x$ , ikkinchi formulada esa  $u$  funksiya  $x$  ning funksiyasi va shuning uchun  $du \neq \Delta u$ .

#### 5.2.4. Asosiy elementar funksiyalarning hosilalari

Asosiy elementar funksiyalarning hosilalarini topishda ekvivalent cheksiz kichik funksiyalardan, teskari va murakkab funksiyalarni differensiallash formulalaridan hamda yig'indi, ayirma, ko'paytma va bo'linmani differensiallash qoidalaridan foydalanamiz.

**1. O'zgarmas funksiya:**  $y = C$  ( $C \in R$ ).

O'zgarmas funksiya  $x \in R$  da o'zining qiymatini saqlagani uchun ixtiyoriy nuqtada uning orttirmasi nolga teng bo'ladi. Shu sababli

$$(C)' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0.$$

**2. Darajali funksiya:**  $y = x^\alpha$ , bunda  $\alpha \in R$ ,  $\alpha \neq 0$ .

Bu funksiya uchun  $x > 0$  da

$$\Delta y = (x + \Delta x)^\alpha - x^\alpha = x^\alpha \left( \left( 1 + \frac{\Delta x}{x} \right)^\alpha - 1 \right).$$

Bundan

$$\frac{\Delta y}{\Delta x} = x^\alpha \frac{\left(1 + \frac{\Delta x}{x}\right)^\alpha - 1}{\Delta x}.$$

Endi  $\Delta x \rightarrow 0$  da  $\left(1 + \frac{\Delta x}{x}\right)^\alpha - 1 \sim \alpha \frac{\Delta x}{x}$  ekanligini hisobga olsak,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = x^\alpha \lim_{\Delta x \rightarrow 0} \frac{\left(1 + \frac{\Delta x}{x}\right)^\alpha - 1}{\Delta x} = x^\alpha \lim_{\Delta x \rightarrow 0} \frac{\alpha \Delta x}{\Delta x \cdot x} = x^\alpha \lim_{\Delta x \rightarrow 0} \frac{\alpha}{x} = \alpha x^{\alpha-1}$$

bo'ladi.

Demak,

$$(x^\alpha)' = \alpha x^{\alpha-1}.$$

Xususan,  $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ ,  $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ .

**3. Ko'rsatkichli funksiya:**  $y = a^x$ , bunda  $a \in R$ ,  $a > 0$ ,  $a \neq 1$ .

Bu funksiyaning orttirmasi  $\Delta y = a^{x+\Delta x} - a^x = a^x(a^{\Delta x} - 1)$  bo'lgani uchun

$$\frac{\Delta y}{\Delta x} = a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x} \text{ bo'ladi. Bundan } \Delta x \rightarrow 0 \text{ da } a^{\Delta x} - 1 \sim \Delta x \ln a \text{ ekanini}$$

hisobga olsak,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x} = a^x \lim_{\Delta x \rightarrow 0} \frac{\Delta x \ln a}{\Delta x} = a^x \ln a$$

bo'ladi.

Demak,

$$(a^x)' = a^x \ln a.$$

Xususan,  $(e^x)' = e^x$ .

**4. Logarifmik funksiya:**  $y = \log_a x$ , bunda  $a \in R$ ,  $a > 0$ ,  $a \neq 1$ .

$y = \log_a x$  funksiya  $x = a^y$  funksiyaga teskari funksiya. Bunda oldingi banddagi formulaga ko'ra,  $x'(y) = a^y \ln a$ .

U holda

$$y'(x) = \frac{1}{x'(y)} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

Demak,

$$(\log_a x)' = \frac{1}{x \ln a}.$$

Xususan,  $(\ln x)' = \frac{1}{x}$ .

### 5. Trigonometrik funksiyalar.

1)  $y = \sin x$  funksiyaning orttirmasi

$$\Delta y = \sin(x + \Delta x) - \sin x = 2 \sin \frac{\Delta x}{2} \cos \left( x + \frac{\Delta x}{2} \right)$$

va

$$\frac{\Delta y}{\Delta x} = \frac{2 \sin \frac{\Delta x}{2} \cos \left( x + \frac{\Delta x}{2} \right)}{\Delta x}.$$

Bu tenglikdan  $\Delta x \rightarrow 0$  da  $\sin \frac{\Delta x}{2} \sim \frac{\Delta x}{2}$  ni hisobga olib, topamiz:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2 \frac{\Delta x}{2} \cos \left( x + \frac{\Delta x}{2} \right)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \cos \left( x + \frac{\Delta x}{2} \right) = \cos(x+0) = \cos x.$$

Demak,

$$(\sin x)' = \cos x.$$

2)  $y = \cos x$  funksiyaning hosilasini murakkab funksiyaning hosilasi formulasidan foydalanib topamiz:

$$(\cos x)' = \left( \sin \left( \frac{\pi}{2} - x \right) \right)' = \cos \left( \frac{\pi}{2} - x \right) \cdot \left( \frac{\pi}{2} - x \right)' = \cos \left( \frac{\pi}{2} - x \right) \cdot (-1) = -\sin x.$$

Demak,

$$(\cos x)' = -\sin x.$$

3)  $y = \operatorname{tg} x$  funksiyaning hosilasini bo'linmaning hosilasi formulasidan foydalanib topamiz:

$$(\operatorname{tg} x)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Demak,

$$(tgx)' = \frac{1}{\cos^2 x}.$$

4)  $y = ctgx$  funksiyaning hosilasini topishda murakkab funksiyaning hosilasi formulasi bilan foydalanamiz:

$$(ctgx)' = \left( tg\left(\frac{\pi}{2} - x\right) \right)' = \frac{1}{\cos^2\left(\frac{\pi}{2} - x\right)} \cdot (-1) = -\frac{1}{\sin^2 x}.$$

Demak,

$$(ctgx)' = -\frac{1}{\sin^2 x}.$$

### 6. Teskari trigonometrik funksiyalar.

1)  $y = \arcsin x$  funksiya  $x = \sin y$  funksiyaga teskari.

Bunda  $x'(y) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ .

U holda

$$y'(x) = \frac{1}{x'(y)} = \frac{1}{\sqrt{1 - x^2}}.$$

Demak,

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$

2)  $y = \arccos x$  funksiyaning hosilasini  $\arcsin x + \arccos x = \frac{\pi}{2}$

formuladan foydalanib topamiz:

$$(\arccos x)' = \left( \frac{\pi}{2} - \arcsin x \right)' = -(\arcsin x)' = -\frac{1}{\sqrt{1 - x^2}}.$$

Demak,

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}.$$

3)  $y = \text{arctg}x$  funksiyaning hosilasini teskari funksiyaning

hosilasidan foydalanib topamiz:

$$(\operatorname{arctgx})' = \frac{1}{(\operatorname{tgy})'} = \cos^2 y = \frac{1}{1 + \operatorname{tg}^2 y} = \frac{1}{1 + x^2}.$$

Demak,

$$(\operatorname{arctgx})' = \frac{1}{1 + x^2}.$$

4)  $\operatorname{arctgx}$  va  $\operatorname{arccctgx}$  funksiyalar  $\operatorname{arctgx} + \operatorname{arccctgx} = \frac{\pi}{2}$  bog'lanishga ega.

Bundan

$$(\operatorname{arccctgx})' = \left( \frac{\pi}{2} - \operatorname{arctgx} \right)' = -(\operatorname{arctgx})' = -\frac{1}{1 + x^2}.$$

Demak,

$$(\operatorname{arccctgx})' = -\frac{1}{1 + x^2}.$$

*1-misol.* Giperbolik funksiyalarning hosilalarini toping.

*Yechish.* Differensiallash qoidalari va  $y = e^x$  funksiyaning hosilasidan foydalanib topamiz:

$$(\operatorname{shx})' = \left( \frac{e^x - e^{-x}}{2} \right)' = \frac{e^x + e^{-x}}{2} = \operatorname{chx}, \text{ ya'ni } (\operatorname{shx})' = \operatorname{chx};$$

$$(\operatorname{chx})' = \left( \frac{e^x + e^{-x}}{2} \right)' = \frac{e^x - e^{-x}}{2} = \operatorname{shx}, \text{ ya'ni } (\operatorname{chx})' = \operatorname{shx};$$

$$(\operatorname{thx})' = \left( \frac{\operatorname{shx}}{\operatorname{chx}} \right)' = \frac{(\operatorname{shx})' \operatorname{chx} - \operatorname{shx} \cdot (\operatorname{chx})'}{\operatorname{ch}^2 x} = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x}, \text{ ya'ni } (\operatorname{thx})' = \frac{1}{\operatorname{ch}^2 x};$$

$$(\operatorname{cthx})' = \left( \frac{\operatorname{chx}}{\operatorname{shx}} \right)' = \frac{\operatorname{sh}^2 x - \operatorname{ch}^2 x}{\operatorname{sh}^2 x} = -\frac{1}{\operatorname{sh}^2 x}, \text{ ya'ni } (\operatorname{cthx})' = -\frac{1}{\operatorname{sh}^2 x}.$$

### 5.2.5. Differensiallash qoidalari va hosilalar jadvali

Keltirib chiqarilgan differensiallash qoidalarini va asosiy elementar funksiyalarning hosilalari formulalarini jadval ko'rinishida yozamiz.

Amalda ko'pincha murakkab funksiyalarning hosilalarini topishga to'g'ri keladi. Shu sababli quyida keltiriladigan formulalarda « $x$ » argument « $u$ » oraliq argumentga almashtiriladi.

### Differentsiallashtirish qoidalarini.

- $(u \pm v)' = u' \pm v'$ ,  $u = u(x)$ ,  $v = v(x)$  -- differentsiallanuvchi funksiyalar;
- $(u \cdot v)' = u'v + uv'$ , xususan  $(Cu)' = Cu'$ ,  $C$  -- o'zgarmas son;
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ , xususan  $\left(\frac{C}{v}\right)' = -\frac{Cv'}{v^2}$ ;
- $y'_x = \frac{1}{x'_y}$ , agar  $y = f(x)$  va  $x = \varphi(y)$ ;

### Asosiy elementar funksiyalarning hosilalar jadvali.

- $(C)' = 0$ ;
- $(u^a)' = au^{a-1} \cdot u'$ , xususan  $\left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$ ,  $(\sqrt{u})' = \frac{1}{2\sqrt{u}} \cdot u'$ ;
- $(a^u)' = a^u \ln a \cdot u'$ , xususan  $(e^u)' = e^u \cdot u'$ ;
- $(\log_a u)' = \frac{1}{u \ln a} \cdot u'$ , xususan  $(\ln u)' = \frac{1}{u} \cdot u'$ ;
- $(\sin u)' = \cos u \cdot u'$ ;
- $(\cos u)' = -\sin u \cdot u'$ ;
- $(\operatorname{tg} u)' = \frac{1}{\cos^2 u} \cdot u'$ ;
- $(\operatorname{ctg} u)' = -\frac{1}{\sin^2 u} \cdot u'$ ;
- $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$ ;
- $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$ ;
- $(\operatorname{arctg} u)' = \frac{1}{1+u^2} \cdot u'$ ;
- $(\operatorname{arccot} u)' = -\frac{1}{1+u^2} \cdot u'$ ;
- $(\operatorname{sh} u)' = \operatorname{ch} u \cdot u'$ ;
- $(\operatorname{ch} u)' = \operatorname{sh} u \cdot u'$ ;
- $(\operatorname{th} u)' = \frac{1}{\operatorname{ch}^2 u} \cdot u'$ ;
- $(\operatorname{cth} u)' = -\frac{1}{\operatorname{sh}^2 u} \cdot u'$ ;

Keltirilgan differensiallash qoidalari va asosiy elementar funksiyalarning hosilalar jadvali bir o'zgaruvchi funksiyasi differensial hisobining asosini tashkil qiladi. Ularni bilgan holda har qanday elementar funksiyaning hosilasini topish mumkin. Bunda yana elementar funksiya hosil bo'ladi.

4-misol.  $f(x) = 5^x + \arcsin x + x \ln x$  funksiyaning hosilasini toping.

*Yechish.* Hosilani topishda differensiallashning 1,2 qoidalari va 3,4,9-formularidan foydalanildi.

$$\begin{aligned} f'(x) &= (5^x + \arcsin x + x \ln x)' = (5^x)' + (\arcsin x)' + (x \ln x)' = 5^x \ln 5 + \\ &+ \frac{1}{\sqrt{1-x^2}} + x' \ln x + x(\ln x)' = 5^x \ln 5 + \frac{1}{\sqrt{1-x^2}} + \ln x + x \cdot \frac{1}{x} = \\ &\frac{1}{x} = 5^x \ln 5 + \frac{1}{\sqrt{1-x^2}} + \ln x + 1. \end{aligned}$$

### 5.2.6. Logarifmik differensiallash

Ayrim hollarda funksiyaning hosilasini topish uchun avval berilgan funksiyani logarifmlash, so'ngra differensiallash maqsadga muvofiq bo'ladi. Bu jarayonga *logarifmik differensiallash* deyiladi.

3-misol.  $y = \frac{(x^3 + 1) \cdot \sqrt[3]{(x-2)^4} \cdot 2^x}{(x-4)^3}$  funksiyaning hosilasini toping.

*Yechish.* Bu hosilani differensiallash qoidalari va formulalari orqali topish mumkin. Ammo bu jarayon katta hajmga ega. Shu sababli logarifmik differensiallashni qo'llaymiz. Buning uchun funksiyani logarifmlaymiz:

$$\ln y = \ln(x^3 + 1) + \frac{4}{5} \ln(x-2) + x \ln 2 - 3 \ln(x-4).$$

Bu tenglikning har ikki qismini  $x$  bo'yicha differensiallaymiz:

$$\frac{1}{y} \cdot y' = \frac{1}{x^3 + 1} \cdot 3x^2 + \frac{4}{5} \cdot \frac{1}{x-2} + \ln 2 - 3 \cdot \frac{1}{x-4}.$$

Endi bu tenglikdan  $y'$  ni topamiz:

$$y' = y \cdot \left( \frac{3x^2}{x^3 + 1} + \frac{4}{5(x-2)} + \ln 2 - \frac{3}{x-4} \right),$$



yoki

$$y' = \frac{(x^3 + 1) \cdot \sqrt[3]{(x-2)^4} \cdot 2^x}{(x-4)^3} \cdot \left( \frac{3x^2}{x^3 + 1} + \frac{4}{5(x-2)} + \ln 2 - \frac{3}{x-4} \right)$$

Shunday funksiyalar borki, ularning hosilalari faqat logarifmik differensiallash orqali topiladi. Bunday funksiyalarning qatoriga *darajali-ko'rsatkichli funksiya* deb ataluvchi  $y = u^v$  funksiya kiradi, bu yerda  $u = u(x)$ ,  $v = v(x)$  -  $x$  ning differensiallanuvchi funksiyalari.

Bu funksiyalarning hosilasini logarifmik differensiallash yordamida topamiz:

$$\ln y = v \cdot \ln u, \quad \frac{1}{y} y' = v' \cdot \ln u + v \cdot \frac{1}{u} \cdot u', \quad y' = y \left( v' \ln u + \frac{vu'}{u} \right)$$

Bundan

$$(u^v)' = u^v \cdot \ln u \cdot v' + v \cdot u^{v-1} \cdot u'. \quad (2.1)$$

(2.1) formulani eslab qolish qoidasini ifodalaymiz: darajali-ko'rsatkichli funksiyaning hosilasi  $u = \text{const}$  shartidagi ko'rsatkichli funksiya hosilasi bilan  $v = \text{const}$  shartidagi darajali funksiya hosilasining yig'indisiga teng.

4- misol.  $y = x^{\cos 3x}$  funksiyaning hosilasini toping.

*Yechish.* (2.1) formula bilan topamiz:

$$y' = x^{\cos 3x} \cdot \ln x \cdot (-\sin 3x) \cdot 3 + \cos 3x \cdot x^{\cos 3x - 1} \cdot 1$$

yoki

$$y' = x^{\cos 3x - 1} \cdot \cos 3x - 3x^{\cos 3x} \cdot \ln x \cdot \sin 3x.$$

### 5.2.7. Parametrik va oshkormas ko'rinishda berilgan funksiyalarni differensiyallash

$t$  o'zgaruvchining  $x = \varphi(t)$  va  $y = \psi(t)$  funksiyalari biror  $T = (\alpha; \beta)$  intervalda aniqlangan bo'lib, bu intervalda  $\varphi'(t)$ ,  $\psi'(t)$  hosilalar va  $x = \varphi(t)$  funksiyaga teskari  $t = \phi(x)$  funksiya mavjud bo'lsin. Agar  $x = \varphi(t)$  funksiya qat'iy monoton bo'lsa,  $t = \phi(x)$  teskari funksiya bir qiymatli, uzluksiz va qat'iy monoton bo'ladi. Shu sababli  $y = \psi(\phi(x))$  murakkab funksiya mavjud bo'ladi. Bunda  $y = f(x)$  funksiya  $x = \varphi(t)$  va

$y = \psi(t)$  tenglamalar bilan *parametrik ko'rinishda* ( $t$  parametrlil) berilgan deyiladi.

$y = f(x)$  funksiya

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \quad t \in T \end{cases}$$

parametrik tenglamalar bilan berilgan bo'lsin. U holda  $t = \phi(x)$  teskari

funksiya mavjud va uning hosilasi  $\phi'(x) = \frac{1}{\varphi'(t)}$  bo'ladi. Shuningdek,

$y = \psi(\phi(x))$  murakkab funksiya hosilasi  $y'_x = \psi'(\phi(x))\phi'(x)$  bo'ladi.

Bundan

$$y'(x) = \frac{\psi'(t)}{\varphi'(t)}$$

yoki

$$y'_x = \frac{y'_t}{x'_t}. \quad (2.2)$$

5- misol.  $\begin{cases} x = 3\cos t, \\ y = 4\sin t \end{cases}$  bo'lsa,  $y'_x$  ni toping.

$$\text{Yechish. } y'_x = \frac{(3\cos t)'_t}{(4\sin t)'_t} = -\frac{3\sin t}{4\cos t} = -\frac{3}{4} \operatorname{ctgt}.$$

Agar funksiya  $y$  ga nisbatan yechilmagan, ya'ni  $x$  va  $y$  o'zgaruvchilar orasidagi bog'lanish  $F(x, y) = 0$  ko'rinishda berilgan bo'lsa, funksiya *oshkormas ko'rinishda* berilgan deyiladi.

Oshkor berilgan har qanday  $y = f(x)$  funksiyani oshkormas ko'rinishda  $f(x) - y = 0$  kabi yozish mumkin, ammo teskarisini hamma vaqt bajarib bo'lmaydi,  $F(x, y) = 0$  tenglamani  $y$  ga nisbatan yechish hamma vaqt ham oson emas, ayrim hollarda esa umuman mumkin emas.

Funksiya oshkormas ko'rinishda berilgan bo'lsa,  $F(x, y)$  funksiya  $x$  ning murakkab funksiyasi, ya'ni bunda  $y = y(x)$  deb qaraladi va  $F(x, y) = 0$  tenglikning chap va o'ng tomoni  $x$  bo'yicha differensialanadi, so'ngra hosil bo'lgan tenglamadan  $y'$  topiladi.

6- misol.  $y - \arctg y - x^3 = 0$  bo'lsa,  $y'$  ni toping.

*Yechish.* Tenglikning har ikkala tomonini  $x$  bo'yicha differensiallaymiz:

$$y' - \frac{1}{1+y^2} y' - 3x^2 = 0.$$

Bundan

$$y' \left( 1 - \frac{1}{1+y^2} \right) = 3x^2, \quad y' \frac{y^2}{1+y^2} = 3x^2$$

yoki

$$y' = 3x^2 \left( 1 + \frac{1}{y^2} \right).$$

## 5.2.8. Yuqori tartibli hosilalar va differensiallar

### *Yuqori tartibli hosilalar*

$f(x)$  funksiya biror  $(a;b)$  intervalda aniqlangan bo'lib, shu intervalda differensiyalanuvchi bo'lsin. U holda  $f'(x)$  hosila  $x \in (a;b)$  ning funksiyasi bo'ladi. Shu sababli bu funksiya uchun hosilaning mavjudligi va uni hisoblash masalalarini qarash mumkin.

$f'(x)$  ga *birinchi tartibli hosila* deyiladi.  $f(x)$  funksiyaning  $f'(x)$  hosilasidan olingan hosilaga *ikkinchi tartibli hosila* deyiladi. Ikkinchi tartibli hosila mavjud bo'lsa, bu hosiladan olingan hosila *uchinchi tartibli hosila* deyiladi va hokazo. Bunday hosilalar ikkinchi tartibidan boshlab *yuqori tartibli hosila* deyiladi.

Yuqori tartibli hosilalar  $y'', y''', y^{(4)}, \dots, y^{(n)}, \dots$

(yoki  $f''(x), f'''(x), f^{(4)}(x), \dots, f^{(n)}(x), \dots$  yoki  $\left( \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \dots, \frac{d^n y}{dx^n}, \dots \right)$ ) kabi

belgilanadi.

7- misol.  $y = x^3 \ln x$  bo'lsa,  $y^{(4)}(3)$  ni toping.

*Yechish.*  $y' = (x^3)' \ln x + x^3 (\ln x)' = 3x^2 \ln x + x^3 \frac{1}{x} = x^2 (3 \ln x + 1);$

$$y'' = (x^2 (3 \ln x + 1))' = (x^2)' (3 \ln x + 1) + x^2 (3 \ln x + 1)' =$$

$$= 2x(3\ln x + 1) + x^2 \cdot \frac{3}{x} = x(6\ln x + 5);$$

$$v''' = (x(6\ln x + 5))' = x'(6\ln x + 5) + x(6\ln x + 5)' = 6\ln x + 5 + x \cdot \frac{6}{x} = 6\ln x + 11;$$

$$y^{(4)} = (6\ln x + 11)' = \frac{6}{x}.$$

Bundan

$$y^{(4)}(3) = \frac{6}{3} = 2.$$

Funksiyaning yuqori tartibli hosilasini topish uchun uning barcha oldingi tartibli hosilalarini topish kerak bo'ladi. Biroq, ayrim funksiyalarning  $n$ -tartibli hosilalarini topish imkonini beruvchi formulalar mavjud. Masalan, quyida keltirilgan formulalar bunday formulalar qatoriga kiradi:

- |  |  |
|--|--|
| 1. $(a^x)^{(n)} = a^x \ln^n a \ (a > 0), \ (e^x)^{(n)} = e^x;$                         | 2. $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right);$           |
| 3. $(x^\alpha)^{(n)} = \alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n}, \ \alpha \in R;$ | 4. $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right);$           |
| 5. $(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n};$                                     | 6. $(u \pm v)^{(n)} = u^{(n)} \pm v^{(n)};$                          |
| 7. $(Cu)^{(n)} = C u^{(n)};$   | 8. $(u \cdot v)^{(n)} = \sum_{k=0}^n C_n^k u^{(k)} \cdot v^{(n-k)}.$ |

Formulalardan ayrimlarining isbotini keltiramiz.

3.  $(x^\alpha)^{(n)} = \alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n}, \ \alpha \in R$  ning isboti.

$$y = x^\alpha, \quad y' = \alpha x^{\alpha-1}, \quad y'' = \alpha(\alpha-1)x^{\alpha-2},$$

$$y''' = \alpha(\alpha-1)(\alpha-2)x^{\alpha-3}, \quad \dots, \quad y^{(n)} = \alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n}.$$

Shunday qilib,

$$(x^\alpha)^{(n)} = \alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n}, \ \alpha \in R.$$

4.  $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$  ning isboti.

$$y = \cos x, \quad y' = -\sin x = \cos\left(x + \frac{\pi}{2}\right), \quad y'' = -\cos x = \cos\left(x + 2 \cdot \frac{\pi}{2}\right),$$

$$y''' = \sin x = \cos\left(x + 3 \cdot \frac{\pi}{2}\right), \dots, \quad y^{(n)} = \cos\left(x + (n-1) \cdot \frac{\pi}{2}\right) = \cos\left(x + n \cdot \frac{\pi}{2}\right)$$

Demak,

$$(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right).$$

5.  $(\ln x)^{(n)} = \frac{(-1)^n (n-1)!}{x^n}$  ning isboti.

$$y = \ln x, \quad y' = \frac{1}{x} = x^{-1}, \quad y'' = -1 \cdot x^{-2}, \quad y''' = (-1)(-2)x^{-3} = (-1)^2 \cdot 1 \cdot 2 \cdot x^{-3}, \dots,$$

$$y^{(n)} = (-1)^{n-1} \cdot 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1)x^{-n} = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}.$$

Shunday qilib,

$$(\ln x)^{(n)} = \frac{(-1)^n (n-1)!}{x^n}.$$

### ***Ikkinchi tartibli hosilaning mexanik ma'nosi***

$M$  moddiy nuqta  $s = s(t)$  qonun bilan to'g'ri chiziqli harakat qilsin.

U holda  $s'(t)$  moddiy nuqtaning  $t$  vaqtdagi tezligini ifodalaydi:

$$s'(t) = v(t).$$

Nuqtaning  $t$  vaqtdagi tezligi  $v(t)$ ,  $t + \Delta t$  vaqtdagi tezligi  $v(t) + \Delta v$  bo'lsin, ya'ni  $\Delta t$  vaqt oralig'ida nuqtaning tezligi  $\Delta v$  ga o'zgarsin.

$\frac{\Delta v}{\Delta t}$  nisbat to'g'ri chiziqli harakatda nuqtaning  $\Delta t$  vaqt oralig'idagi o'rtacha tezlanishini ifodalaydi. Bu nisbatning  $\Delta t \rightarrow 0$  dagi limiti  $M$  nuqtaning berilgan  $t$  vaqtdagi tezlanishi deyiladi va u  $a(t)$  bilan belgilanadi:  $\lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = a(t)$ , ya'ni  $v'(t) = a(t)$ .

Shu sababli

$$a(t) = s''(t).$$

Demak, moddiy nuqta harakat qonunidan  $t$  vaqt bo'yicha olingan ikkinchi tartibli hosila to'g'ri chiziqli harakatda nuqtaning  $t$  vaqtdagi

tezlanishiga teng. Bu tasdiq *ikkinchi tartibli hosilaning mexanik ma'nosini* ifodalaydi.

**Parametrik va oshkormas ko'rinishda berilgan  
funksiyalarning yuqori tartibli hosilalari**

$y = f(x)$  funksiya

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \quad t \in T \end{cases}$$

parametrik tenglamalar bilan berilgan bo'lsin.

U holda

$$y'_x = \frac{y'_t}{x'_t},$$

Bundan murakkab va teskari funksiyalarni differensiallash qoidalariga ko'ra,

$$y''_x = (y'_x)'_t \cdot t'_x = \frac{(y''_t)_t}{x'_t},$$

$$y''_x = \frac{(y''_t)_t}{x'_t}. \quad (2.3)$$

Shu kabi

$$y'''_x = \frac{(y'''_t)_t}{x'_t}, \quad y^{(4)}_x = \frac{(y^{(4)}_t)_t}{x'_t}$$

va boshqa hosilalar topiladi.

8- misol.  $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}$  bo'lsa, ikkinchi tartibli hosilani toping.

*Yechish.* (2.2) formula bilan topamiz:

$$y'_x = \frac{(a(1 - \cos t))'_t}{(a(t - \sin t))'_t} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t} = \operatorname{ctg} \frac{t}{2}.$$

Bundan (2.3) formulaga ko'ra,

$$y''_x = \frac{\left(\operatorname{ctg} \frac{t}{2}\right)'_t}{(a(t - \sin t))'_t} = \frac{-\frac{1}{2 \sin^2 \frac{t}{2}}}{a(1 - \cos t)} = -\frac{1}{1 - \cos t} \cdot \frac{1}{a(1 - \cos t)} = -\frac{1}{a(1 - \cos t)^2}.$$

$y''$  hosilani (2.3) formula ustida almashtirishlar bajarib, quyidagicha yozish mumkin

$$y'' = \frac{y'_i x'_i - x'_i y'_i}{(x'_i)^3} \quad (2.4)$$

$y = f(x)$  funksiya  $F(x, y) = 0$  tenglama bilan oshkormas ko'rinishda berilgan bo'lsin.

Bu tenglama  $x$  bo'yicha differensiallansa va  $y'$  ga nisbatan yechilsa, birinchi tartibli hosila topiladi. Topilgan birinchi tartibli hosila  $x$  bo'yicha differensiallansa, ikkinchi tartibli hosila kelib chiqadi. Bu hosilada  $x, y, y'$  lar qatnashadi. Ikkinchi tartibli hosilaga topilgan  $y'$  ni qo'yib,  $y''$  ni  $x$  va  $y$  orqali ifodasi aniqlanadi.

9- misol.  $b^2 x^2 + a^2 y^2 = a^2 b^2$  bo'lsa, ikkinchi tartibli hosilani toping.

*Yechish.* Oshkormas ko'rinishdagi funktsiyani differensiallash qoidasidan foydalanib topamiz:

$$(b^2 x^2 + a^2 y^2)' = (a^2 b^2)', \quad b^2 \cdot 2x + a^2 \cdot 2yy' = 0.$$

Bundan

$$y' = -\frac{b^2 x}{a^2 y}.$$

U holda

$$\begin{aligned} y'' &= -\frac{b^2}{a^2} \left( \frac{x}{y} \right)' = -\frac{b^2}{a^2} \cdot \frac{x'y' - y'x}{y^2} = \frac{b^2}{a^2} \cdot \frac{xy' - y}{y^2} = \frac{b^2}{a^2} \cdot \frac{-x \cdot \frac{b^2 x}{a^2 y} - y}{y^2} = \\ &= -\frac{b^2}{a^2} \cdot \frac{x^2 b^2 + a^2 y^2}{a^2 y^3} = -\frac{b^2 a^2 b^2}{a^4 y^3} = -\frac{b^4}{a^2 y^3}. \end{aligned}$$

### ***Yuqori tartibli differensiallar***

Biror  $(a; b)$  intervalda differensiyallanuvchi  $y = f(x)$  funktsiyaning  $dy = y'(x)dx$  differensiyali *birinchi tartibli differensial* deyiladi.

U holda

$$d(dy) = d(f'(x)dx) = (f'(x)dx) dx = f''(x)dx \cdot dx$$

differensialga *ikkinchi tartibli differensial* deyiladi va

$$d^2 y = f''(x)dx^2 \quad (2.5)$$

kabi yoziladi, bu yerda  $dx^2$  bilan  $(dx)^2$  belgilanadi.

Ikkinchi tartibli differensialdan olingan differensial *uchinchi tartibli differensial* deyiladi va hokazo. *n*-tartibli differensial deb (*n*-1)-tartibli

differensialdan olingan differensialga aytiladi va  $d^n y = f^{(n)}(x) dx^n$  kabi yoziladi.

Bundan  $y^{(n)}(x) = \frac{d^n y}{dx^n}$ , ya'ni  $y = f(x)$  funksiyaning *n*-tartibli hosilasi funksiya *n*-tartibli differensialning argument differensialining *n*-darajasiga nisbatiga teng bo'lishi kelib chiqadi.

*10-misol.*  $y = x^4 + 3x - 1$  bo'lsa,  $d^3 y$  ni toping.

*Yechish.*  $y' = 4x^3 + 3$ ,  $y'' = 12x^2$ ,  $y''' = 24x$ .

Bundan

$$d^3 y = y'''(x) dx^3 = 24x dx^3.$$

Yuqorida keltirilgan formulalar *x* erkli o'zgaruvchi bo'lganda o'rinli bo'ladi. Agar  $y = f(x)$  funksiya *x* boshqa bir erkli o'zgaruvchining funksiyasi bo'lsa, yuqori tartibli differensiallar invariantlik xossasiga bo'ysunmaydi.

Buni ikkinchi tartibli differensial uchun ko'rsatamiz.

Ko'paytmaning differensial formulasi ko'ra,

$$d^2 y = d(f'(x) dx) = d(f'(x)) dx + f'(x) d(dx) = f''(x) dx \cdot dx + f'(x) d^2 x,$$

ya'ni

$$d^2 y = f''(x) dx^2 + f'(x) d^2 x. \quad (2.6)$$

(2.5) va (2.6) formulalarni solishtiramiz: murakkab funksiya uchun ikkinchi tartibli differensial o'zgaradi, ya'ni bunda ikkinchi qo'shiluvchi  $f'(x) d^2 x$  hosil bo'ladi.

Agar bunda *x* erkli o'zgaruvchi bo'lsa, u holda

$$d^2 x = d(dx) = d(1 \cdot dx) = dx \cdot d1 = dx \cdot 0 = 0$$

va (2.5) ifoda (2.6) formulaga o'tadi.

*11-misol.*  $y = x^2$  va  $x = \sin t$  bo'lsa,  $d^2 y$  ni toping.

*Yechish.*  $y' = 2x$ ,  $y'' = 2$ ,  $dx = \cos t dt$ ,  $d^2 x = -\sin t dt^2$

(2.6) formulaga formula bilan topamiz:

$$\begin{aligned} d^2 y &= 2dx^2 - 2x \sin t dt^2 = 2(\cos t dt)^2 - 2 \sin t \cdot \sin t dt^2 = \\ &= 2 \cos^2 t dt^2 - 2 \sin^2 t dt^2 = 2 \cos 2t dt^2. \end{aligned}$$



Boshqa yechim:  $y = x^2$  va  $x = \sin t$ .

Bundan

$$y = \sin^2 t, \quad y' = 2 \sin t \cos t = \sin 2t, \quad y'' = 2 \cos 2t.$$

U holda

$$d^2 y = y'' dt^2 = 2 \cos 2t dt^2.$$

### 5.2.9. Mashqlar

1. Differensiallash qoidalari va formulalaridan foydalanib, berilgan funksiyalarning hosilasini toping:

$$1) y = 3x^4 - \frac{1}{3}x^3 + \ln 2;$$

$$2) y = \frac{1}{6}x^6 + 3x^i - 2x;$$

$$3) y = \frac{2}{\sqrt{x}} + 3x^2 \sqrt{x} - \frac{6}{\sqrt[3]{x^2}};$$

$$4) y = \sqrt{x} - \frac{3}{x} + \frac{1}{3x^3};$$

$$5) y = \frac{xe^x - e^{-x}}{x^2};$$

$$6) y = \frac{2^x + 3^x}{2^x - 3^x};$$

$$7) y = \frac{x \ln x}{\ln x - 1};$$

$$8) y = \frac{\ln x + e^x}{\ln x - e^x};$$

$$9) y = \frac{1 + \cos x}{1 - \cos x};$$

$$10) y = \frac{1 + \operatorname{tg} x}{1 - \operatorname{tg} x};$$

$$11) y = \operatorname{tg} x - \operatorname{ctg} x;$$

$$12) y = \frac{x \sin x - \cos x}{x \cos x + \sin x};$$

$$13) y = \frac{x \operatorname{ch} x - \operatorname{sh} x}{x \operatorname{sh} x - \operatorname{ch} x};$$

$$14) y = \operatorname{th} x + \operatorname{cth} x;$$

$$15) y = \log_x e;$$

$$16) y = 4 \sin^2 x - 3 \lg x + 4 \cos^2 x;$$

$$17) y = \sqrt{4 - 3x^2};$$

$$18) y = \arcsin \sqrt{x};$$

$$19) y = \cos^4 x - \sin^4 x;$$

$$20) y = \frac{1}{6} \ln \frac{x-3}{x+3};$$

$$21) y = \sqrt{1-x^2} + x \arcsin x;$$

$$22) y = \ln(e^{2x} + 1) - 2 \operatorname{arctg} e^x;$$

$$23) y = \frac{1}{2} \ln \frac{1+3^x}{1-3^x};$$

$$24) y = \log_x x^x;$$

$$25) y = \frac{\operatorname{tg} 3x + \ln \cos^2 3x}{3};$$

$$26) y = e^{-3x} (\sin 3x + \cos 3x);$$

27)  $y = \sqrt{e^x - 1} - \operatorname{arctg} \sqrt{e^x - 1}$ ;

28)  $y = \ln \operatorname{ctg} \left( \frac{\pi}{4} + \frac{x}{2} \right)$ ;

29)  $y = 3 \arccos \frac{x-3}{\sqrt{5}} + \sqrt{6x-4-x^2}$ ;

30)  $y = \frac{2-x}{4(x^2+2)} - \frac{1}{4\sqrt{2}} \operatorname{arctg} \frac{x}{\sqrt{2}} + \ln \sqrt{x^2+2}$ .

2. Berilgan  $x = \varphi(y)$  funksiyalar uchun  $y'$  hosilani toping:

1)  $x = \frac{1-y}{1+y}$ ;      2)  $x = e^{-y}$ ;      3)  $x = 2 \sin y$ ;      4)  $x = 3 \operatorname{ctg} y$ .

3. Quyidagi sonlarni differensial yordamida taqriban hisoblang:

1)  $\sqrt[3]{33}$ ;      2)  $\lg 10,21$ ;      3)  $\operatorname{ctg} 45^{\circ}10'$ ;      4)  $3,013^3$ .

4. Quyidagi funksiyalarning berilgan nuqtadagi taqribiy qiymatini differensial yordamida hisoblang:

1)  $y = \sqrt{x^2 - 7x + 10}$ ,  $x = 0,98$ ;

2)  $y = \sqrt{\frac{2-x}{2+x}}$ ,  $x = 0,15$ ;

3)  $y = \sqrt{\frac{x^2-3}{x^2+5}}$ ,  $x = 2,037$ ;

4)  $y = \sqrt{2x - \sin \frac{\pi x}{2}}$ ,  $x = 1,02$ .

5. Berilgan funksiyalarning birinchi tartibli differensialini toping:

1)  $y = x(\ln x - 1)$ ;

2)  $y = \frac{\ln x}{x}$ ;

3)  $y = \cos^2 2x$ ;

4)  $y = a \sin^3 x$ .

5)  $y = 3^{\cos x}$ ;

6)  $y = \ln^3 \cos x$ .

6. Berilgan hosilalar uchun  $y'''$  ni toping:

1)  $y = (x^2 - 1)^3$ ;      2)  $y = e^{2x} \cos x$ ;      3)  $y = (1 + x^2) \operatorname{arctg} x$ ;      4)  $y = x^2(\ln x - 1)$ .

7. Berilgan funksiyalar uchun  $y^{(n)}(0)$  ni toping:

1)  $y = \sin 5x \cos 2x$ ;      2)  $y = x \cos x$ ;      3)  $y = x^2 \sin x$ ;      4)  $y = x^2 e^x$ .

8. Oshkormas funksiyalarning hosilasini toping:

1)  $b^2 x^2 + a^2 y^2 = a^2 b^2$ ;      2)  $y^3 = x^3 + 3xy$ ;      3)  $e^{xy} = xy$ ;

4)  $\cos(xy) = x^2$ ;      5)  $e^y + xy = e$ ;      6)  $x \sin y + y \sin x = 0$ .

9. Berilgan funksiyalar uchun  $\frac{d^2 y}{dx^2}$  ni toping:

1)  $\begin{cases} x = t^2 + 1, \\ y = t^3 - 1; \end{cases}$

2)  $\begin{cases} x = a \cos t, \\ y = a \sin t; \end{cases}$

3)  $\begin{cases} x = \ln(1+t^2), \\ y = t - \operatorname{arctg} t; \end{cases}$

4)  $\begin{cases} x = \arcsin t, \\ y = \sqrt{1-t^2}. \end{cases}$

10. Berilgan egri chiziqqa  $M_0(x_0, y_0)$  nuqtada o'tkazilgan urinma va normal tenglamalarini tuzing:

$$1) y = \frac{x^3}{3}, M_0\left(-1, -\frac{1}{3}\right);$$

$$2) y = \sin x, M_0(\pi, 0);$$

3)  $y = x^3 + x^2 - 1$  egri chiziqqa  $y = x^2$  parabola bilan kesishish nuqtasida;

$$4) \frac{x^2}{9} + \frac{y^2}{25} = 1, M_0\left(\frac{9}{5}, 4\right); \quad 5) \begin{cases} x = \frac{1+t}{t^2}, \\ y = \frac{1}{t^2} - \frac{1}{t} \end{cases} M_0(2, 2); \quad 6) \begin{cases} x = \sin t, \\ y = \cos 2t, \end{cases} M_0\left(\frac{1}{2}, \frac{1}{2}\right)$$

## 5.3. DIFFERENSIAL HISOBNING ASOSIY TEOREMALARI

### 5.3.1. Ferma teoremasi

Differensiallanuvchi funksiyalarning nazariy va amaliy ahamiyatga ega bo'lgan teoremlari bilan tanishamiz.

**1-teorema (Ferma teoremasi).**  $f(x)$  funksiya  $(a; b)$  intervalda aniqlangan bo'lib, bu intervalning biror  $c$  nuqtasida o'zining eng katta (eng kichik) qiymatiga erishsin. Agar funksiya  $c$  nuqtada chekli hosilaga ega bo'lsa, u holda

$$f'(c) = 0$$

bo'ladi.

*Isboti.* Aytaylik,  $y = f(x)$  funksiya  $c \in (a; b)$  nuqtada o'zining eng katta qiymatiga ega bo'lsin. U holda  $\forall x \in (a; b)$  uchun  $f(x) \leq f(c)$ , ya'ni  $f(x) - f(c) \leq 0$  bo'ladi.

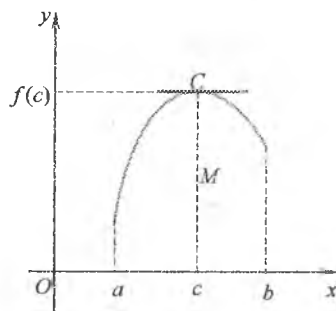
$y = f(x)$  funksiya  $c$  nuqtada hosilaga ega. Shu sababli bu nuqtada funksiyaning o'ng va chap hosilalari mavjud va

$$f'_+(c) = \lim_{x \rightarrow c+0} \frac{f(x) - f(c)}{x - c} \leq 0 \quad (x > c),$$

$$f'_-(c) = \lim_{x \rightarrow c-0} \frac{f(x) - f(c)}{x - c} \geq 0 \quad (x < c),$$

$$f'_+(c) = f'_-(c) = f'(c)$$

bo'ladi.



5-shakl.

Bu munosabatlardan  $f'(c)=0$  bo'lishi keilib chiqadi.

Funksiya  $c$  nuqtada eng kichik qiymatga ega bo'lgan hol uchun teorema shu kabi isbotlanadi.

Ferma teoremasining geometrik talqini quyidagicha bo'ladi:  $y=f(x)$  funksiya  $c$  nuqtada eng katta (eng kichik) qiymatga erishsa va  $f'(c)$  hosila mavjud bo'lsa,  $f(x)$  funksiya grafigiga  $M(c;f(c))$  nuqtada o'tkazilgan urinma  $Ox$  o'qqa parallel bo'ladi (5-shakl).

$[a;b]$  kesma uchun Ferma teoremasi hamma vaqt ham o'rinli bo'lmaydi. Masalan,  $f(x)=x$  funksiya  $[0;1]$  kesmada o'zining eng katta ( $x=1$  da) va eng kichik ( $x=0$  da) qiymatiga erishadi. Bu nuqtalarda hosila  $f'(x)=1 \neq 0$ .

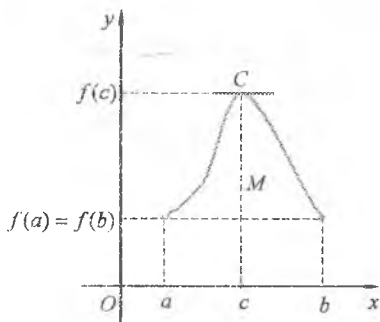
### 5.3.2. Roll teoremasi

**2-teorema (Roll teoremasi).**  $f(x)$  funksiya  $[a;b]$  kesmada aniqlangan, uzluksiz va  $f(a)=f(b)$  bo'lsin. Agar funksiya  $(a;b)$  intervalda differensiallanuvchi bo'lsa, u holda shunday  $c \in (a;b)$  nuqta topiladiki,

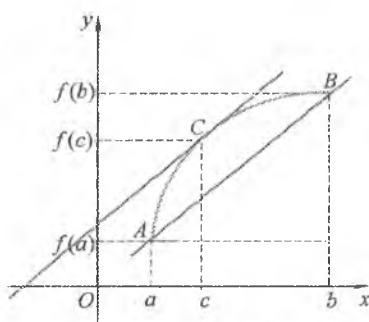
$$f'(c) = 0$$

bo'ladi.

*Isboti.* Shartga ko'ra,  $f(x)$  funksiya  $[a;b]$  kesmada aniqlangan va uzluksiz. U holda Veersht rassning 2-teoremasiga ko'ra, funksiya shu kesmada o'zining eng katta qiymati  $M$  ga va eng kichik qiymati  $m$  ga ega bo'ladi. Bunda  $M=m$  bo'lsa,  $f(x)$  funksiya  $[a;b]$  kesmada o'zgarmas va shu sababli  $\forall x \in (a;b)$  uchun  $f'(x)=0$  bo'ladi.



6-shakl.



7-shakl.

Agar  $M \neq m$  bo'lsa,  $f(x)$  funksiya  $M$  va  $m$  qiymatlardan biriga biror  $c \in (a; b)$  nuqtada ega bo'ladi. Bunda Ferma teoremasiga asosan,  $f'(c) = 0$  bo'ladi.

Roll teoremasi ushbu geometrik talqinga ega:  $[a; b]$  kesmada uzluksiz,  $(a; b)$  intervalda differensiallanuvchi va kesmaning chetki nuqtalarida teng qiymatlar qabul qiluvchi funksiya grafigida shunday  $(c; f(c))$  nuqta topiladi va bu nuqtada funksiya grafigiga o'tkazilgan urinma  $Ox$  o'qiga parallel bo'ladi (6-shakl).

*1-misol.* Roll teoremasi o'rinli bo'lishini tekshiring:

1)  $f(x) = x^2 - 3x - 4$  funksiya uchun  $[0; 3]$  kesmada; 2)  $f(x) = \sqrt[3]{x^2} - 1$  funksiya uchun  $[-1; 1]$  kesmada.

*Yechish.* 1)  $f(x) = x^2 - 3x - 4$  funksiya  $[0; 3]$  kesmada uzluksiz, differensiallanuvchi va uning chetki nuqtalarida bir xil qiymatga ega:

$f(0) = f(3) = -4$ . Shu sababli, bu funksiya uchun Roll teoremasi o'rinli bo'ladi.  $x$  ning  $f'(x) = 0$  bo'lgan qiymatini topamiz:  $f'(x) = 4x - 3 = 0$ .

Bundan  $x = \frac{3}{4}$ .

2)  $f(x) = \sqrt[3]{x^2} - 1$  funksiya  $[-1; 1]$  kesmada uzluksiz,  $f(-1) = f(1) = 0$ ,  $f'(x) = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$ . Bu hosila  $x = 0 \in (-1; 1)$  nuqtada mavjud emas.

Demak, bu funksiya uchun Roll teoremasi o'rinli bo'lmaydi.

### 5.3.3. Lagranj teoremasi

**3-teorema (Lagranj teoremasi).**  $f(x)$  funksiya  $[a; b]$  kesmada aniqlangan va uzluksiz bo'lsin. Agar funksiya  $(a; b)$  intervalda chekli hosilaga ega bo'lsa, u holda shunday  $c \in (a; b)$  nuqta topiladiki,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad (3.1)$$

bo'ladi.

*Isboti.* Teoremani isbotlash uchun yordamchi

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

funksiyadan foydalanamiz. Shartga ko'ra,  $f(x)$  funksiya  $[a;b]$  kesmada aniqlangan, uzluksiz va  $(a;b)$  intervalda hosilaga ega bo'lgani uchun  $F(x)$  funksiya ham  $[a;b]$  kesmada aniqlangan, uzluksiz va  $(a;b)$  intervalda hosilaga ega bo'ladi.

Bunda

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \quad F(a) = F(b) = 0. \quad (3.2)$$

Demak,  $F(x)$  funksiya Roll teoremasining barcha shartlarini qanoatlantiradi. U holda biror  $c \in (a;b)$  nuqta uchun  $F'(c) = 0$ ,  $f'(c) - \frac{f(b) - f(a)}{b - a} = 0$  bo'ladi.

Bundan

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Teoremaning geometrik talqinini beramiz.

$\frac{f(b) - f(a)}{b - a}$  qiymat funksiya grafigining  $A(a; f(a))$  va  $B(b; f(b))$  nuqtalari orqali o'tivchi kesuvchining burchak koeffitsiyentini aniqlaydi. Teoremaga ko'ra, shunday  $c \in (a;b)$  topiladiki,  $C(c; f(c))$  nuqtada funksiya grafigiga o'tkazilgan urinma  $AB$  kesuvchiga parallel bo'ladi (7-shakl).

(3.1) tenglikdan

$$f(b) - f(a) = f'(c)(b - a) \quad (3.3)$$

kelib chiqadi. Bu formulaga *Lagranj formulasi* yoki *chekli ayirmalar formulasi* deyiladi.

Agar  $a = x$ ,  $b = x + \Delta x$  desak, bu formulani

$$f(x + \Delta x) - f(x) = f'(c)\Delta x \quad (3.4)$$

ko'rinishda yozish mumkin.

$c \in (a;b)$  bo'lgani uchun  $c = a + \theta(b - a)$ ,  $0 < \theta < 1$ , deyish mumkin.

U holda (3.4) tenglik

$$f(x + \Delta x) - f(x) = f'(x + \theta\Delta x)\Delta x$$

ko'rinishni oladi.

Langranj teoremasi yordamida  $\Delta y \approx dy$  taqribiy tenglikning aniqligini baholash mumkin. Buning uchun  $f(x)$  funksiya ikkinchi

tartibli uzluksiz  $f''(x)$  hosilaga ega bo'lsin deb, topamiz:

$$\begin{aligned}\Delta y - dy &= (f(x + \Delta x) - f(x)) - f'(x)\Delta x = f'(c)\Delta x - f'(x)\Delta x = \\ &= (f'(c) - f'(x))\Delta x = f''(c_1)(c - x)\Delta x, \text{ bu yerda } c_1 \in (c; x).\end{aligned}$$

Demak,  $\Delta y - dy = f''(c_1)(c - x)\Delta x$ .  $M = \max_{|x, x+\Delta x|} |f''(x)|$  bo'lsin.

$|c - x| < \Delta x$  va  $f''(c_1) \leq M$  tengsizliklarni hisobga olib, topamiz:

$$|\Delta y - dy| \leq M |\Delta x|^2.$$

Lagranj teoremasidan quyidagi natijalar kelib chiqadi.

**1-natija.** Agar biror intervalda funksiyaning hosilasi nolga teng bo'lsa, funksiya shu intervalda o'zgarmas bo'ladi.

**2-natija.** Agar biror intervalda ikkita funksiya teng hosilalarga ega bo'lsa, funksiyalar bir-biridan o'zgarmas qo'shiluvchiga farq qiladi.

*2-misol.*  $\arcsin x + \arccos x = \frac{\pi}{2}$ ,  $x \in [-1; 1]$  ekanini isbotlang.

*Yechish.*  $f(x) = \arcsin x + \arccos x$  deb olsak,  $(-1; 1)$  oraliqda

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$$

U holda 1-natijaga ko'ra,  $f(x) = C$ , ya'ni  $\arcsin x + \arccos x = C$ .  $C$  ning qiymatini topish uchun  $x$  ga  $(-1; 1)$  intervaldagi qiymatlardan birini, masalan,  $x = 0$  ni qo'yamiz:  $\arcsin 0 + \arccos 0 = C$  yoki  $\frac{\pi}{2} = C$ .

Bundan

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

### 5.3.4. Koshi teoremasi

**4-teorema (Koshi teoremasi).**  $f(x)$  va  $g(x)$  funksiyalar  $[a; b]$  kesmada aniqlangan va uzluksiz bo'lsin. Agar funksiyalar  $(a; b)$  intervalda differensiallanuvchi va  $\forall x \in (a; b)$  uchun  $g'(x) \neq 0$  bo'lsa, u holda shunday  $c \in (a; b)$  nuqta topiladi va

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad (3.5)$$

bo'ladi.

*Isboti.* Teoremaning  $g'(x) \neq 0$  shartiga ko'ra, (3.5) tenglik ma'noga ega bo'lishi uchun  $g(b) \neq g(a)$  bo'lishi kerak.  $f(x)$  va  $g(x)$  funksiyalardan ushbu

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot (g(x) - g(a))$$

funksiyani tuzamiz. Bu funksiya  $[a; b]$  kesmada aniqlangan, uzluksiz va  $(a; b)$  intervalda hosilaga ega.

Bundan tashqari,

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(x), \quad F(a) = F(b) = 0.$$

Demak,  $F(x)$  funksiya Roll teoremasining barcha shartlarini qanoatlantiradi va biror  $c \in (a; b)$  nuqta uchun  $F'(c) = 0$ , ya'ni

$$f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(c) = 0$$

bo'ladi.

Bundan

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

### 5.3.5. Lopital teoremasi

**5-teorema**  $\left( \frac{0}{0} \right)$  ko'rinishdagi aniqmasliklarni ochishning Lopital qoidasi).

$x_0$  nuqtaning biror atrofida  $f(x)$  va  $g(x)$  funksiyalar uzluksiz, differensiallanuvchi va  $g'(x) \neq 0$  bo'lsin. Agar  $\lim_{x \rightarrow x_0} f(x) = 0$ ,  $\lim_{x \rightarrow x_0} g(x) = 0$  va

$\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = k$  (chekli yoki cheksiz) limit mavjud bo'lsa, u holda

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} \quad (3.6)$$

bo'ladi.

*Isboti.*  $f(x)$  va  $g(x)$  funksiyalar uchun  $x_0$  nuqtaning biror atrofida yotuvchi  $[x_0; x]$  kesmada Koshi teoremasini qo'llaymiz.



Bunda

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)}, \quad f(x_0) = g(x_0) = 0$$

hisobga olinsa,

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad (3.7)$$

hosil bo'ladi, bu yerda  $c$  nuqta  $x$  va  $x_0$  nuqtalar orasida yotuvchi biror son.

$x \rightarrow x_0$  da  $c$  ham  $x_0$  ga intiladi.

(3.7) tenglikda limitga o'tamiz:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{c \rightarrow x_0} \frac{f'(c)}{g'(c)}$$

$$\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = k \text{ ekanidan } \lim_{c \rightarrow x_0} \frac{f'(c)}{g'(c)} = k.$$

Shu sababli

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

**Izohlar:** 1. 1-teorema  $f(x)$  va  $g(x)$  funksiyalar  $x = x_0$  da aniqlanmagan, ammo  $\lim_{x \rightarrow x_0} f(x) = 0$  va  $\lim_{x \rightarrow x_0} g(x) = 0$  bo'lganda ham o'rinli bo'ladi. Bunda  $f(x_0) = \lim_{x \rightarrow x_0} f(x) = 0$  va  $g(x_0) = \lim_{x \rightarrow x_0} g(x) = 0$  deb olish yetarli.

2. 1-teorema  $x \rightarrow \infty$  da ham o'rinli bo'ladi. Haqiqatan ham,  $x = \frac{1}{z}$  deb, topamiz:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{z \rightarrow 0} \frac{f\left(\frac{1}{z}\right)}{g\left(\frac{1}{z}\right)} = \lim_{z \rightarrow 0} \frac{\left(f\left(\frac{1}{z}\right)\right)'}{\left(g\left(\frac{1}{z}\right)\right)'} = \lim_{z \rightarrow 0} \frac{f'\left(\frac{1}{z}\right) \cdot \left(-\frac{1}{z^2}\right)}{g'\left(\frac{1}{z}\right) \cdot \left(-\frac{1}{z^2}\right)} = \lim_{z \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

3.  $f'(x)$  va  $g'(x)$  funksiyalar 1-teoremaning shartlarini qanoatlantirsa, teorema takror qo'llanishi mumkin:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow x_0} \frac{f''(x)}{g''(x)}$$

3- misol.  $\lim_{x \rightarrow 1} \frac{x^2 - 1 + \ln x}{e^x - e}$  limitni toping.

*Yechish.*  $f(x) = x^2 - 1 + \ln x$ ,  $g(x) = e^x - e$  funksiyalar  $x=1$  nuqta atrofida aniqlangan.  $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} g(x) = 0$ , ya'ni  $\frac{0}{0}$  ko'rinishdagi aniqmaslik berilgan.

U holda 1-teoreмага ko'ra,

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 1} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 1} \frac{2x + \frac{1}{x}}{e^x} = \frac{3}{e}.$$

Demak,

$$\lim_{x \rightarrow 1} \frac{x^2 - 1 + \ln x}{e^x - e} = \frac{3}{e}.$$

$\frac{\infty}{\infty}$  ko'rinishdagi aniqmasliklarni ochish haqidagi teoremani isbotsiz keltiramiz.

**6-teorema**  $\left( \frac{\infty}{\infty} \text{ ko'rinishdagi aniqmasliklarni ochishning Lopital qoidasi} \right)$ .

$x_0$  nuqtaning biror atrofida  $f(x)$  va  $g(x)$  funksiyalar uzluksiz, differensiallanuvchi va  $g'(x) \neq 0$  bo'lsin. Agar  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = \infty$

bo'lib,  $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$  limit mavjud bo'lsa, u holda

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

bo'ladi.

4- misol.  $\lim_{x \rightarrow a} \frac{\ln(x-a)}{\ln(e^x - e^a)}$  limitni toping.

$$\begin{aligned} \text{Yechish. } \lim_{x \rightarrow a} \frac{\ln(x-a)}{\ln(e^x - e^a)} &= \left( \frac{\infty}{\infty} \right) = \lim_{x \rightarrow a} \frac{\frac{1}{x-a}}{\frac{e^x - e^a}{e^x(x-a)}} = \left( \frac{0}{0} \right) = \\ &= \lim_{x \rightarrow a} \frac{e^x}{e^x(x-a) + e^a} = \lim_{x \rightarrow a} \frac{1}{1+(x-a)} = \frac{1}{1+(a-a)} = \frac{1}{1+0} = 1. \end{aligned}$$

$\frac{0}{0}$  va  $\frac{\infty}{\infty}$  ko'rinishdagi aniqmasliklarga asosiy aniqmasliklar deyiladi.

$0 \cdot \infty$  yoki  $\infty - \infty$  ko'rinishdagi aniqmasliklar algebraik almashtirishlar yordamida asosiy aniqmasliklarga keltiriladi.

$0^0$ ,  $\infty^0$  yoki  $1^\infty$  ko'rinishdagi aniqmasliklar

$$f(x)^{g(x)} = e^{g(x)\ln f(x)}$$

formula yordamida  $0 \cdot \infty$  aniqmaslikka keltiriladi.

5- misol.  $\lim_{x \rightarrow 0} x^3 \ln x$  limitni toping.

$$\text{Yechish. } \lim_{x \rightarrow 0} x^3 \ln x = (0 \cdot \infty) = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x^3}} = \left( \frac{\infty}{\infty} \right) = \lim_{x \rightarrow 0} \frac{1}{-3 \frac{1}{x^4}} = -\frac{1}{3} \lim_{x \rightarrow 0} x^4 = 0.$$

6- misol.  $\lim_{x \rightarrow 0} \left( \frac{1}{x} - \operatorname{ctgx} \right)$  limitni toping.

$$\begin{aligned} \text{Yechish. } \lim_{x \rightarrow 0} \left( \frac{1}{x} - \operatorname{ctgx} \right) &= (\infty - \infty) = \lim_{x \rightarrow 0} \left( \frac{\sin x - x \cos x}{x \sin x} \right) = \left( \frac{0}{0} \right) = \\ &= \lim_{x \rightarrow 0} \frac{\cos x - \cos x + x \sin x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{x \sin x}{\sin x + x \cos x} = \left( \frac{0}{0} \right) = \\ &= \lim_{x \rightarrow 0} \frac{\sin x + x \cos x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0. \end{aligned}$$

7- misol.  $\lim_{x \rightarrow 0} x^{\sin x}$  limitni toping.

$$\begin{aligned} \text{Yechish. } \lim_{x \rightarrow 0} x^{\sin x} &= (0^0) = \lim_{x \rightarrow 0} e^{\ln x^{\sin x}} = e^{\lim_{x \rightarrow 0} \sin x \ln x} = e^{\lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{\sin x}}} = \\ &= e^{-\lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}} = e^{-\lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos x}} = e^{-\lim_{x \rightarrow 0} (\operatorname{tg} x) \frac{\sin x}{x}} = e^0 = 1. \end{aligned}$$

### 5.3.6. Teylor teoremasi

7-teorema (Teylor teoremasi).  $f(x)$  funksiya  $x_0$  nuqtaning biror atrofida aniqlangan bo'lib, bu atrofda  $(n+1)$ -tartibligacha hosilalarga ega va  $f^{(n+1)}(x)$  hosila  $x_0$  nuqtada uzluksiz bo'lsin.

U holda

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \quad (3.8)$$

bo'ladi, bunda  $c = x_0 + \theta(x-x_0)$ ,  $0 < \theta < 1$ .

(3.8) tenglikka *Taylor formulasi* deyiladi.

*Isboti.* Avval

$$\varphi(x, x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

$$R_{n+1}(x) = f(x) - \varphi(x, x_0)$$

belgilashlar kiritamiz.

Bunda biror  $c$  son uchun  $R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$  bo'lishi ko'rsatilsa, teorema isbot bo'ladi.

Endi  $x_0$  nuqtaning atrofida  $x$  ( $x > x_0$  bo'lsin) nuqtani tanlaymiz va  $[x_0; x]$  kesmada

$$F(t) = f(x) - \varphi(x, t) - \frac{(x-t)^{n+1} R_{n+1}(x)}{(x-x_0)^{n+1}}, \quad t \in [x_0; x]$$

yordamchi funksiyani tanlaymiz.

$F(t)$  funksiya  $[x_0; x]$  kesmada usluksiz va differensiallanuvchi va

$$\begin{aligned} F'(t) &= -f'(t) + \frac{f'(t)}{1!} - \frac{f''(t)}{1!}(x-t) + \frac{f''(t)}{2!}2(x-t) - \frac{f'''(t)}{3!}(x-t)^2 + \dots + \\ &+ \frac{f^{(n)}(t)}{n!}n(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{(n+1)(x-t)^n R_{n+1}(x)}{(x-x_0)^{n+1}} = \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{(n+1)(x-t)^n R_{n+1}(x)}{(x-x_0)^{n+1}}. \end{aligned} \quad (3.9)$$

$t = x_0$  da

$$F(x_0) = f(x) - \varphi(x, x_0) - R_{n+1}(x) = R_{n+1}(x) - R_{n+1}(x) = 0;$$

$t = x$  da

$$F(x) = f(x) - f(x) - \frac{f'(x)}{1!}(x-x) - \dots - \frac{f^{(n)}(x)}{n!}(x-x)^n - \frac{(x-x)^{n+1} R_{n+1}(x)}{(x-x_0)^{n+1}} = 0.$$

Demak,  $F(t)$  funksiya  $[x_0; x]$  kesmada Roli teoremasining barcha shartlarini qanoatlantiradi. U holda shunday  $c$  ( $x_0 < c < x$ ) nuqta topiladi va  $F'(c) = 0$  bo'ladi.

(3.9) tenglikka ko'ra,

$$-\frac{f^{(n+1)}(t)}{n!}(x-c)^n + \frac{(n+1)(x-c)^n R_{n+1}(x)}{(x-x_0)^{n+1}} = 0.$$

Bundan

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}.$$

$$\varphi(x, x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

ko'phadga  $n$ -tartibli *Taylor ko'phadi*,

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}$$

funksiyaga Taylor formulasing *Lagranj ko'rinishdagi qoldiq hadi* deyiladi.

$n=0$  da Taylor formulasidan  $f(x) = f(x_0) + f'(c)(x-x_0)$  yoki  $f(x) - f(x_0) = f'(c)(x-x_0)$  tenglik, ya'ni Lagranj formulasi kelib chiqadi. Demak, Lagranj formulasi Taylor formulasing xususiy holi bo'ladi.

$x_0 = 0$  da Taylor formulasing xususiy hollaridan yana biri

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots = \frac{f^{(n+1)}(cx)}{(n+1)!}x^{n+1}, \quad 0 < c < 1$$

hosil bo'ladi.

Bu formulaga *Makloren formulasi* deyiladi.

Ayrim funksiyalarning Makloren formulasiga yoyilmasini keltiramiz:

$$1. e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \quad x \in R;$$

$$2. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \sin cx \frac{x^{2n+2}}{(2n+2)!}, \quad x \in R;$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \cos cx \frac{x^{2n+1}}{(2n+1)!}, \quad x \in R;$$

$$4. (1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \\ + \frac{m(m-1)\dots(m-n)}{(n+1)!}(1+cx)^{m-n+1}x^{n+1}, \quad x \in (-1;1);$$

Xususan,  $n = m$  da

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-(n-2))}{(n-1)!}x^{n-1} + x^n.$$

Formulardan ba'zilarining isbotini keltiramiz.

1.  $f(x) = e^x$  bo'lsin.

U holda

$$f(x) = f'(x) = f''(x) = \dots = f^{(n+1)}(x) = e^x,$$

$$f(0) = f'(0) = f''(0) = \dots = f^{(n+1)}(0) = 1.$$

Makloren formulasi quyidagi ko'rinishga keladi:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1}. \quad (3.10)$$

2.  $f(x) = \sin x$  bo'lsin.

U holda

$$f^{(n)}(x) = \sin\left(x + n \cdot \frac{\pi}{2}\right), \quad f^{(n)}(0) = \sin\left(n \cdot \frac{\pi}{2}\right) = \begin{cases} 0, & n \text{ juft bo'lsa,} \\ (-1)^{\frac{n-1}{2}}, & n \text{ toq bo'lsa.} \end{cases}$$

Bundan

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \sin \theta x \frac{x^{2n+2}}{(2n+2)!}.$$

4.  $f(x) = (1+x)^m$  bo'lsin.

U holda

$$f^{(n)}(x) = m(m-1)\dots(m-n+1)(1+x)^{m-n+1},$$

$$f^{(n)}(0) = m(m-1)\dots(m-n+1).$$

Bundan

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n +$$

$$+ \frac{m(m-1)\dots(m-n)}{(n+1)!} (1+cx)^{m-n+1} x^{n+1}, x \in (-1;1).$$

Taylor formulasi funksiyalar qiymatlari va limitlarini berilgan aniqlikda hisoblash imkonini beradi. Masalan,  $f(x)$  funksiyaning  $x=a$  nuqtadagi qiymatini xatoligi  $\varepsilon$  dan katta bo'lmagan aniqlikda hisoblash uchun Taylor ko'phadini shunday  $k$  darajasigacha olinadiki, bunda  $k$  son  $|R_n(a)| < \varepsilon$  tengsizlikni qanoatlaniradigan  $n$  larning eng kichigi qilib tanlanadi.

$n$	$e_n$
0	1.00000000000000
1	2.00000000000000
2	2.50000000000000
3	2.66666666666667
4	2.70833333333333
5	2.71666666666667
6	2.71805555555556
7	2.7182539682540
8	2.7182787698413
9	2.7182815255732
10	2.7182818011464

8- misol.  $e$  sonini 0,001 aniqlikda hisoblang.

*Yechish.*  $f(x) = e^x$  funksiyani qaraymiz.

Shartga ko'ra,  $x = a = 1$ ,  $\varepsilon = 0,001$ .

(3.10) formulaga binoan  $n$  ning

$$R_n(1) = \frac{e^c}{(n+1)!} < \varepsilon = 0,001 \text{ shartni qanoatlantiruvchi eng kichik qiymati}$$

$n = 6$ , bunda  $0 < c < 1$ .

Demak,

$$e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{6!}$$

$$= 2 + 0,5 + 0,16667 + 0,04167 + 0,00833 + 0,00139 = 2,718.$$

Shu kabi  $e$  sonining istalgan aniqlikdagi qiymatlari jadvalini topish mumkin.

### 5.3.7. Mashqlar

1. Funksiya uchun berilgan kesmada Roll teoremasi o'rinli bo'lishini tekshiring. Agar o'rinli bo'lsa, teoremadagi  $c$  ning tegishli qiymatini toping:

1)  $f(x) = 4x - x^3 + 5$ ,  $[0;2]$ ;

2)  $f(x) = \sin 2x$ ,  $\left[\frac{\pi}{2}; \pi\right]$ ;

3)  $f(x) = 2 - \sqrt[3]{x^2}$ ,  $[-1;1]$ ;

4)  $f(x) = 3 - |x|$ ,  $[-2;2]$ .

2. Funksiya uchun berilgan kesmada Lagranj formulasidagi  $c$  ning tegishli qiymatini toping:

1)  $f(x) = \frac{1}{3}x^3 - x + 1$ ,  $[0;1]$ ;

2)  $f(x) = e^x$ ,  $[0;1]$ ;

3)  $f(x) = \ln x$ ,  $[1;e]$ ;

4)  $f(x) = x^3 - 6x + 1$ ,  $[0;1]$ .

3. Berilgan funksiya grafigining urinmasi  $AB$  vatarga parallel bo'lgan nuqtasini toping:

1)  $f(x) = x^2 + 3x$ ,  $A(-2; -2)$ ,  $B(1; 4)$ ;

2)  $f(x) = \sqrt{x+1}$ ,  $A(0; 1)$ ,  $B(3; 2)$ .

4. Funktsiyalar uchun berilgan kesmada Koshi formulasini yozing va bu formuladagi  $c$  ning tegishli qiymatini toping:

1)  $f(x) = \sin 2x$ ,  $g(x) = \cos 2x$ ,  $\left[0; \frac{\pi}{4}\right]$ ;

2)  $f(x) = x^4 - 3$ ,  $g(x) = x^3 + 2$ ,  $[0; 2]$ .

5. Funktsiyaning o'zgarma bo'lishlik alomatidan foydalanib, quyidagilarni isbotlang:

1)  $\arccos \frac{1-x^2}{1+x^2} = 2 \arctg x$ ,  $0 \leq x < +\infty$ ;

2)  $\arcsin \frac{2x}{1+x^2} = \begin{cases} -\pi - 2 \arctg x, & x \leq -1, \\ 2 \arctg x, & -1 \leq x < 1, \\ -\pi + 2 \arctg x, & x \geq 1. \end{cases}$

6. Loopital qoidasidan foydalanib, limitlarni toping:

1)  $\lim_{x \rightarrow 1} \frac{\sin \pi x}{\ln x}$ ;

2)  $\lim_{x \rightarrow 0} \frac{x - \arctg x}{x^3}$ ;

3)  $\lim_{x \rightarrow 0} \frac{\ln \operatorname{tg} 2x}{\ln \sin x}$ ;

4)  $\lim_{x \rightarrow +0} \frac{\ln x}{\operatorname{ctg} x}$ ;

5)  $\lim_{x \rightarrow +\infty} \frac{\log_3 x}{3^x}$ ;

6)  $\lim_{x \rightarrow +\infty} \frac{\pi - 2 \arctg x}{\ln \left(1 + \frac{1}{x}\right)}$ ;

7)  $\lim_{x \rightarrow 0} \frac{e^{x^2} - x^2 - 1}{\sin^4 x}$ ;

8)  $\lim_{x \rightarrow 0} \frac{\ln \cos(3x^2 - x)}{\sin 2x^2}$ ;

9)  $\lim_{x \rightarrow \infty} x \operatorname{tg} \frac{3}{x}$ ;

10)  $\lim_{x \rightarrow 0} (1 - e^{3x}) \operatorname{ctg} x$ ;

11)  $\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \operatorname{tg} x)$ ;

12)  $\lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{\arctg x} \right)$ ;

13)  $\lim_{x \rightarrow \frac{\pi}{2}-0} (\pi - 2x)^{\cos x}$ ;

14)  $\lim_{x \rightarrow 0} x^{\frac{1}{\ln(e^x - 1)}}$ ;

15)  $\lim_{x \rightarrow 3} \left( 2 - \frac{x}{3} \right)^{\frac{6}{x-3}}$ ;

16)  $\lim_{x \rightarrow 1} (\cos 3x)^{\frac{2}{x-1}}$ ;

17)  $\lim_{x \rightarrow \frac{\pi}{2}} (\operatorname{tg} x)^{1 - \sin x}$ ;

18)  $\lim_{x \rightarrow +\infty} (x + 3x)^{\frac{1}{x}}$ .



7. Ko'phadni  $(x - x_0)$  ning darajasi bo'yicha yoying:

1)  $P(x) = x^3 + 5x^2 - 3x + 1, x_0 = -2;$

2)  $P(x) = x^4 - 2x^3 + 5x - 6, x_0 = 2.$

8. Funksiyaning berilgan nuqtada uchinchi tartibli Teylor formulasini yozing:

1)  $f(x) = \sqrt{1+x}, x_0 = 3;$

2)  $f(x) = \frac{1}{x}, x_0 = -2.$

9. Funksiyalarni Makloren formulasi yordamida  $x$  ning darajalari bo'yicha yoying:

1)  $f(x) = xe^x;$

2)  $f(x) = chx.$

10. Berilganlarni 0,001 aniqlikda hisoblang:

1)  $\sin 36^\circ;$

2)  $\cos 32^\circ;$

3)  $\sqrt[3]{e};$

4)  $\lg 10,09.$

11. Limitlarni Makloren formulasi yordamida toping:

1)  $\lim_{x \rightarrow 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}};$

2)  $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4};$

3)  $\lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x}};$

4)  $\lim_{x \rightarrow 0} \frac{1 + x \cos x - \sqrt{1 + 2x}}{\ln(1+x) - x}$

## 5.4. FUNKSIYAILARNI HOSILALAR YORDAMIDA TEKSHIRISH

Differensial hisobning tatbiqlaridan biri funksiyalarni tekshirish va grafigini chizishga hosilaning qo'llanilishi hisoblanadi. Quyida bu tatbiqlarni ifodalovchi teoremlarni keltiramiz.

### 5.4.1. Funksiyaning monotonlik shartlari

**1-teorema** (*funksiya monoton bo'lishining zaruriy sharti*). Agar  $(a; b)$  intervalda differensiallanuvchi  $f(x)$  funksiya shu intervalda o'suvchi (kamayuvchi) bo'lsa, u holda  $\forall x \in (a; b)$  da

$$f'(x) \geq 0 \quad (f'(x) \leq 0)$$

bo'ladi.

Isboti.  $f(x)$  funksiya  $(a;b)$  intervalda o'suvchi bo'lsin.  $(a;b)$  intervalning ixtiyoriy  $x$  va  $x + \Delta x$  nuqtalarini olamiz.

U holda  $\Delta x > 0$  bo'lsa,  $f(x + \Delta x) > f(x)$  va  $\Delta x < 0$  bo'lsa,  $f(x + \Delta x) < f(x)$  bo'ladi.

Ikkala holda ham

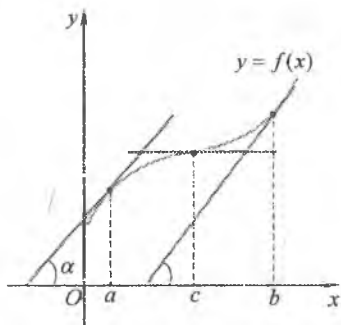
$$\frac{f(x + \Delta x) - f(x)}{\Delta x} > 0.$$

Teoremaning shartiga ko'ra  $f(x)$  funksiya  $(a;b)$  intervalda differensiallanuvchi.

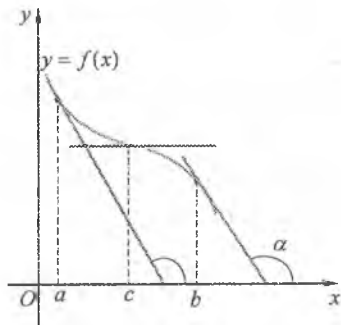
Shu sababli

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0.$$

$f(x)$  funksiya  $(a;b)$  intervalda kamayuvchi bo'lganda teorema shu kabi isbotlanadi.



8-shakl.



9-shakl.

Bu teorema ushbu geometrik talqinga ega: biror intervalda differensiallanuvchi bo'lgan o'suvchi (kamayuvchi) funksiya grafigiga o'tkazilgan urinmalar  $Ox$  o'qning musbat yo'nalishi bilan o'tkir (o'tmas) burchak tashkil qiladi yoki ayrim nuqtalarda  $Ox$  o'qiga parallel bo'ladi (8-shakl) ((9-shakl)).

**2-teorema** (*funksiya monoton bo'lishining yetarli sharti*). Agar  $(a;b)$  intervalda differensiallanuvchi  $f(x)$  funksiya uchun  $\forall x \in (a;b)$  da  $f'(x) > 0$  ( $f'(x) < 0$ ) bo'lsa,  $f(x)$  funksiya  $(a;b)$  intervalda o'sadi (kamayadi).

*Isboti.*  $(a;b)$  intervalda  $f'(x) > 0$  bo'lsin.  $\forall x_1, x_2 \in (a;b), x_1 < x_2$  nuqtalarni olamiz.

$f(x)$  funksiya uchun  $[x_1, x_2]$  kesmada Lagranj teoremasining shartlari bajariladi. Shu sababli Lagranj formulasiga binoan, biror  $c \in (x_1, x_2)$  da  $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$  bo'ladi.

Teoremaning shartiga ko'ra,  $\forall x \in (a;b)$  da  $f'(x) > 0$ , shu jumladan,  $c \in (x_1, x_2)$  da  $f'(c) > 0$ .  $x_2 - x_1 > 0$  va shuning uchun  $f'(c)(x_2 - x_1) > 0$ . Bundan  $f(x_2) - f(x_1) > 0$  yoki  $f(x_2) > f(x_1)$ . Shunday qilib,  $f(x)$  funksiya  $\forall x \in (a;b)$  da o'sadi.

$f'(x) < 0$  bo'lganda teorema shu kabi isbotlanadi.

*Eslatmalar.* 1. Funksiya o'suvchi va kamayuvchi bo'lgan intervallarga funksiyaning monotonlik intervallari deyiladi.

2.  $(a;b)$  intervalda o'suvchi (kamayuvchi) va  $[a;b]$  kesmada uzluksiz bo'gan  $f(x)$  funksiya  $[a;b]$  kesmada o'suvchi (kamayuvchi) bo'ladi.

*1-misol.*  $f(x) = x^3 - 12x + 5$  funksiyaning monotonlik intervallarini toping.

*Yechish.*  $D(f) = R$ ,  $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$ . U holda:  $f'(x) > 0$  dan  $x^2 - 4 > 0$  yoki  $|x| > 2$ ;  $f'(x) < 0$  dan  $x^2 - 4 < 0$  yoki  $|x| < 2$ .

Demak,  $f(x)$  funksiya  $(-\infty; -2) \cup (2; +\infty)$  intervalda o'sadi,  $(-2; 2)$  intervalda kamayadi.

## 5.4.2. Funksiyaning ekstremumlari

**1-ta'rif.** Agar  $x_0$  nuqtaning shunday  $\delta$  atrofi topilsa va bu atrofning barcha  $x \neq x_0$  nuqtalarida  $f(x) < f(x_0)$  ( $f(x) > f(x_0)$ ) tengsizlik bajarilsa,  $x_0$  nuqtaga  $f(x)$  funksiyaning *qat'iy lokal maksimum* (*qat'iy lokal minimum*) nuqtasi deyiladi.

Funksiyaning maksimum va minimum nuqtalariga *ekstremum* nuqtalar deyiladi. Funksiyaning ekstremum nuqtadagi qiymati *funksiyaning ekstremumi* deb ataladi.

Ekstremum tushunchasi funksiya aniqlanish sohasining biror atrofi bilan bog'liq. Shu sababli funksiya ekstremumga aniqlanish sohasining faqat ichki nuqtalarida erishadi. Shu bilan birga, funksiya o'zining aniqlanish sohasida bir nechta minimumga yoki

maksimumga erishishi va bunda maksimumlardan ayrimlari qandaydir minimumdan kichik bo'lishi mumkin (10-shakl).

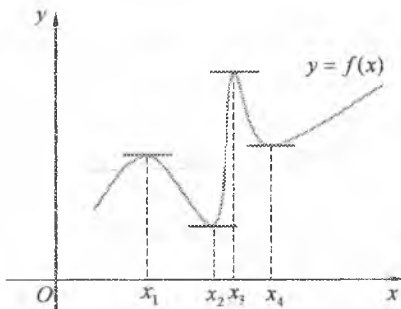
**3-teorema (ekstremum mavjud bo'lishining zaruriy sharti).** Agar  $x_0$  nuqtada differensiallanuvchi  $f(x)$  funksiya shu nuqtada ekstremumga ega bo'lsa, u holda  $f'(x_0)=0$  bo'ladi.

*Ishoti.*  $x_0$  nuqta  $f(x)$  funksiyaning ekstremum nuqtasi bo'lsin. U holda shunday  $(x_0 - \delta, x_0 + \delta)$  interval topiladi va bu intervalda  $f(x)$  funksiya o'zining eng katta yoki eng kichik qiymatiga ega bo'ladi.

U holda Ferma teoremasiga ko'ra,  $f'(x_0)=0$  bo'ladi.

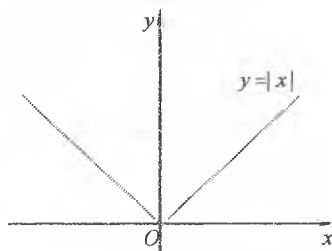
Bu teorema quyidagicha geometrik talqiniga ega. Agar  $x$  nuqta  $f(x)$  funksiyaning ekstremum nuqtasi bo'lsa (masalan, 10-shaklda  $x_1$  nuqta),

funksiya grafigiga shu nuqtada urinma o'tkazish mumkin va bu urinma  $Ox$  o'qiga parallel bo'ladi.



10-shakl.

**Izohlar.** 1.  $f(x)$  funksiya  $x_0$  nuqtada uzluksiz bo'lib, differensiallanuvchi bo'lmaganida ham, ekstremumga ega bo'lishi mumkin. Masalan, uzluksiz  $y=|x|$  funksiya  $x=0$  nuqtada hosilaga ega emas, ammo  $x=0$  minimum nuqta (11-shakl).



11-shakl.

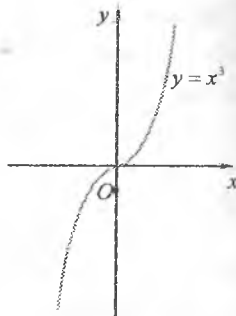
Shunday qilib,  $f(x)$  funksiya  $x_0$  nuqtada ekstremumga ega bo'lsa, bu nuqtada  $f'(x)$  hosila nolga teng yoki mavjud bo'lmaydi.

$f'(x)$  hosilasi nolga teng bo'lgan yoki mavjud bo'lmagan nuqtaga *birinchi tur kritik nuqta* deyiladi.

2. Hamma birinchi tur kritik nuqta ham ekstremum nuqta bo'lavermaydi. Masalan,  $f(x)=x^3$  funksiya uchun  $x=0$  da  $f'(x)=3x^2=0$ . Demak,  $x=0$  birinchi tur kritik nuqta, ammo u ekstremum nuqta emas (12-shakl).

**4-teorema** (*ekstremum mavjud bo'lishining yetarli sharti*). Agar  $f(x)$  funksiya  $x_0$  birinchi tur kritik nuqtaning biror  $\delta$  atrofida differensiallanuvchi bo'lib,  $x$  nuqta  $x_0$  nuqtadan chapdan o'ngga o'tganida  $f'(x)$  hosila: ishorasini musbatdan manfiyga o'zgartirsa  $x_0$  nuqta maksimum nuqta bo'ladi; manfiydan musbatga o'zgartirsa  $x_0$  nuqta minimum nuqta bo'ladi.

*Isboti.*  $x_0$  – birinchi tur kritik nuqta,  $x \in (x_0 - \delta; x_0)$  da  $f'(x) > 0$  va  $x \in (x_0, x_0 + \delta)$  da  $f'(x) < 0$  bo'lsin. U holda 1-teoremaga ko'ra, funksiya  $(x_0 - \delta; x_0)$  intervalda o'sadi va  $(x_0; x_0 + \delta)$  intervalda kamayadi. Demak,  $f(x)$  funksiyaning  $x_0$  nuqtadagi qiymati uning  $\forall x \in (x_0 - \delta; x_0 + \delta)$  nuqtadagi qiymatidan katta bo'ladi, ya'ni  $f(x)$  funksiya  $x_0$  nuqtada maksimumga erishadi.



12-shakl.

$x \in (x_0 - \delta; x_0)$  da  $f'(x) < 0$  va  $x \in (x_0, x_0 + \delta)$  da  $f'(x) > 0$  bo'lgan hol uchun teorema shu kabi isbotlanadi.

**Izoh.** Agar  $x$  nuqta  $x_0$  nuqtadan chapdan o'ngga o'tganida ishorasini o'zgartirmasa  $x_0$  nuqtada ekstremum mavjud bo'lmaydi.

Funksiyani ekstremumga tekshirish – bu funksiyaning barcha ekstremumlarini topish demakdir. Ekstremum mavjud bo'lishining zaruriy va yetarli shartlaridan funksiyaning ekstremumga tekshirishning quyidagi qoidasi kelib chiqadi:

1°.  $y = f(x)$  funksiyaning birinchi tur kritik nuqtalari topiladi;

2°. Bu nuqtalardan funksiyaning aniqlanish sohasiga tegishli bo'lganlari tanlanadi;

3°. tanlangan nuqtadan chapdan o'ngga o'tishda  $f'(x)$  hosilaning ishorasi tekshiriladi;

4°. 4- teoremaga asosan funksiyalarning ekstremum nuqtalari (agar ular bor bo'lsa) aniqlanadi va funksiyaning bu nuqtalardagi qiymatlari hisoblanadi.

2- misol.  $f(x) = \sqrt[3]{x^2} - \frac{x}{3}$  funksiyaning ekstremumlarini toping.

*Yechish.* Ravshanki,

$$D(f) = R, \quad f'(x) = \frac{2}{3 \cdot \sqrt[3]{x}} - \frac{1}{3} = \frac{1}{3} \cdot \frac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}$$

Hosila  $x_1 = 0$  nuqtada mavjud emas va  $x_2 = 8$  nuqtada nolga teng. Bu kritik nuqtalar berilgan funksiyaning aniqlanish sohasini uchta

intervallarga ajratadi. Hosilaning har bir birinchi tur kritik nuqtadan chapdan o'ngga o'tishdagi ishoralarini chizmada



13-shakl.

belgilaymiz (13-shakl). Bunda strelkalar funksiyaning tegishli intervalda o'suvchi yoki kamayuvchi ekanligini bildiradi.

Demak,  $x_1 = 0$  – minimum nuqta,  $y_{\min} = f(0) = 0$  va  $x_2 = 8$  – maksimum nuqta,  $y_{\max} = f(8) = \frac{4}{3}$ .

Funksiyaning eng katta va eng kichik qiymatini topish matematika, fizika, kimyo, iqtisodiyot va boshqa fanlarning ko'plab masalalarini yechishda keng qo'llaniladi. Masalan: minimal xarajat sarflab yukni tashish haqidagi transport masalasi, maksimal daromad olish maqsadida ishlab chiqarishni tashkil etish masalasi, eng katta va eng kichik qiymatlarni izlash usullarini takomillashtirish va rivojlantirishga olib keluvchi optimal yechimlarni izlash haqidagi boshqa masalalar. Bunday masalalarni yechish bilan matematikaning maxsus bo'limi – chiziqli programmalashtirish shug'ullanadi.

Biz soddaroq masalalardan birini ko'rib chiqamiz.

**3-misol.** Tomoni 12 uzunlik birligiga teng kvadrat tunukaning burchaklaridan bir xil o'lchamli kvadratlar kesib olingan va usti ochiq quti yasalgan. Qutining sig'imi eng katta bo'lishi uchun tomoni necha uzunlik birligiga teng kvadratlar kesilishi kerak?

**Yechish.** Kesib olinadigan kvadratlar tomoni  $x$  bo'lsin.

U holda qutining asosi  $12 - 2x$  va balandligi  $x$  ga teng bo'ladi.

Qutining hajmini topamiz:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3, \quad x \in [0; 6].$$

Bu funksiyaning maksimumini aniqlaymiz.

$$V'(x) = 144 - 96x + 12x^2 = 12(2 - x)(6 - x)$$

hosila  $x = 2$  da ishorasini musbatdan manfiyga almashtiradi, ya'ni  $x = 2$  maksimum nuqta bo'ladi.

Demak, kesib olinadigan kvadratlar tomoni  $x = 2$  (uz.b), bunda  $V(2) = 128$  (kub.b).

### 5.4.3. Kesmada uzluksiz funksiyaning eng katta va eng kichik qiymatlari

$y = f(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lsin. U holda Veersht rassning ikkinchi teoremasiga ko'ra, funksiya bu kesmada o'zining eng katta va eng kichik qiymatlariga ega bo'ladi. Bu qiymatlarga funksiya yoki ekstremum nuqtalarda yoki  $[a; b]$  kesmaning chetki nuqtalarida erishadi.

Bundan  $[a; b]$  kesmada uzluksiz  $y = f(x)$  funksiyaning eng katta va eng kichik qiymatlarini topishning quyidagi qoidasi kelib chiqadi:

1°.  $y = f(x)$  funksiyaning  $(a; b)$  intervaldagi birinchi tur kritik nuqtalari topiladi;

2°. funksiyaning topilgan kritik nuqtalardagi va  $[a; b]$  kesmaning chetki nuqtalaridagi qiymatlari hisoblanadi;

3°. hisoblangan qiymatlar orasidan eng kattasi va eng kichigi tanlanadi.

**Izohlar.** 1. Agar  $y = f(x)$  funksiya  $[a; b]$  kesmada faqat bitta kritik nuqtaga ega bo'lib, u maksimum (minimum) nuqta bo'lsa, bu nuqtada funksiya o'zining eng katta (eng kichik) qiymatiga ega bo'ladi, ya'ni  $\max_{x \in [a; b]} f(x) = f(x_0)$  ( $\min_{x \in [a; b]} f(x) = f(x_0)$ ) bo'ladi.

2. Agar  $y = f(x)$  funksiya  $[a; b]$  kesmada kritik nuqtaga ega bo'lmasa, bu funksiyaning kesmada monoton o'sishini yoki monoton kamayishini bildiradi. Bunda  $y = f(x)$  funksiya  $[a; b]$  kesmadagi eng katta va eng kichik qiymatlariga kesmaning chetki nuqtalarida erishadi.

**4- misol.**  $y = x^3 - 3x$  funksiyaning  $[0, 2]$  kesmada eng katta va eng kichik qiymatlarini toping.

*Yechish.*

1°.  $f'(x) = 3x^2 - 3 = 0$  dan  $x_1 = -1$ ,  $x_2 = 1$ ,  $x_2 \in [0, 2]$ ;

2°.  $f(1) = -2$ ,  $f(0) = 0$ ,  $f(2) = 2$ ;

3°.  $\max_{x \in [0, 2]} f(x) = f(2) = 2$ ;  $\min_{x \in [0, 2]} f(x) = f(1) = -2$ .

### 5.4.4. Funksiya grafigining botiqligi, qavariqligi va egilish nuqtalari

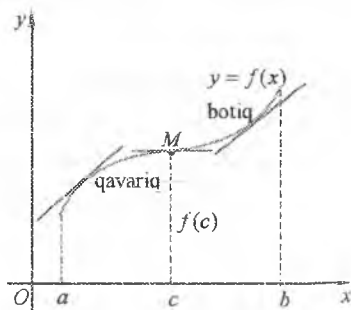
$y = f(x)$  funksiya  $(a; b)$  intervalda differensiallanuvchi bo'lsin. U holda  $y = f(x)$  funksiya grafigining  $\forall M(x; f(x))$ ,  $x \in (a, b)$  nuqtada

urinishi mavjud bo'ladi.

**2-ta'rif.** Agar  $(a;b)$  intervalda  $y = f(x)$  funksiyaning grafigi unga intervalning ixtiyoriy nuqtasida o'tkazilgan urinmadan yuqorida (pastda) yotsa, *funksiya grafigi  $(a;b)$  intervalda botiq (qavariq)* deyiladi (14-shakl).

Funksiya grafigining botiq qismini qavariq qismidan ajratuvchi  $M(c; f(c))$  nuqta *funksiya grafigining egilish nuqtasi* deb ataladi.

**5-teorema.** Agar  $y = f(x)$  funksiya  $(a;b)$  intervalda ikkinchi tartibli hosilaga ega va  $\forall x \in (a;b)$  da  $f''(x) < 0$  ( $f''(x) > 0$ ) bo'lsa, u holda  $y = f(x)$  funksiya grafigi  $(a;b)$  intervalda qavariq (botiq) bo'ladi.



14-shakl.

*Isboti.*  $\forall x \in (a;b)$  da  $f''(x) < 0$  bo'lsin. Funksiya grafigida  $x_0 \in (a;b)$  absissali ixtiyoriy  $M$  nuqta olamiz (15-shakl). Funksiyaning grafigi bu urinmadan pastda yotishini ko'rsatamiz. Buning uchun  $x \in (a;b)$  nuqtada  $y = f(x)$  egri chiziqning  $y$  ordinatasi bilan urinmaning  $y_{ur}$  ordinasini solishtiramiz.

Urunma tenglamasi

$$y_{ur} = f(x_0) + f'(x_0)(x - x_0).$$

bo'lgani uchun

$$y - y_{ur} = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

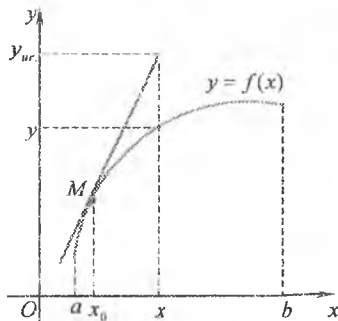
Lagranj teoremasiga ko'ra  $f(x) - f(x_0) = f'(c)(x - x_0)$ , bu yerda  $c$  nuqta  $x_0$  bilan  $x$  ning orasida yotadi.

Shu sababli

$$y - y_{ur} = f'(c)(x - x_0) - f'(x_0)(x - x_0)$$

yoki

$$y - y_{ur} = (f'(c) - f'(x_0))(x - x_0).$$



15-shakl.



$f'(c) - f'(x_0)$  ayirmaga Lagranj teoremasini takror qo'llaymiz:

$f'(c) - f'(x_0) = f''(c_1)(x - x_0)$ , bu yerda  $c_1$  nuqta  $c$  bilan  $x_0$  ning orasida yotadi. Demak,  $y - y_{ur} = f''(c_1)(c - x_0)(x - x_0)$ .

Bu tengsizlikni tekshiramiz:

1) agar  $x > x_0$  bo'lsa, u holda  $x - x_0 > 0$ ,  $c - x_0 > 0$  bo'ladi va  $f''(c_1) < 0$ ;

2) agar  $x < x_0$  bo'lsa, u holda  $x - x_0 < 0$ ,  $c - x_0 < 0$  bo'ladi va  $f''(c_1) < 0$ .

Har ikkala holda ham  $y - y_{ur} = f''(c_1)(c - x_0)(x - x_0) < 0$ , ya'ni  $y < y_{ur}$ .

Demak,  $\forall x \in (a; b)$  da urunmaning ordinatasi funksiya grafigining ordinatasidan katta bo'ladi va  $(a; b)$  intervalda funksiya grafigi qavariq.

$f''(x) > 0$  da funksiya grafigi botiq bo'lishi shu kabi isbotlanadi.

Funksiya grafigining egilish nuqtasini topish quyidagi teoremalarga asoslanadi.

**6-teorema (egilish nuqta mavjud bo'lishining zaruriy sharti).** Agar  $y = f(x)$  funksiya  $(a; b)$  intervalda uzluksiz ikkinchi tartibli hosilaga ega va  $M(x_0; f(x_0))$  nuqta funksiya grafigining egilish nuqtasi bo'lsa, u holda  $f''(x_0) = 0$  bo'ladi.

*Isboti.* Teskarisini faraz qilamiz:  $f''(x_0) \neq 0$ , aniqlik uchun  $f''(x_0) > 0$ ;  $x_0$  nuqtaning biror atrofida  $f''(x) > 0$  bo'lsin. U holda 5-teoremaga ko'ra, funksiya grafigi bu atrofda botiq bo'ladi. Bu  $x_0$  nuqta egilish nuqtaning absissasi bo'ladi mulohazasiga zid. Demak, qilingan faraz noto'g'ri va  $f''(x) = 0$ .

$f''(x) = 0$  bo'ladigan nuqtalarning hammasi ham funksiya grafigining egilish nuqtasi bo'lavermaydi. Masalan,  $f(x) = x^4$  funksiya grafigining  $M(0; 0)$  nuqtasi egilish nuqta emas, ammo  $x = 0$  da  $f''(x) = 12x^2 = 0$ .

Demak,  $f''(x_0) = 0$  shart egilish nuqta mavjud bo'lishining zaruriy sharti bo'ladi.

**7-teorema (egilish nuqta mavjud bo'lishining yetarli sharti)**  $y = f(x)$  funksiya  $x_0$  nuqtaning biror  $\delta$  atrofida ikkinchi tartibli hosilaga ega bo'lsin. Agar  $\delta$  atrofning  $x_0$  nuqtadan chap va o'ng qismlarida  $f''(x)$  hosila har xil ishoraga ega bo'lsa, u holda  $M(x_0; f(x_0))$  nuqta funksiya grafigining egilish nuqtasi bo'ladi.

*Isboti.*  $x \in (x_0 - \delta; x_0)$  da  $f''(x) > 0$  va  $x \in (x_0, x_0 + \delta)$  da  $f''(x) < 0$  bo'lsin. U holda 5-teoremaga ko'ra,  $x_0$  nuqtadan chapda funksiya

grafigi botiq va o'ngda qavariq bo'ladi. Demak,  $M(x_0; f(x_0))$  nuqta funksiya grafigining egilish nuqtasi bo'ladi.

$x \in (x_0 - \delta; x_0)$  da  $f''(x) < 0$  va  $x \in (x_0, x_0 + \delta)$  da  $f''(x) > 0$  bo'lgan holda teorema shu kabi isbotlanadi.

**Izoh.**  $y = f(x)$  funksiya  $x_0$  nuqtaning biror  $\delta$  atrofida ikkinchi tartibli hosilaga ega bo'lib, uning  $f''(x_0)$  hosilasi mavjud bo'lmaganida ham egilish nuqtaga ega bo'lishi mumkin. Shu sababli egilish nuqtalarni ikkinchi tartibli hosila nolga teng bo'lgan yoki uzilishga ega bo'lgan nuqtalardan izlash kerak bo'ladi.

$f''(x)$  hosilasi nolga teng bo'lgan yoki mavjud bo'lmagan nuqtaga *ikkinchi tur kritik nuqta* deyiladi.

5- misol.  $y = \frac{x}{1-x^2}$  funksiya grafigini botiq va qavariqlikka tekshiring.

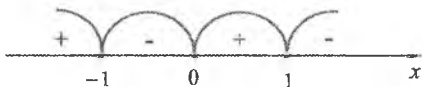
*Yechish.*  $D(f) = (-\infty; -1) \cup (-1; 1) \cup (1; \infty)$ .

$$y' = \left( \frac{x}{1-x^2} \right)' = \frac{x^2 + 1}{(1-x^2)^2}, \quad y'' = \left( \frac{x^2 + 1}{(1-x^2)^2} \right)' = \frac{2x(x^2 + 3)}{(1-x^2)^3}$$

Ikkinchi tartibli hosila  $x_2 = 0$  nuqtada nolga teng va  $x_1 = -1$ ,  $x_3 = 1$  nuqtalarda mavjud emas.

$f''(x)$  hosilaning ishorasini oraliqlar usuli bilan tekshiramiz:

Demak, funksiyaning grafigi  $(-1; 0)$  va  $(1; \infty)$  intervallarda qavariq,  $(-\infty; -1)$  va  $(0; 1)$  intervallarda botiq bo'ladi.  $O(0; 0)$  nuqta funksiya grafigining egilish nuqtasi bo'ladi.



### 5.4.5. Funksiya grafigining asimptotalari

*Egri chiziqning asimptotasi* deb shunday to'g'ri chiziqqa aytiladiki, egri chiziqda yotuvchi  $M$  nuqta egri chiziq bo'ylab harakat qilib koordinata boshidan cheksiz uzoqlashgani sari  $M$  nuqtadan bu to'g'ri chiziqqacha bo'lgan masofa nolga intiladi (16-shakl).

Uch turdagi, ya'ni vertikal, gorizontal va og'ma asimptotalar mavjud.

Agar  $\lim_{x \rightarrow x_0+0} f(x)$  yoki  $\lim_{x \rightarrow x_0-0} f(x)$  limitlardan hech bo'lmaganda bittasi cheksiz ( $+\infty$  yoki  $-\infty$ ) bo'lsa,  $x = x_0$  to'g'ri chiziq  $y = f(x)$  funksiya grafigining asimptotasi bo'ladi. Bunday asimptota *vertikal asimptota* deb ataladi.

Masalan,  $f(x) = \frac{1}{x}$  funksiya grafigi uchun  $x = 0$  to'g'ri chiziq vertikal asimptota, chunki  $\lim_{x \rightarrow x_0+0} f(x) = +\infty$  va  $\lim_{x \rightarrow x_0-0} f(x) = -\infty$ .

Shunday qilib, vertikal asimptotalarni izlash uchun  $x$  ning unga yaqin qiymatlarida  $f(x)$  funksiya modul bo'yicha cheksiz o'sadigan  $x_0$  qiymatini topish kerak. Odatda, bu  $x_0$  ikkinchi tur uzilish nuqtasi bo'ladi.

Agar shunday  $k$  va  $b$  sonlari mavjud bo'lib,  $x \rightarrow \infty$  ( $x \rightarrow -\infty$ ) da  $f(x)$  funksiya

$$f(x) = kx + b + \alpha(x), \quad \lim_{x \rightarrow \infty} \alpha(x) = 0$$

ko'rinishda ifodalansa,  $y = kx + b$  to'g'ri chiziq  $y = f(x)$  funksiya grafigining asimptotasi bo'ladi. Bunday asimptota *og'ma asimptota* deb ataladi.

**8-teorema.**  $y = f(x)$  funksiya grafigi  $y = kx + b$  og'ma asimptotaga ega bo'lishi uchun

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = k, \quad \lim_{x \rightarrow \infty} (f(x) - kx) = b$$

bo'lishi zarur va yetarli.

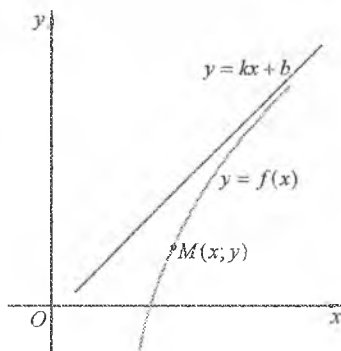
*Isboti. Zarurligi.*  $y = f(x)$  funksiya grafigi  $y = kx + b$  og'ma asimptotaga ega bo'lsin.

U holda og'ma asimptotaning ta'rifiga ko'ra,  $y = kx + b + \alpha(x)$ ,  $\lim_{x \rightarrow \infty} \alpha(x) = 0$  bo'ladi.

Bundan

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \left( k + \frac{b}{x} + \frac{\alpha(x)}{x} \right) = k, \quad \lim_{x \rightarrow \infty} (f(x) - kx) = \lim_{x \rightarrow \infty} (b - \alpha(x)) = b$$

kelib chiqadi.



16-shakl.

*Yotarliligi.*  $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = k$ ,  $\lim_{x \rightarrow +\infty} (f(x) - kx) = b$  bo'lsin.

U holda  $\lim_{x \rightarrow +\infty} (f(x) - kx) = b$  dan  $f(x) - kx = b + \alpha(x)$ ,  $\lim_{x \rightarrow +\infty} \alpha(x) = 0$  kelib chiqadi. Demak,  $f(x) = kx + b + \alpha(x)$  bo'ladi. Bu esa  $y = kx + b$  to'g'ri chiziq  $f(x)$  funksiya grafigining asimptotasi ekanini bildiradi.

Agar  $\lim_{x \rightarrow +\infty} \frac{f(x)}{x}$ ,  $\lim_{x \rightarrow +\infty} (f(x) - kx)$  limitlardan hech bo'lmaganda bittasi mavjud bo'lmasa yoki cheksiz bo'lsa,  $f(x)$  funksiya grafigi og'ma asimptotaga ega bo'lmaydi.

Agar  $k=0$  bo'lsa,  $b = \lim_{x \rightarrow +\infty} f(x)$  bo'ladi. Bunda  $y = b$  to'g'ri chiziqqa  $f(x)$  funksiya grafigining *gorizontal asimptotasi* deyiladi.

*Izoh.*  $y = f(x)$  funksiya grafigining asimptotalari  $x \rightarrow +\infty$  da va  $x \rightarrow -\infty$  da har xil bo'lishi mumkin. Shu sababli  $\lim_{x \rightarrow +\infty} \frac{f(x)}{x}$ ,  $\lim_{x \rightarrow +\infty} (f(x) - kx)$  limitlarni aniqlashda  $x \rightarrow +\infty$  va  $x \rightarrow -\infty$  hollarini alohida qarash lozim.

*5- misol.*  $y = \frac{x^2 - 3}{x}$  funksiya grafigining asimptotalarini toping.

*Yechish.*  $\lim_{x \rightarrow 0+} \frac{x^2 - 3}{x} = -\infty$ ,  $\lim_{x \rightarrow 0-} \frac{x^2 - 3}{x} = +\infty$ .

Demak,  $x = 0$  to'g'ri chiziq vertikal asimptota.

$$k = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{x^2 - 3}{x^2} = 1,$$

$$b = \lim_{x \rightarrow +\infty} [f(x) - kx] = \lim_{x \rightarrow +\infty} \left( \frac{x^2 - 3}{x} - x \right) = \lim_{x \rightarrow +\infty} \frac{-3}{x} = 0,$$

$$k = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} \frac{x^2 - 3}{x^2} = 1,$$

$$b = \lim_{x \rightarrow -\infty} [f(x) - kx] = \lim_{x \rightarrow -\infty} \left( \frac{x^2 - 3}{x} - x \right) = \lim_{x \rightarrow -\infty} \frac{-3}{x} = 0$$

Bundan

$$y = kx + b = x.$$

Demak,  $y = x$  to'g'ri chiziq og'ma asimptota.

### 5.4.6. Funksiyani tekshirish va grafigini chizishning umumiy sxemasi

Funksiyani tekshirish va grafigini chizish ma'lum tartibda (sxema asosida) bajariladi. Shunday sxemalardan birini keltiramiz.

1°. Funksiyaning aniqlanish sohasini topish.

2°. Funksiya grafigining koordinata o'qlari bilan kesishadigan nuqtalarini (agar ular mavjud bo'lsa) aniqlash.

3°. Funksiyaning ishorasi o'zgarmaydigan oraliqlarni ( $f(x) > 0$  yoki  $f(x) < 0$  bo'ladigan oraliqlarni) aniqlash.

4°. Funksiyaning juft-toqligini tekshirish.

5°. Funksiya grafigining asimptotalarini topish.

6°. Funksiyaning monotonlik oraliqlarini aniqlash va ekstremumlarini topish.

7°. Funksiyaning qavariqlik va botiqlik oraliqlarini hamda egilish nuqtalarini aniqlash.

8°. 1° - 7° bandlardagi tekshirishlar asosida funksiyaning grafigini chizish.

Funksiya grafigini chizish uchun keltirilgan sxemaning hamma bandlari, albatta, bajarilishi shart emas. Soddaroq hollarda keltirilgan bandlardan ayrimlarini, masalan 1°, 2°, 6°ni bajarish yetarli bo'ladi. Agar funksiya grafigi juda tushunarli bo'lmasa 1° - 7° bandlardan keyin funksiyaning davriyligini tekshirish, funksiyaning bir nechta qo'shimcha nuqtalarini topish va funksiyaning boshqa xususiyatlarini aniqlash bo'yicha qo'shimcha tekshirishlar o'tkazish mumkin. •

7- misol.  $y = \frac{x^2 + 1}{x^2 - 1}$  funksiyaning tekshirish va grafigini chizish.

*Yechish.* 1°. Funksiyaning aniqlanish sohasi:

$$D(f) = (-\infty; -1) \cup (-1; 1) \cup (1; +\infty).$$

2°.  $x = 0$  da  $y = -1$  bo'ladi. Funksiya Oy o'qini  $(0; -1)$  nuqtada kesadi.  $y \neq 0$  bo'lgani uchun funksiya Ox o'qini kesmaydi.

3°. Funksiya  $(-\infty; -1)$  va  $(1; +\infty)$  intervallarda musbat ishorali va  $(-1; 1)$  intervalda manfiy ishorali.

4°. Funksiya uchun  $x \in D(f)$  da  $f(-x) = f(x)$  bo'ladi. Demak, u juft.

$$5^{\circ} \lim_{x \rightarrow -1-0} \frac{x^2+1}{x^2-1} = +\infty, \quad \lim_{x \rightarrow -1+0} \frac{x^2+1}{x^2-1} = -\infty,$$

$$\lim_{x \rightarrow 1-0} \frac{x^2+1}{x^2-1} = -\infty, \quad \lim_{x \rightarrow 1+0} \frac{x^2+1}{x^2-1} = +\infty.$$

Demak,  $x = -1$  va  $x = 1$  to'g'ri chiziqlar vertikal asimptotalar bo'ladi.

$$k = \lim_{x \rightarrow \pm\infty} \frac{x^2+1}{x(x^2-1)} = 0 \quad (x \rightarrow +\infty \text{ da ham } x \rightarrow -\infty \text{ da ham } k=0),$$

$$b = \lim_{x \rightarrow \pm\infty} \left( \frac{x^2+1}{x^2-1} - 0 \cdot x \right) = 1.$$

Demak,  $y = 1$  to'g'ri chiziq  $x \rightarrow +\infty$  da ham  $x \rightarrow -\infty$  da ham gorizontal asimptota bo'ladi.

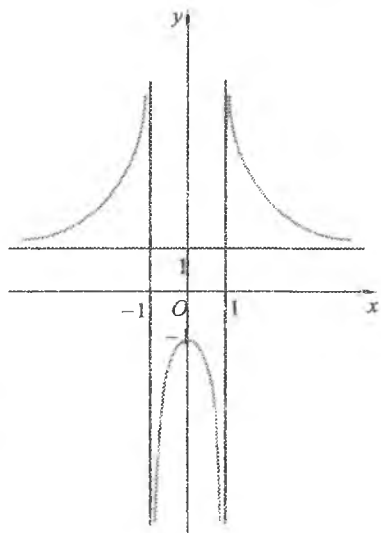
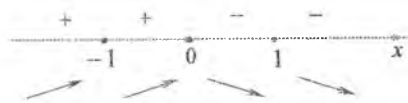
6°. Funksiyaning monotonlik oraliqlarini aniqlaymiz va ekstremumlarini topamiz.

$$y' = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} = -\frac{4x}{(x^2-1)^2}.$$

Birinchi tartibli hosila  $x = -1$  va  $x = 1$  da mavjud emas va  $x = 0$  da nolga teng. Bu nuqtalar berilgan funksiyaning aniqlanish sohasini to'rtta

$$(-\infty; -1), (-1; 0), (0; 1), (1; +\infty)$$

intervallarga ajratadi. Hosilaning bu intervallardagi va har bir birinchi tur kritik nuqtadan chapdan o'ngga o'tishdagi ishoralarini chizmada belgilaymiz:



17-shakl.

Demak, funksiya  $(-\infty; -1)$  va  $(-1; 0)$  intervallarda o'sadi,  $(0; 1)$  va  $(1; +\infty)$  intervallarda kamayadi.  $x = 0$  maksimum nuqta,  $y_{\max} = f(0) = -1$ .

7°. Funksiyaning qavariqlik va botiqlik oraliqlarini hamda egilish nuqtalarini aniqlaymiz.

$$y'' = \left( -\frac{4x}{(x^2-1)^2} \right)' = -4 \frac{(x^2-1)^2 - x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{4(1+3x^2)}{(x^2-1)^3}.$$

Ikkinchi tartibli hosila  $x_1 = -1$  va  $x_2 = 1$  nuqtalarda mavjud emas.

$y''$  hosilaning  $(-\infty; -1)$ ,  $(-1; 1)$ ,  $(1; +\infty)$  intervallardagi ishoralarini tekshiramiz:

Demak, funksiyaning grafigi  $(-1; 1)$  intervalda qavariq,  $(-\infty; -1)$  va  $(1; +\infty)$  intervallarda botiq bo'ladi.



Funksiya grafigining egilish nuqtasi yo'q.

8°. 1° - 7° bandlar asosida funksiya grafigini chizamiz (17-shakl).

8- misol.  $y = \frac{x^3}{(x+1)^2}$  funksiyaning tekshiring va grafigini chizing.

*Yechish.* Misolni Maple paketida bajaramiz.

> with(plots) :

> restart :

> readlib(extrema) :

> f :=  $\frac{x^3}{(x+1)^2}$  :

>  $\lim_{x \rightarrow -1^+} f;$

$-\infty$

>  $\lim_{x \rightarrow -1^-} f;$

$-\infty$

>  $\lim_{x \rightarrow \infty} f;$

$\infty$

>  $\lim_{x \rightarrow -\infty} f;$

$-\infty$

>  $\lim_{x \rightarrow \infty} \frac{f}{x};$

1

$$> \lim_{x \rightarrow \infty} (f - x);$$

-2

$$> g := x - 2;$$

$$g := x - 2$$

$$> \text{solve}(\{f=0\}, x);$$

$$\{x=0\}, \{x=0\}, \{x=0\}$$

$$> \text{solve}(\{f > 0\}, x);$$

$$\{0 < x\}$$

$$> \text{solve}(\{f < 0\}, x);$$

$$\{x < -1\}, \{-1 < x, x < 0\}$$

$$> \frac{d}{dx} f;$$

$$\frac{3x^2}{(x+1)^2} - \frac{2x^3}{(x+1)^3}$$

$$\{x = -3\}, \{x = 0\}, \{x = 0\}$$

$$> \text{extrema}(f, \{ \}, \{x\}, s'); s;$$

$$\left[ 0, -\frac{27}{4} \right]$$

$$\{x = -3\}, \{x = 0\}$$

$$> \text{solve}\left(\left|\frac{d}{dx} f > 0\right|, x\right);$$

$$\{x < -3\}, \{-1 < x, x < 0\}, \{0 < x\}$$

$$> \text{solve}\left(\left|\frac{d}{dx} f < 0\right|, x\right);$$

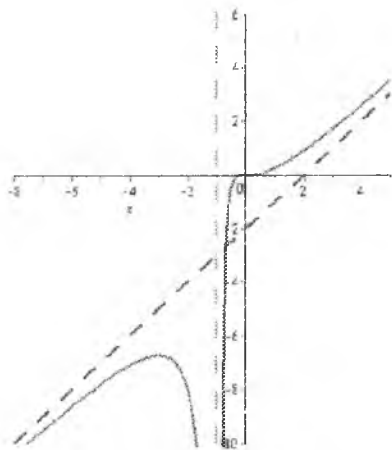
$$\{-3 < x, x < -1\}$$

$$> \frac{d^2}{dx^2} f;$$

$$\frac{6x}{(x+1)^2} - \frac{12x^2}{(x+1)^3} + \frac{6x^3}{(x+1)^4}$$

$$> \text{simplify}( \ );$$

$$\frac{6x}{(x+1)^4}$$



18-shakl.



$$> \text{solve}\left(\left[\frac{d^2}{dx^2} f = 0\right], x\right);$$

$$\{x = 0\}$$

$$> \text{solve}\left(\left[\frac{d^2}{dx^2} f > 0\right], x\right);$$

$$\{0 < x\}$$

$$> \text{solve}\left(\left[\frac{d^2}{dx^2} f < 0\right], x\right);$$

$$\{x < -1\}, \{-1 < x, x < 0\}$$

>

```
plot([f(x), g(x), [-1, 1, t=-10..10]], x=-8..5, -10..6, color
      = [red, blue, green], linestyle = [1, 6, 6], thickness = 2, discont
      = true, grid = [50, 50]);
```

## 5.4.7. Mashqlar

1. Funksiyalarning monotonlik intervallarini va ekstremumlarini toping:

1)  $f(x) = x^3 - 9x^2 + 15x;$

2)  $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x;$

3)  $f(x) = \frac{x^2}{4 - x^2};$

4)  $f(x) = \frac{4x}{x^2 + 4};$

5)  $f(x) = x\sqrt{1 - x^2};$

6)  $f(x) = 3\sqrt[3]{x^2 - x^2};$

7)  $f(x) = xe^{-x};$

8)  $f(x) = ch^2 x;$

9)  $f(x) = \ln(x^2 + 1);$

10)  $f(x) = \frac{x}{\ln x};$

11)  $f(x) = x - 2\sin x, 0 \leq x \leq 2\pi;$

12)  $f(x) = x + 2\cos^2 x, 0 \leq x \leq \pi.$

2. Funksiyalarning berilgan kesmadagi eng katta va eng kichik qiymatlarini toping:

1)  $f(x) = x^3 - 3x, [0; 2];$

2)  $f(x) = x^3 + 3x^2 - 9x - 10, [-4; 0];$

3)  $f(x) = x + \cos 2x, \left[0; \frac{\pi}{3}\right];$

4)  $f(x) = x^3 \ln x, [1; e].$

3. Jism  $S = 21t + 3t^2 - t^3$  qonun bilan harakatlanmoqda. Jismning eng katta tezligini toping.

4. Ko'ndalang kesimi to'g'ri to'rtburchakdan iborat to'sinning bukilishga qarshiligi ko'ndalang kesimning eni bilan bo'yi kvadratining ko'paytmasiga proporsional.  $D$  diametrlilik xodadan kesilgan to'sinning bukilishga qarshiligi eng katta bo'lishi uchun to'sinning o'lchamlari qanday bo'lishi kerak?

5. Uzunligi  $l$  ga teng simdan to'g'ri to'rtburchak yasalgan. To'g'ri to'rtburchakning yuzasi eng katta bo'lishi uchun uning o'lchamlari qanday bo'lishi kerak?

6.  $\frac{x^2}{16} + \frac{y^2}{9} = 1$  ellipsga to'g'ri to'rtburchak ichki chizilgan. To'g'ri

to'rtburchakning mumkin bo'lgan eng katta yuzasini toping.

7.  $R$  radiusli sharga yon sirti eng katta bo'lgan silindr ichki chizilgan. Silindrning balandligi qanday bo'lishi kerak?

8. Silindrning hajmi  $V$  ga teng. Eng kichik to'la sirtga ega bo'lgan silindrning balandligini toping?

9. Kemada yoqilg'i sarfi uning tezligining kubiga proporsional bo'lib,  $u$  20 km/soat tezlikda soatiga 40 so'm sarflaydi. Kemaning boshqa xarajatlari soatiga 270 so'mni tashkil qiladi. Kemaning 1 km. yo'l uchun umumiy xarajati eng kam bo'lishi uchun uning tezligi qanday bo'lishi kerak?

10.  $A$  zavoddan  $B$  shahar orqali o'tuvchi temir yo'lgacha bo'lgan qisqa masofa  $a$  km. Agar yuk tashish narxi avtomobil yo'lida temir yo'ldagiga qaraganida ikki baravar kam bo'lsa, yukni  $A$  zavoddan  $B$  shaharga eng arzon yetkazish uchun  $A$  zavoddan avtomobil yo'li temir yo'lga qanday burchak o'tkazilishi kerak?

11. Funktsiyalar grafigining botiqlik-qavariqlik intervallarini va egilish nuqtalarini toping:

1)  $f(x) = x^4 - 4x^3 + 6x$ ;

2)  $f(x) = (x-5)^5 + 4x - 13$ ;

3)  $f(x) = 2x - 3\sqrt[3]{x^2}$ ;

4)  $f(x) = 1 + \sqrt[3]{(x-3)^5}$ ;

5)  $f(x) = x - \ln(1+x)$ ;

6)  $f(x) = \ln(1+x^2)$ ;

7)  $f(x) = \frac{1}{1+x^2}$ ;

8)  $f(x) = x^3 - \frac{3}{x}$ .

12. Funktsiyalar grafigining asimptotalarini toping:

1)  $f(x) = \frac{x}{x^2-1}$ ;

2)  $f(x) = \frac{\sqrt{1+x^2}}{x}$ ;

3)  $f(x) = \sqrt[3]{x^3-3x}$ ;

4)  $f(x) = -x \operatorname{arctg} x$ ;

5)  $f(x) = \frac{e^x}{x+2}$ ;

6)  $f(x) = \frac{\ln^2 x}{x}$ ;

7)  $f(x) = 3x - \frac{\sin x}{x}$ ;

8)  $f(x) = \sqrt{\frac{x^3}{x-1}}$ .

13. Funktsiyalarni tekshiring va grafigini chizing:

$$1) f(x) = x - x^3;$$

$$2) f(x) = x^3 - 3x + 2;$$

$$3) f(x) = \frac{x-2}{x^2};$$

$$4) f(x) = \frac{x^2}{1-x^2};$$

$$5) f(x) = x + \frac{1}{x};$$

$$6) f(x) = x - \frac{1}{x};$$

$$7) f(x) = x - \frac{1}{x^2};$$

$$8) f(x) = \frac{x}{3} - \frac{9}{2x^2}.$$

## 5.5. HOSILANING GEOMETRIK TATBIQLARI

### 5.5.1. Yassi egri chiziq yoyining differensial

$[a; b]$  kesmada differensiallanuvchi  $y = f(x)$  funksiya berilgan va uning grafigi  $\overline{AB}$  yoydan iborat bo'lsin (19-shakl).  $[a; b]$  kesmani  $x_1, x_2, \dots, x_{n-1}$  nuqtalar bilan  $n$  ta bo'lakka bo'lamiz. Bu nuqталarga  $\overline{AB}$  yoyning  $M_1, M_2, \dots, M_{n-1}$  nuqtalari mos keladi va ularni siniq chiziq bilan tutashtiramiz. Bu siniq chiziqqa  $\overline{AB}$  yoyga *ichki chizilgan siniq chiziq* deyiladi. Siniq chiziqning perimetrini  $s_n$  bilan belgilaymiz, ya'ni

$$s_n = \sum_{i=1}^n |M_{i-1}M_i| \quad (M_0 = A, M_n = B).$$

$M_{i-1}M_i$  bo'g'inlardan eng kattasining uzunligini  $\lambda_i$  bilan belgilaymiz, bunda  $M_{i-1}M_i$  bo'g'inlar soni cheksiz ortganida, ya'ni  $n \rightarrow \infty$  da  $\lambda_i \rightarrow 0$ .

$\overline{AB}$  yoyga ichki chizilgan siniq chiziq perimetrining  $\lambda_i \rightarrow 0$  dagi chekli limiti  $M_1, M_2, \dots, M_{n-1}$  nuqtalarni tanlash usuliga bog'liq bo'lmagan holda mavjud bo'lsa, bu limitga *chiziq yoyining uzunligi* deyiladi.

Yoy uzunligini  $s$  bilan belgilab, topamiz:

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n |M_{i-1}M_i|.$$

Yoy uzunligini uning biror nuqtasidan, masalan,  $A$  nuqtadan hisoblaymiz.

$M(x; y)$  nuqtada  $\overline{AM}$  yoy uzunligi  $s$  ga,  $M'(x + \Delta x; y + \Delta y)$  nuqtada  $\overline{AM'}$  yoy uzunligi  $s + \Delta s$  ga teng bo'lsin, bu yerda

$\Delta s = \overset{\sim}{MM'}$  yoy uzunligi (20-shakl).

$MNM'$  uchburchakdan  $MM'$  uzunlikni topamiz:

$$MM' = \sqrt{\Delta x^2 + \Delta y^2}.$$

Geometrik mulohazalardan

$$\lim_{M' \rightarrow M} \frac{\overset{\sim}{MM'}}{MM'} = 1, \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} = 1$$

kelib chiqadi, ya'ni chiziqning cheksiz kichik yoyi va unga tortilgan to'g'ri chiziq ekvivalent bo'ladi. Ushbu ekvivalentlikni hisobga olib,  $MM'$  uzunlikni ifodalovchi formulada almashtirish bajaramiz:

$$\frac{MM'}{\Delta s} \cdot \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

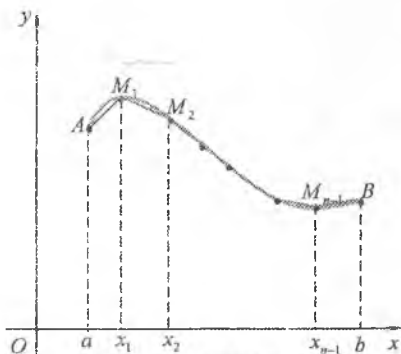
Oxirgi tenglikda limitga o'tib,  $s = s(x)$  funksiyaning hosilasi uchun formula topamiz:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \quad (5.1)$$

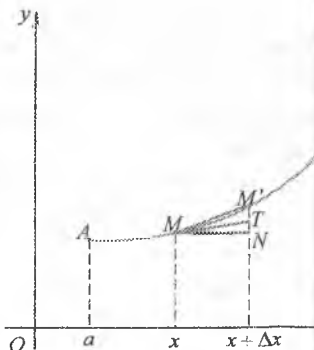
Bundan

$$ds = \sqrt{dx^2 + dy^2}. \quad (5.2)$$

(5.1) formula yassi egri chiziq yoyi differensialini ifodalaydi va ushbu geometrik ma'noga ega: yoy differensialini chiziq urinishning tegishli kesmasi bilan ifodalash mumkin (20-shaklda  $MT$  kesma).



19-shakl.



20-shakl.

Agar egri chiziq  $x = x(t), y = y(t), t \in T$  parametrik tenglamalar bilan berilgan bo'lsa,  $dx = x'(t)dt, dy = y'(t)dt$  bo'ladi va (5.2) ifoda ushbu ko'rinishni oladi:

$$ds = \sqrt{x'(t)^2 + y'(t)^2} dt. \quad (5.3)$$

Agar egri chiziq qutb koordinatalarida  $r = r(\varphi)$  tenglama bilan berilgan bo'lsa,  $x = r \cos \varphi, y = r \sin \varphi$  ifodalarni  $\varphi$  parametrli tenglamalar deb, topamiz:

$$\frac{dx}{d\varphi} = \frac{dr}{d\varphi} \cos \varphi - r \sin \varphi, \quad \frac{dy}{d\varphi} = \frac{dr}{d\varphi} \sin \varphi + r \cos \varphi,$$

$$\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = r^2 + \left(\frac{dr}{d\varphi}\right)^2;$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} d\varphi. \quad (5.4)$$

## 5.5.2. Yassi egri chiziqning egriligi

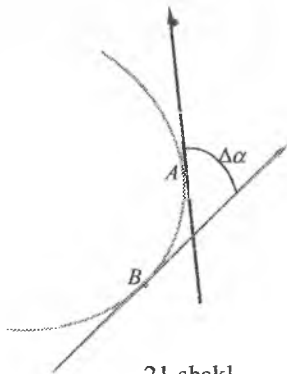
### *Egrilik*

Egri chiziq shaklini xarakterlovchi elementlardan biri uning egilganlik darajasidir.

O'z-o'zini kesib o'tmaydigan va har bir nuqtasida tayin urinmaga ega bo'lgan yassi egri chiziqni qaraymiz. Egri chiziqda ikkita  $A$  va  $B$  nuqtalarni olamiz. Bu nuqtalarda egri chiziqqa o'tkazilgan urinmalar orasidagi burchakni  $\Delta\alpha$  bilan belgilaymiz (21-shakl).

$\Delta\alpha$  burchakka  $\overline{AB}$  yoyning qo'shnilik burchagi deyiladi.

Bir xil uzunlikka ega ikkita yoydan qo'shnilik burchagi katta bo'lgani ko'proq egilgan bo'ladi. Har xil uzunlikka ega yoylarning egilganlik darajasini qo'shnilik burchagi orqali baholab bo'lmaydi. Bunda  $\overline{AB}$  yoy uzunligida  $\Delta\alpha$  burchakning o'rtacha qancha ulushi to'g'ri kelishi muhim hisoblanadi.



21-shakl.

$\overset{\frown}{AB}$  yoyning o'rtacha egriligi deb, tegishli qo'shnilik burchagining  $\overset{\frown}{AB}$  yoy uzunligiga nisbatining absolut qiymatiga aytiladi:

$$K_{or} = \left| \frac{\Delta\alpha}{\Delta s} \right| \quad (5.5)$$

Bitta egri chiziqning turli qismlarida o'rtacha egrilik har xil bo'lishi mumkin.

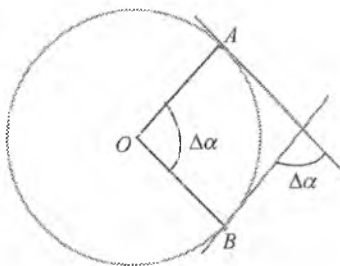
Egri chiziqning bevosita  $A$  nuqtadagi egilganlik darajasini baholash uchun egri chiziqning berilgan nuqtadagi egriligi tushunchasini kiritamiz.

Berilgan egri chiziqning  $A$  nuqtadagi egriligi deb  $\overset{\frown}{AB}$  yoy o'rtacha egriligining  $B \rightarrow A$  dagi ( $s \rightarrow 0$  dagi) limitiga aytiladi:

$$K = \lim_{B \rightarrow A} \left| \frac{\Delta\alpha}{\Delta s} \right| \quad \left( K = \lim_{s \rightarrow 0} \left| \frac{\Delta\alpha}{\Delta s} \right| \right) \quad (5.6)$$

$\Delta\alpha$  burchak urinish nuqtasi egri chiziq yoyi bo'ylab  $\Delta s$  masofaga ko'chishida urinmaning burilish burchagini ifodalaydi. Shu sababli egri chiziqning egriligiga quyidagicha ta'rif bersa ham bo'ladi: *egri chiziqning egriligi* bu – urinish nuqtasi egri chiziq bo'ylab harakatlanganida urinma aylanishining burchak tezligidir.

Misol uchun to'g'ri chiziq va aylananing o'rtacha egriligi va egriligini topaylik.



22-shakl.

1. To'g'ri chiziqning istalgan nuqtasiga o'tkazilgan urinma to'g'ri chiziqning o'zi bilan ustma-ust tushadi. Shu sababli  $\Delta\alpha = 0$ , uning ixtiyoriy nuqtasida o'rtacha egrilik va egrilik nolga teng bo'ladi.

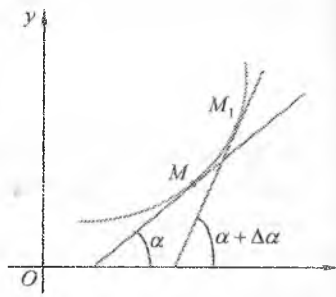
2. Aylana uchun  $\Delta\alpha$  qo'shnilik burchagi uning  $OA$  va  $OB$  radiuslari orasidagi burchakka,  $\overset{\frown}{AB}$  yoy uzunligi  $\Delta s = R\Delta\alpha$  ga teng bo'ladi, bu yerda  $R$  – aylana radiusi (22-shakl).

U holda

$$K_{or} = \frac{\Delta\alpha}{R\Delta\alpha} = \frac{1}{R}, \quad K = \frac{1}{R}$$

ya'ni aylananing egriligi o'zgarmaydi va radiusining teskarisiga teng bo'ladi.

Uzlüksiz ikkinchi tartibli hosilaga ega  $y = f(x)$  funksiya bilan aniqlanuvchi yassi egri chiziqning  $M(x; y)$  nuqtadagi egriligini hisoblaymiz. Egri chiziqqa  $M(x; y)$  va  $M_1(x + \Delta x; y + \Delta y)$  nuqtalarda urinma o'tkazamiz, urinmalarning og'ish burchaklarini  $\alpha$  va  $\alpha + \Delta\alpha$  bilan belgilaymiz (23-shakl).



$\overline{MM_1}$  yoyga mos keluvchi qo'shnilik burchagi  $\Delta\alpha$  ga teng (qavariq yoy uchun  $\Delta\alpha < 0$ , botiq yoy uchun  $\Delta\alpha > 0$ ).

$\alpha = \alpha(x)$  va  $s = s(x)$  bo'lgani uchun

23-shakl.

$$K = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta\alpha}{\Delta s} \right| = \left| \frac{d\alpha}{ds} \right| = \left| \frac{d\alpha}{dx} \right| \quad (5.7)$$

$\operatorname{tg} \alpha = y'$ , va demak,  $\alpha = \arctg y'$  bo'ladi. Bu tenglikni differensiallab, topamiz:

$$\frac{d\alpha}{dx} = \frac{y''}{1 + y'^2} \quad (5.8)$$

(5.8) va (5.1) ifodalarni (5.7) tenglamaga qo'yib,  $y = f(x)$  egri chiziqning  $M(x; y)$  nuqtadagi egriligini topish formulasini hosil qilamiz:

$$K = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}} \quad (5.9)$$

Agar egri chiziq  $x = x(t), y = y(t), t \in T$  parametrik tenglamalar bilan berilgan bo'lsa,

$$y'_x = \frac{y'_t}{x'_t}, \quad y'' = \frac{x'_t y''_t - x''_t y'_t}{(x'^2_t + y'^2_t)^{\frac{3}{2}}}$$

bo'ladi va (5.9) formula ushbu ko'rinishni oladi:

$$K = \frac{|x'_t y''_t - x''_t y'_t|}{(x'^2_t + y'^2_t)^{\frac{3}{2}}} \quad (5.10)$$

Agar egri chiziq qutb koordinatalarida  $r=r(\varphi)$  tenglama bilan berilgan bo'lsa,  $x=r\cos\varphi$ ,  $y=r\sin\varphi$  ifodalarni  $\varphi$  parametri tenglamalar deb, topamiz:

$$K = \frac{|r^2 + 2r'^2 - rr''|}{(r'^2 + r^2)^{\frac{3}{2}}}. \quad (5.11)$$

*1-misol.*  $y=-x^3$  egri chiziqning  $x=\frac{1}{2}$  nuqtasidagi egriligini toping.

*Yechish.* Hosilalarni topamiz:  $y'=-3x^2$ ,  $y''=-6x$ . Hosilalarning  $x=\frac{1}{2}$  nuqtadagi qiymatlarini hisoblaymiz:

$$y' = -\frac{3}{4}, \quad y'' = -3.$$

U holda

$$K = \frac{|-3|}{\left(1 + \left(-\frac{3}{4}\right)^2\right)^{\frac{3}{2}}} = \frac{3}{\frac{125}{64}} = \frac{192}{125}.$$

*2-misol.*  $x=t^2, y=2t^3$  egri chiziqning  $t=1$  parametr ga mos nuqtasidagi egriligini toping.

*Yechish.*  $x'(t)=2t$ ,  $x''(t)=2$ ,  $y'(t)=6t^2$ ,  $y''(t)=12t$  hosilalarning qiymatlarini  $t=1$  da topamiz:  $x'=2$ ,  $x''=2$ ,  $y'=6$ ,  $y''=12$ .

Bundan

$$K = \frac{|2 \cdot 12 - 2 \cdot 6|}{(2^2 + 6^2)^{3/2}} = \frac{3}{20\sqrt{10}}.$$

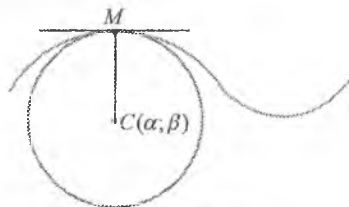
### *Egrilik radiusi, markazi va aylanasi, evolyuta, evolventa*

Chiziqning berilgan  $M$  nuqtadagi egriligi  $K$  ga teskari  $R$  miqdorga shu chiziqning qaralayotgan nuqtadagi *egrilik radiusi* deyiladi:

$$R = \frac{1}{K} \quad (5.12)$$

yoki

$$R = \frac{(1+y'^2)^{\frac{3}{2}}}{|y''|}. \quad (5.13)$$



24-shakl.



Egri chiziqning  $M$  nuqtadagi normaliga egri chiziqning botiqligi yoʻnalishida  $MC = R$  kesmani qoʻyamiz (24-shakl).

$C$  nuqtaga berilgan egri chiziqning  $M$  nuqtadagi *egrilik markazi* deyiladi.

Markazi  $C$  nuqtada boʻlgan  $R$  radiusii aylana berilgan egri chiziqning  $M$  nuqtadagi *egrilik aylanasini* deb ataladi (24-shakl).

Egrilik aylanasining bu taʼrifidan berilgan  $M$  nuqtada egri chiziqning egriligi va egriklik aylanasining egriligi bir-biriga teng boʻlishi kelib chiqadi.

Egrilik markazining koordinatalarini  $\alpha$  va  $\beta$  bilan belgilab, ularni hisoblash uchun formulalar chiqaramiz.

$y = f(x)$  egri chiziqqa  $M(x; y)$  nuqtada oʻtkazilgan normal tenglamasini tuzamiz:

$$Y - y = -\frac{1}{y'}(X - x), \quad (5.14)$$

bu yerda  $X, Y$  – normalning oʻzgaruvchi koordinatalari.

$C(\alpha; \beta)$  nuqta normalga tegishli boʻlgani uchun uning koordinatalari (5.14) tenglamani qanoatlantiradi:

$$\beta - y = -\frac{1}{y'}(\alpha - x). \quad (5.15)$$

$C(\alpha; \beta)$  nuqta  $M(x; y)$  nuqtadan egriklik radiusi  $R$  ga teng masofada yotadi, shu sababli

$$(\alpha - x)^2 + (\beta - y)^2 = R^2. \quad (5.16)$$

(5.15) va (5.16) tenglamalarni birgalikda yechib,  $\alpha$  va  $\beta$  larni topamiz:

$$\alpha = x \pm \frac{y'}{\sqrt{1+y'^2}} R, \quad \beta = y \mp \frac{1}{\sqrt{1+y'^2}} R. \quad (5.17)$$

Bu formulalarga (5.13) tenglikdan  $R$  ni qoʻyamiz:

$$\alpha = x \pm \frac{y'(1+y'^2)}{|y''|}, \quad \beta = y \mp \frac{1+y'^2}{|y''|}. \quad (5.18)$$

Bu formulalarda qaysi ishoralar olinishini aniqlashtirish uchun  $y'' > 0$  va  $y'' < 0$  boʻlgan hollar alohida qaraladi. Bunda  $y'' > 0$  boʻlsa, mos nuqtada egri chiziq botiq va  $\beta > y$  boʻladi. Shu sababli bunda quyi

ishoralar olinadi, ya'ni egrilik markazining koordinatalari

$$\alpha = x - \frac{y'(1+y'^2)}{|y''|}, \quad \beta = y + \frac{1+y'^2}{|y''|} \quad (5.19)$$

formulalar bilan aniqlanadi. Bu formulalar  $y'' < 0$  bo'lganida ham o'rinli bo'ladi.

Agar egri chiziq  $x = x(t), y = y(t), t \in T$  parametrik tenglamalar bilan berilgan bo'lsa, (5.19) ifoda ushbu ko'rinishni oladi:

$$\alpha = x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'}, \quad \beta = y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'}. \quad (5.20)$$

3-misol.  $r = 3(1 + \cos \varphi)$  kardioidaning istalgan nuqtasida egrilik radiusini toping.

*Yechish.* Hosilalarni topamiz:  $r' = -3 \sin \varphi$ ,  $r'' = -3 \cos \varphi$ .

U holda

$$\begin{aligned} R &= \frac{1}{K} = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{|r^2 + 2r'^2 - rr''|} = \\ &= \frac{(9(1 + 2 \cos \varphi + \cos^2 \varphi) + 9 \sin^2 \varphi)^{\frac{3}{2}}}{|9(1 + 2 \cos \varphi + \cos^2 \varphi) + 18 \sin^2 \varphi + 9(\cos \varphi + \cos^2 \varphi)|} = \\ &= \frac{27(1 + 2 \cos \varphi + \cos^2 \varphi + \sin^2 \varphi)^{\frac{3}{2}}}{9|1 + 2 \cos \varphi + \cos^2 \varphi + 2 \sin^2 \varphi + \cos \varphi + \cos^2 \varphi|} = \\ &= \frac{3(2(1 + \cos \varphi))^{\frac{3}{2}}}{3(1 + \cos \varphi)} = 2(2(1 + \cos \varphi))^{\frac{1}{2}} = 4 \cos \frac{\varphi}{2}. \end{aligned}$$

Egri chiziqning barcha egrilik markazlari to'plamiga *evolyuta* deyiladi.

Berilgan chiziq o'z evolyutasiga nisbatan *evolventa* (yoki *yoyilma*) deb yuritiladi.

Agar egri chiziq  $y = f(x)$  tenglama bilan berilgan bo'lsa, (5.19) tenglamani uning evolyutasi uchun parametrik ( $x$  parametrli) tenglama deb qarash mumkin.

Egri chiziq parametrik tenglamalar bilan berilgan holda (5.20) tenglama evolyutaning parametrik tenglamalarini (bu tenglamalarning o'ng tomoniga kiruvchi kattaliklar  $t$  parametrga bog'liq bo'ladi) ifodalaydi.

4-misol.  $y = x^2$  parabola evolyutasi tenglamasini toping.

Yechish. Hosilarni topamiz:  $y' = 2x$ ,  $y'' = 2$ .

Egrilik markazining koordinatalarini aniqlaymiz:

$$\alpha = x - \frac{2x(1+4x^2)}{2} = -4x^3, \quad \beta = x^2 + \frac{1+4x^2}{2} = \frac{6x^2+1}{2}.$$

Bu tenglamalarni  $y = x^2$  parabola evolyutasining parametrik tenglamalari deb qarash mumkin. Tenglamadan  $x$  parametrimni chiqarib, topamiz:

$$\alpha^3 = \frac{16}{27} \left( \beta - \frac{1}{2} \right)^3.$$

Bu tenglama yarim kubik parabola tenglamasi bo'lad.

### 5.5.3. Skalyar argumentning vektor funksiyasi

#### Fazodagi egri chiziqning parametrik tenglamalari

Egri chiziq tekislikdagi kabi fazoda ham parametrik tenglamalar bilan berilishi mumkin.

$t$  argumentning bir xil aniqlanish sohasiga ega bo'lgan uchta funksiyasini qaraymiz:

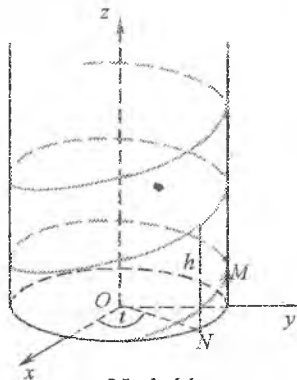
$$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \quad t \in T. \end{cases} \quad (5.21)$$

Bunda  $t \in T$  ning har bir qiymatiga tayin  $x, y, z$  qiymatlar va fazoviy  $M(x; y; z)$  nuqta mos keladi.  $t$  o'zgarishi bilan  $M$  nuqta fazoda biror  $\gamma$  egri chiziqni chizadi. Bunda egri chiziq (5.21) parametrik

tenglamalar bilan berilgan va  $t$  ga parametr deymiz.

Biz, fazodagi to'g'ri chiziqning quyidagi parametrik tenglamalari bilan avvaldan tanishmiz:

$$\begin{cases} x = x_0 + pt, \\ y = y_0 + qt, \\ z = z_0 + rt, \quad t \in R. \end{cases}$$



Yana bitta misol keltiramiz. Ushbu parametrik tenglamalar bilan berilgan egri chiziqni qaraymiz:

$$\begin{cases} x = a \cos t, \\ y = a \sin t, \\ z = bt. \end{cases}$$

Bu egri chiziqqa *vint chizig'i* deyiladi (25-shakl).

$t$  parametrning istalgan qiymatida:

$$x^2 + y^2 = a^2 (\cos^2 t + \sin^2 t) = a^2.$$

Bu tenglik *vint chizig'ining*  $x^2 + y^2 = a^2$  silindrdagi joylashishini bildiradi. Bundan,  $M$  nuqta *vint chizig'i* bo'ylab harakatlanganida uning  $Oxy$  tekisligidagi proektsiyasi radiusi  $a$  ga teng aylanada harakatlanishi kelib chiqadi, bunda  $t$  parametr  $N$  nuqtaning qutb burchagi bo'ladi.  $t$  parametr  $2\pi$  ga o'sganida  $N$  nuqta aylanani to'liq aylanib o'tadi, *vint chizig'i*  $M$  nuqtasining  $z$  applikatsiyasi esa  $h = 2\pi b$  kattalikka o'zgaradi. Bu kattalikka *vint chizig'ining qadami* deyiladi.

### *Skalyar argumentning vektor funksiyasi*

Fazodagi egri chiziq (5.21) parametrik tenglamalari bilan berilgan bo'lsin. Bunda  $x(t)$ ,  $y(t)$ ,  $z(t)$  funksiyalarning aniqlanish sohasi  $T$  ga tegishli har bir  $t$  parametrga biror  $M(x, y, z)$  nuqta, har bir  $M$  nuqtaga esa boshi koordinata boshida va oxiri  $M$  nuqtada bo'lgan  $\vec{r} = \overline{OM}$  radius vektor mos keladi (26-shakl). Bu vektorning koordinata o'qlaridagi proektsiyalari  $M$  nuqtaning koordinatalari bo'ladi va shu sababli vektor (5.21) formulalar bilan aniqlanadi.

Shunday qilib,  $t \in T$  parametrning har bir qiymatiga biror

$$\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad (5.22)$$

vektor mos keladi. Bu vektorni *t skalyar argumentning vektor funksiyasi* (yoki *vektor funksiya*) deb ataymiz va  $\vec{r}(t)$  bilan belgilaymiz.

$\vec{r}(t)$  radius vektorning oxiri chizadigan  $L$  chiziqqa  $\vec{r}(t)$  vektorning *godografi* deyiladi.

$\vec{r}(t)$  vektor funksiyaning berilishi uchta skalyar funksiyalarning berilishiga teng kuchli, uning koordinata o'qlaridagi proektsiyalari  $x(t)$ ,  $y(t)$ ,  $z(t)$  bo'ladi.

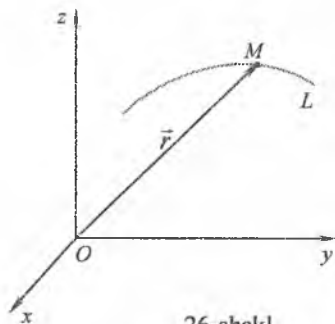
Vektor funksiya uchun limit, uzluksizlik va hosila tushunchalarini kiritamiz.

**1-ta'rif.** Agar  $\lim_{t \rightarrow t_0} x(t) = x_0$ ,  $\lim_{t \rightarrow t_0} y(t) = y_0$ ,  $\lim_{t \rightarrow t_0} z(t) = z_0$  bo'lsa,  $\vec{r}_0 = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$  vektorga  $\vec{r} = x(t) \vec{i} + y(t) \vec{j} + z(t) \vec{k}$  vektor funksiyaning  $t \rightarrow t_0$  dagi limiti deyiladi va  $\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{r}_0$  deb yoziladi.

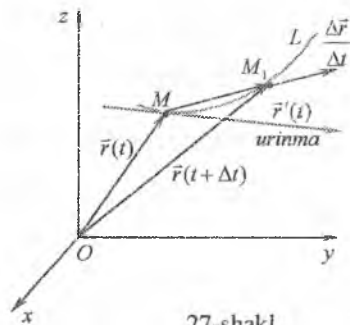
$\vec{r}(t)$  vektor funksiya  $t = t_0$  da va  $t_0$  nuqtani o'z ichiga olgan biror intervalda aniqlangan bo'lsin.

**2-ta'rif.** Agar  $t_0$  nuqtada  $\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{r}(t_0)$  bo'lsa,  $\vec{r}(t)$  vektor funksiya  $t_0$  nuqtada uzluksiz deyiladi.

$\vec{r}(t) = x(t) \vec{i} + y(t) \vec{j} + z(t) \vec{k}$  vektor funksiya  $M(x, y, z)$  nuqtaning radius vektori, ya'ni  $\vec{r} = \overline{OM}$  bo'lsin (27-shakl).  $t$  parametrning o'zgarishi bilan  $M$  nuqta  $L$  godografni chizadi. Parametrning tanlangan tayin  $t$  qiymatiga  $M$  nuqta va  $\vec{r}(t)$  vektor mos kelsin.



26-shakl.



27-shakl.

Perimetrning boshqa  $t + \Delta t$  qiymatini olamiz. Unga  $M_1$  nuqta va  $\vec{r}(t + \Delta t)$  vektor mos keladi.  $\vec{r}(t + \Delta t) = \overline{OM_1}$  va  $\vec{r}(t) = \overline{OM}$  vektorlarning ayirmasi  $\Delta \vec{r} = \overline{MM_1}$  vektorni qaraymiz:

$$\Delta \vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t).$$

Uni  $\vec{r}(t)$  vektorning  $t$  nuqtadagi orttirmasi deymiz.

$\frac{\Delta \vec{r}}{\Delta t}$  nisbat  $\Delta \vec{r}$  vektorga kollinear, chunki  $\frac{1}{\Delta t}$  skalyar ko'paytuvchi.

**3-ta'rif.**  $\Delta \vec{r}$  vektor funksiya orttirmasining argumentning mos  $\Delta t$  orttirmasiga nisbatining  $\Delta t \rightarrow 0$  dagi limitiga  $\vec{r}(t)$  vektor funksiyaning  $t$  nuqtadagi  $t$  skalyar argument bo'yicha hosilasi deyiladi va  $\vec{r}'(t)$  yoki  $\frac{d\vec{r}(t)}{dt}$  kabi belgilanadi.

Shunday qilib,

$$r'(t) = \frac{d\vec{r}(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{d\vec{r}}{\Delta t}. \quad (5.23)$$

Limit ta'rifiga ko'ra,  $r'(t)$  – vektor.

$\vec{r}(t)$  funksiya hosilasini uning koordinatalar o'qlaridagi proeksiyalari bilan bog'laymiz.

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

va

$$\vec{r}(t + \Delta t) = x(t + \Delta t)\vec{i} + y(t + \Delta t)\vec{j} + z(t + \Delta t)\vec{k}$$

bo'lgani uchun

$$\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t) = (x(t + \Delta t) - x(t))\vec{i} + (y(t + \Delta t) - y(t))\vec{j} + (z(t + \Delta t) - z(t))\vec{k}$$

yoki

$$\Delta\vec{r} = \Delta x\vec{i} + \Delta y\vec{j} + \Delta z\vec{k}.$$

Shunday qilib,

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j} + \frac{\Delta z}{\Delta t}\vec{k}.$$

Bundan

$$\begin{aligned} r'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}\vec{i} + \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}\vec{j} + \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t}\vec{k} = \\ &= x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}. \end{aligned}$$

Demak,

$$\vec{r}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}. \quad (5.24)$$

(5.24) ifodadan vektor funksiyani differensiallash qoidalarini taqdim etishda vektor funksiya hosilasining geometrik va mexanik ma'nolariga differensial hisobning asosiy differensiallash qoidalariga taqdim etilgan hosilaning geometrik va mexanik ma'nolariga o'xshash bo'lishligi ko'rsatib chiqiladi.

2.  $\vec{r}'(t)$  vektor  $\vec{r}(t)$  radius vektor godografiga  $t$  parametriga nisbatan yo'nalishida o'tkazilgan urinma bo'ylab yo'nalgan bo'lib (5.25) shaklida (vektor funksiya hosilasining geometrik ma'nosi).

3. Vektor funksiyaning  $t$  vaqt bo'yicha  $\vec{r}'(t)$  hosilasi moddiy nuqtaning  $t$  ondagi harakati tezligi  $\vec{v}(t)$  ga teng (vektor funksiya hosilasining mexanik ma'nosi).

**1. Skalyar argumentning vektor funksiyasini differensiallashning asosiy qoidalari.**

$$1^{\circ}. \frac{d}{dt}(\vec{r}_1 + \vec{r}_2 - \vec{r}_3) = \frac{d\vec{r}_1}{dt} + \frac{d\vec{r}_2}{dt} - \frac{d\vec{r}_3}{dt};$$

$$2^{\circ}. \frac{d\vec{c}}{dt} = 0, \vec{c} \text{ - o'zgarmas vektor};$$

$$3^{\circ}. \frac{d}{dt}(\lambda\vec{r}) = \lambda \frac{d\vec{r}}{dt} + \vec{r} \frac{d\lambda}{dt}, \lambda = \lambda(t) \text{ - } t \text{ argumentning skalyar funksiyasi};$$

$$4^{\circ}. \frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) = \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 + \vec{r}_1 \cdot \frac{d\vec{r}_2}{dt};$$

$$5^{\circ}. \frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \frac{d\vec{r}_1}{dt} \times \vec{r}_2 + \vec{r}_1 \times \frac{d\vec{r}_2}{dt}.$$

(5.24) ifoda va keltirilgan o'xshashliklar asosida fazodagi egri chiziqning urinmasi, normal tekisligi, yoyining differensiali va egriligi uchun quyidagi tenglamalar keltirib chiqariladi.

1. (5.21) parametrik tenglamalar bilan berilgan fazodagi egri chiziqqa parametrning  $t = t_0$  qiymatiga mos  $M_0(x_0; y_0; z_0)$  nuqtasida o'tkazilgan urinma (to'g'ri chiziq)

$$\frac{x - x_0}{x'(t_0)} = \frac{y - y_0}{y'(t_0)} = \frac{z - z_0}{z'(t_0)} \quad (5.25)$$

tenglama bilan aniqlanadi.

2. Fazodagi egri chiziq urinish nuqtasida urinmaga perpendikular o'tkazilgan tekislikka *egri chiziqqa o'tkazilgan normal tekislik* deyiladi.

Urinish nuqtasi  $M_0(x_0; y_0; z_0)$  bo'lgan normal tekislik

$$x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0) = 0 \quad (5.26)$$

tenglama bilan ifodalanadi.

3. Fazoviy egri chiziq yoyining differensiali

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \quad (5.27)$$

formula bilan topiladi.

#### 4. Fazodagi egri chiziqning egriligi

$$K = \frac{\sqrt{(y'z'' - y''z')^2 + (x'z'' - x''z')^2 + (x'y'' - x''y')^2}}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} \quad (5.28)$$

formula bilan aniqlanadi.

5-misol.  $x=t^3$ ,  $y=t^2$ ,  $z=t$  egri chiziqqa  $t=-1$  parametrga mos nuqtada o'tkazilgan urinma va normal tekislik tenglamalarini toping.

*Yechish.* Urinish nuqtasining koordinatalarini topamiz:

$$x=-1, y=1, z=-1.$$

Hosilalarni topamiz va ularning urinish nuqtasidagi qiymatlarini hisoblaymiz:

$$\begin{aligned} x'(t) &= 3t^2, & y'(t) &= 2t, & z'(t) &= 1; \\ x'(-1) &= 3, & y'(-1) &= -2, & z'(-1) &= 1. \end{aligned}$$

Urinish nuqtasining koordinatalari va hosilalarning hisoblangan qiymatlari asosida izlanayotgan tenglamalarni topamiz:

a) urinma tenglamasi

$$\frac{x+1}{3} = \frac{y-1}{-2} = \frac{z+1}{1};$$

b) normal tekislik tenglamasi

$$\begin{aligned} 3(x+1) - 2(y-1) + z+1 &= 0, \\ 3x - 2y + z + 6 &= 0. \end{aligned}$$

7-misol. Asosining radiusi  $R=4$  ga teng silindrda joylashgan qadami  $h=6\pi$  ga teng vint chizig'i tenglamasini tuzing. Uning yoyi differensialini va egriligini toping.

*Yechish.*  $t=2\pi$  da  $z=h=3t$ .

Demak, vint chizig'ining tenglamasi

$$x=4\cos t, y=4\sin t, z=3t.$$

Bundan

$$x'(t) = -4\sin t, y'(t) = 4\cos t, z'(t) = 3, x''(t) = -4\cos t, y''(t) = -4\sin t, z''(t) = 0.$$

Hosilalarning hisoblangan qiymatlari asosida izlanayotgan tenglamalarni topamiz:

a) vint chizig'i yoyining differensial

$$ds = \sqrt{x'^2 + y'^2 + z'^2} dt = \sqrt{16\sin^2 t + 16\cos^2 t + 9} dt =$$



$$= \sqrt{16(\sin^2 t + \cos^2 t) + 9} dt = 5 dt.$$

b) vint chizig'ining egriligi

$$y'z'' - y''z' = 4 \cos t \cdot 0 - (-4 \sin t) \cdot 3 = 12 \sin t;$$

$$x'z'' - x''z' = (-4 \sin t) \cdot 0 - (-4 \cos t) \cdot 3 = 12 \cos t;$$

$$x'y'' - x''y' = (-4 \sin t) \cdot (-4 \sin t) - (-4 \cos t) \cdot 4 \cos t = 16;$$

$$x'^2 + y'^2 + z'^2 = 16 \sin^2 t + 16 \cos^2 t + 9 = 25;$$

$$K = \frac{\sqrt{144 \sin^2 t + 144 \cos^2 t + 256}}{25^{\frac{3}{2}}} = \frac{20}{125} = \frac{4}{25}.$$

### 5.5.4. Mashqlar

1. Egri chiziqning egriligini toping:

1)  $y = \cos 2x$ ,  $x = \frac{\pi}{2}$  nuqtada;      2)  $y = e^{-x^2}$ ,  $Oy$  o'q bilan kesishish nuqtasida.

3)  $x = 9 \cos t$ ,  $y = 3 \sin t$  ellipsning istalgan nuqtasida;

4)  $x = 4(t - \sin t)$ ,  $y = 4(1 - \cos t)$  sikloidaning istalgan nuqtasida;

5)  $r = 2(1 - \cos \varphi)$ ,  $\varphi = \pi$  nuqtada;      6)  $r^2 = a^2 \sin 2\varphi$ ,  $\varphi = \frac{\pi}{4}$  nuqtada.

2. Egri chiziqning egrilik radiusini toping:

1)  $\frac{x^2}{25} + \frac{y^2}{9} = 1$ ,  $x = 0$  nuqtada;      2)  $y = \sqrt[3]{x}$ ,  $x = 8$  nuqtada.

3)  $x = t^2$ ,  $y = t - \frac{1}{3}t^3$ ,  $t = 1$  nuqtada;      4)  $r = a\varphi$ , istalgan nuqtada.

3. Egri chiziq egrilik markazining koordinatalarini toping:

1)  $y = \frac{1}{x}$ ,  $x = 1$  nuqtada;      2)  $x = t - \sin t$ ,  $y = 1 - \cos t$ ,  $t = \frac{\pi}{2}$  nuqtada.

4. Egri chiziq evolyutasi tenglamasini toping:

1)  $y^2 = 2(x+1)$ ;      2)  $y^2 - 2x = 0$ ;

3)  $x = 2t$ ,  $y = t^2 - 2$ ;      4)  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$ .

5. Vektor funksiyaning godografini toping:

1)  $\vec{r}(t) = (2t-1)\vec{i} + (-3t+2)\vec{j} + 4t\vec{k}$ ,  $t \in \mathbb{R}$ ;      2)  $\vec{r}(t) = 4ch\vec{t}\vec{i} - \vec{j} + 3sh\vec{t}\vec{k}$ ,  $t \in \mathbb{R}$ ;

3)  $\vec{r}(t) = 2\cos^3 t\vec{i} + 2\sin^3 t\vec{j} + \vec{k}$ ,  $t \in [0; 2\pi]$ ;      4)  $\vec{r}(t) = 6\cos t\vec{i} + 5\sin t\vec{j} + 3\vec{k}$ ,  $t \in [0; 2\pi]$ .

6. Egri chiziqqa o'tkazilgan urinma va normal tekislik tenglamalarini toping:

1)  $\vec{r}(t) = t^2\vec{i} + t^3\vec{j} + t^6\vec{k}$ ,  $t = 1$  nuqtada;

2)  $\vec{r}(t) = \frac{1}{2}t^2\vec{i} + \frac{1}{3}t^3\vec{j} + \frac{1}{4}t^4\vec{k}$ ,  $t = 2$  nuqtada;

3)  $\vec{r}(t) = \sin^2 t\vec{i} + 2\sin t \cos t\vec{j} + 3\cos^2 t\vec{k}$ ,  $t = \frac{\pi}{4}$  nuqtada;

4)  $\vec{r}(t) = e^t(\cos t + \sin t)\vec{i} + e^t(\sin t - \cos t)\vec{j} + e^t\vec{k}$ ,  $t = 0$  nuqtada.

7. Egri chiziq yoyi differensialini toping:

1)  $\vec{r}(t) = 3\cos^2 t\vec{i} + 5\sin t \cos t\vec{j} + 4\sin^2 t\vec{k}$ ;      2)  $\vec{r}(t) = 5\cos t\vec{i} + 5\sin t\vec{j} + 12t\vec{k}$ .

## 5.6. TENGLAMALARNI TAQRIBIY YECHISH

### 5.6.1. Asosiy tushunchalar

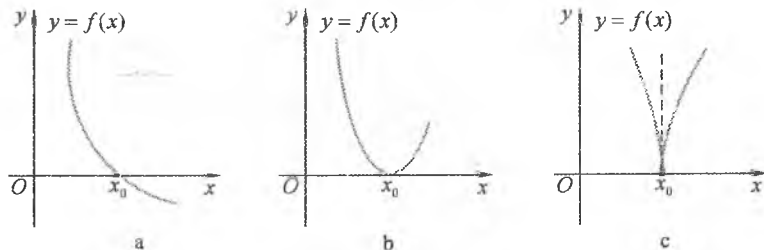
Bir o'zgaruvchining har qanday tenglamasini

$$f(x) = 0 \tag{6.1}$$

ko'rinishda yozish mumkin, bu yerda  $f(x) - x$  o'zgaruvchining biror funksiyasi.

(6.1) tenglamaning *ildizi* deb shunday  $x = x_0$  qiymatga aytiladiki, bunda tenglamaning chap tomoni nolga aylanadi, ya'ni  $f(x_0) = 0$  bo'ladi.

(6.1) tenglamani *yechish* deb uning ildizlarini topishga aytiladi.



28-shakl.

(6.1) tenglamaning ildizi geometrik jihatdan yo  $y = f(x)$  funksiya grafigining  $Ox$  o'q bilan kesishish nuqtasining absissasini (28-a shakl),

yo bu grafikning shu o'q bilan urinish nuqtasini (28-b shakl) yoki grafik va o'qning umumiy nuqtasini (28-c shakl) ifodalaydi.

Ildizning bu interpretatsiyalaridan bir o'zgaruvchi tenglamasini yechishning geometrik usuli kelib chiqadi: (6.1) tenglamani yechish uchun uning grafigini chizish va grafikning  $Ox$  ( $y=0$ ) o'q bilan kesishish nuqtalarini topish kerak.

$f(x)=0$  tenglamani taqribiy yechish, odatda, ikki bosqichda amalga oshiriladi.

Birinchi bosqichda tenglamaning ildizlari ajratiladi, ya'ni tenglama yagona haqiqiy ildiziga ega bo'lgan chekli oraliq aniqlanadi.

Ikkinchi bosqichda ildizlar aniqlashtiriladi, ya'ni ular berilgan aniqlikda topiladi.

(6.1) tenglamaning ildizlarini ajratish uchun ushbu kriteriya qo'llaniladi: agar  $f(x)$  funksiya  $[a;b]$  kesmada uzluksiz, monoton va kesmaning chetki nuqtalarida har xil ishorali qiymatlarga ega bo'lsa, u holda (6.1) tenglama shu kesmada yagona haqiqiy ildizga ega bo'ladi.

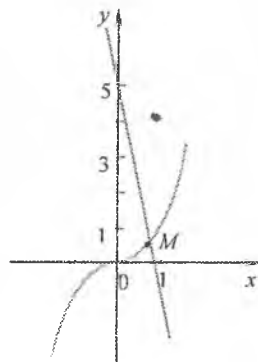
$f(x)$  funksiyaning kesmada monoton bo'lishining yetarli sharti uning birinchi tartibli hosilasining bu kesmada ishorasini saqlashi hisoblanadi (agar  $f(x) > 0$  bo'lsa, funksiya kesmada o'sadi, agar  $f(x) < 0$  bo'lsa, kamayadi).

Agar  $y=f(x)$  funksiyaning  $Ox$  o'q bilan kesishish nuqtalari joylashgan oraliqlarni aniq ko'rsatuvchi grafigini chizish mumkin bo'lsa, (6.1) tenglamaning ildizlarini grafik usulda ajratsa bo'ladi.

$f_1(x)$  va  $f_2(x)$  funksiya grafiklarini chizish oson bo'lgan hollarda (6.1) tenglamani

$$f_1(x) = f_2(x) \quad (6.2)$$

ekvivalent ko'rinishda ifodalash orqali ildizlarni ajratsa bo'ladi. Bunda (6.2) tenglamaning ildizi  $y=f_1(x)$  va  $y=f_2(x)$  grafiklarning kesishish nuqtasi bo'ladi.



29-shakl.

*1-misol.*  $x^3 + 6x - 5 = 0$  tenglama ildizlarini ajrating.

*Yechish.* Misolning shartiga ko'ra,

$$f(x) = x^3 + 6x - 5.$$

$x \in R$  da  $f'(x) = 3x^2 + 6 > 0$ ,  $x \in R$  bo'lgani uchun funksiya ixtiyoriy oraliqda o'sadi.  $x$  ning manfiy qiymatlarida  $f(x) < 0$  va  $x$  ning yetarlicha katta qiymatlarida, masalan,  $x \geq 1$  da  $f(x) > 0$ .

Demak, funksiya  $[0;1]$  kesmada bitta haqiqiy ildizga ega.

Berilgan tenglamaning ildizini grafik usulda ham ajratish mumkin. Tenglamani  $x^3 = -6x + 5$ , ya'ni (6.2) ko'rinishga keltiramiz.  $y = x^3$  va  $y = -6x + 5$  funksiya grafiklarini chizamiz (29-shakl). Chizmadan ko'rinadiki, keltirilgan grafiklar absissasi  $(0;1)$  intervalda yotuvchi  $M$  nuqtada kesishishadi.

### 5.6.2. Ildizlarni aniqlashtirishning vatarlar va urimlar usullari

#### Vatarlar usuli

Biror (analitik yoki grafik) usul bilan (6.1) tenglamaning ildizi ajratilgan, ya'ni tenglama yagona  $X$  ildizga ega bo'lgan  $[a;b]$  kesma aniqlangan bo'lsin. Bu kesma shunday tanlanadiki, bunda quyidagi uch shart bajaradi:

1)  $f(a)$  va  $f(b)$  sonlari har xil ishorali:  $f(a) \cdot f(b) < 0$ ;

2)  $[a;b]$  kesmada  $f'(x)$  hosila nolga teng emas va ishorasini saqlaydi;

3)  $[a;b]$  kesmada  $f''(x)$  hosila nolga teng emas va ishorasini saqlaydi.

(6.1) tenglama yechimining birinchi yaqinlashishi sifatida  $y = f(x)$  funksiya grafigi yoyining chetki  $A(a, f(a))$  va  $B(b, f(b))$  nuqtalarini tutashtiruvchi vatar bilan  $Ox$  o'q kesishish nuqtasining absissasini olamiz (30-shakl).

Ikki nuqtadan o'tuvchi to'g'ri chiziq formulasi asosida  $AB$  to'g'ri chiziq tenglamasini tuzamiz:

$$\frac{x-a}{b-a} = \frac{y-f(a)}{f(b)-f(a)} \quad (6.3)$$

(6.3) tenglamaga  $y=0$  ni qo'yib, ildizning birinchi yaqinlashishini topamiz:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \quad (6.4)$$

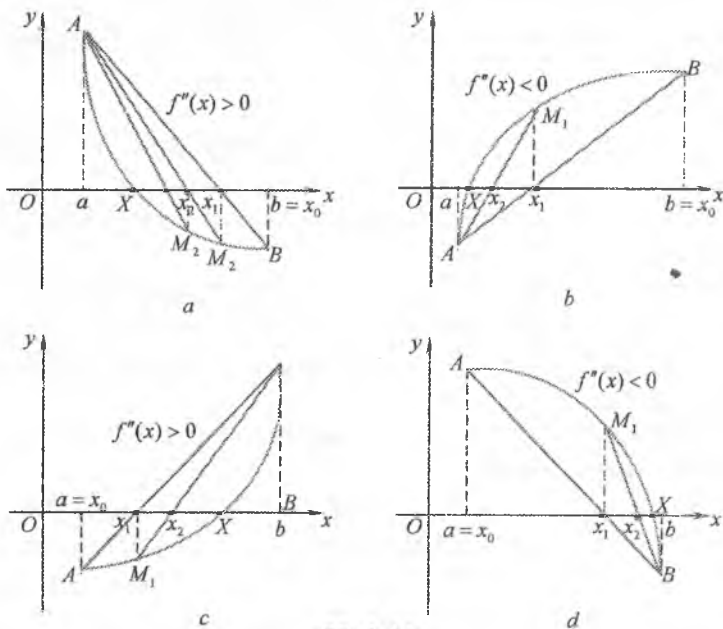
(6.1) tenglamaning izlanayotgan  $X$  ildizining ikkinchi  $x_2$  yaqinlashishini topish uchun (6.4) ga monand formula  $f(x)$  funksiya chetki nuqtalarida har xil ishoralarga ega bo'lgan  $[a; x_1]$  va  $[x_1; b]$  kesmalardan biriga qo'llaniladi. Bunda, agar  $f(a)f''(x) > 0$  shart bajarilsa (1-hol)  $[a; x_1]$  kesmada

$$x_2 = \frac{af(x_1) - x_1f(a)}{f(x_1) - f(a)} \quad (6.4)$$

kabi topiladi (30,a,b-shakl), agar  $f(a)f''(x) < 0$  shart bajarilsa (2-hol)  $[x_1; b]$  kesmada

$$x_2 = \frac{bf(x_1) - x_1f(b)}{f(x_1) - f(b)} \quad (6.5)$$

kabi aniqlanadi (30,c,d-shakl).



30-shakl.

Bu jarayonni davom ettirib, ildizning keyingi  $x_3, x_4, \dots$  yaqinlashishlari topiladi. Agar  $(n-1)$ - yaqinlashish mavjud

bo'lsa,  $n$ -yaqinlashish 1-hol uchun

$$x_n = \frac{af(x_{n-1}) - x_{n-1}f(a)}{f(x_{n-1}) - f(a)}, \quad x_0 = b, \quad n = 1, 2, 3, \dots \quad (6.6)$$

formula bilan, 2-hol uchun

$$x_n = \frac{bf(x_{n-1}) - x_{n-1}f(b)}{f(x_{n-1}) - f(b)}, \quad x_0 = a, \quad n = 1, 2, 3, \dots \quad (6.7)$$

formula bilan topiladi.

Geometrik mulohazalarga ko'ra,  $x_n$  ( $n = 1, 2, 3, \dots$ ) sonlarning ketma-ketligi ildizga yaqinlashadi, ya'ni

$$\lim_{n \rightarrow \infty} x_n = X. \quad (6.8)$$

(6.6) va (6.7) formulalar bilan hisoblashlar oxirgi kesma uchun talab qilingan aniqlik olinganga qadar davom ettiriladi.

Vatarlar usulining absolut xatoligi

$$|X - x_n| \leq \frac{|f(x_n)|}{\mu} \quad (6.9)$$

tengsizlik bilan baholanadi, bu yerda  $\mu = \min_{a \leq x \leq b} |f'(x)|$ ,  $f'(x) \neq 0$ .

### Urinmalar usuli

( $a; b$ ) oraliqda (6.1) tenglama yagona  $X$  ildizga ega bo'lsin.  $y = f(x)$  funksiya grafigiga  $A(a; f(a))$  nuqtada urinma o'tkazamiz.

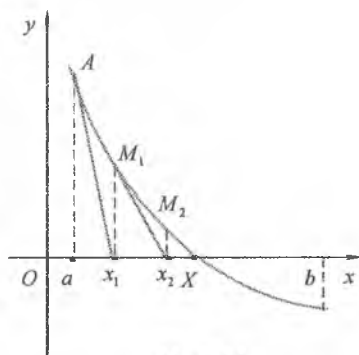
Bu urinmaning tenglamasini tuzamiz:

$$y - f(a) = f'(a)(x - a). \quad (6.10)$$

Bu tenglamada  $y = 0$  deb urinmaning  $Ox$  o'q bilan kesishish nuqtasining absissasini topamiz:

$$x_1 = a - \frac{f(a)}{f'(a)}, \quad f'(a) \neq 0. \quad (6.11)$$

(6.11) formula ildizning birinchi yaqinlashishini beradi. Ikkinchi yaqinlashishni (6.11) ga monand



31-shakl.

formulani  $(x, b)$  oraliqqa qo'llab, topamiz:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \quad (6.12)$$

Ildizning keyingi yaqinlashishlari shu kabi topiladi.

Agar  $(n-1)$ -yaqinlashish mavjud bo'lsa,  $n$ -yaqinlashish

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n = 1, 2, 3 \quad (6.13)$$

formula bilan topiladi.

Bu tenglamada nolinchi yaqinlashish  $x_0$  sifatida  $[a; b]$  kesmaning

$$f(x_0)f''(x) > 0 \quad (6.14)$$

shartni qanoatlantiruvchi nuqtasining absissasi olinadi (31-shakl). Demak, bu usulni  $A$  nuqtadan boshlash shart emas.

Urinmalar usulida ham (6.8) va (6.9) munosabatlar o'rinli bo'ladi.

### *Vatarlar va urinmalar usullarining birgalikda qo'llanilishi*

Vatarlar usuli bilan urinmalar usuli ildizga turli tomondan yaqinlashishni beradi. Shu sababli, odatda, ularni birgalikda qo'llash qulay bo'ladi. Bunda ildizni aniqlashtirish tez bajariladi va hisoblashni nazorat qilish mumkin bo'ladi.

Vatarlar va urinmalar usullarini birgalikda qo'llanilishi masalasini qaraymiz. Bunda ham vatarlar usulidagi uchta shart bajarilsin deymiz.

Bu shartlar bajarilganida to'rtta hol bo'lishi mumkin (30-shakl):

- |                             |                             |
|-----------------------------|-----------------------------|
| a) $f'(x) < 0, f''(x) > 0;$ | b) $f'(x) > 0, f''(x) < 0;$ |
| c) $f'(x) > 0, f''(x) > 0;$ | d) $f'(x) < 0, f''(x) < 0.$ |

Aniqlik uchun  $f'(x) > 0, f''(x) > 0$  bo'lgan holni qaraymiz (32-shakl).

$A(a; f(a))$  va  $B(b; f(b))$  nuqtalar orqali vatar o'tkazamiz.

Bu vatarning  $Ox$  o'q bilan kesishish nuqtasini topamiz:

$$a_1 = a - \frac{f(a)}{f(b) - f(a)}(b - a).$$

Endi  $f(x)$  funksiya grafigiga  $B(b; f(b))$  nuqtaga urinma o'tkazamiz.

Bu urinmaning  $Ox$  o'q bilan kesishish nuqtasini topamiz:

$$b_1 = b - \frac{f(b)}{f'(b)}.$$

Shunday qilib,  $a < a_1 < X < b_1 < b$ .

Bu jarayonni davom ettirib, taqribiy qiymatlarning ikkita ketma-ketligini topamiz:

$$a < a_1 < a_2 < \dots < X \quad \text{va} \quad b > b_1 > b_2 > \dots > X,$$

bu yerda

$$a_n = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}, n=1,2,\dots, a_0 = a; \quad (6.15)$$

$$b_n = b_{n-1} - \frac{f(b_{n-1})}{f'(b_{n-1})}, n=1,2,\dots, b_0 = b. \quad (6.16)$$

$\{a_n\}$  va  $\{b_n\}$  ketma-ketliklar monoton va chegaralangan. Demak, ular limitga ega.  $\lim_{n \rightarrow \infty} a_n = \alpha$ ,

$\lim_{n \rightarrow \infty} b_n = \beta$  bo'lsin. Agar 1) va

3) shartlar bajarilsa,  $\alpha = \beta = X$   $f(x) = 0$  teng-lamaning yagona yechimi bo'ladi.

Agar ildizning aniqligi  $\varepsilon$  oldindan berilgan bo'lsa,

u holda  $a_n$  va  $b_n$  lar  $|a_n - b_n| < \varepsilon$

yoki  $\varepsilon < |a_n - b_n| < 2\varepsilon$  shartlardan

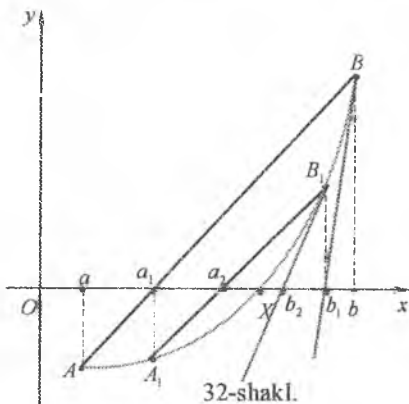
biri bajarilishiga qadar davom

ettiriladi. Bunda birinchi shart bajarilsa  $X \approx a_n$  (yoki  $X \approx b_n$ ) bo'ladi,

ikkinchi shart bajarilsa  $X \approx \frac{a_n + b_n}{2}$  bo'ladi.

*1-misol.*  $2 - x - \lg x = 0$  tenglamaning haqiqiy ildizini vatarlar va urinmalar usullarini birgalikda qo'llab,  $\varepsilon = 0,000001$  aniqlikda hisoblang.

*Yechish.* Izlanayotgan ildizni ajratish uchun berilgan tenglamani  $\lg x = 2 - x$  ko'rinishga keltiramiz va  $y = \lg x$ ,  $y = 2 - x$  funksiyalarning





grafiklarini chizamiz (33-shakl). Chizmadan izlanayotgan ildiz  $[1,6;1,8]$  kesmaning ichki nuqtasi bo'lishi kelib chiqadi.

1) va - 3) shartlarning bajarilishini tekshiramiz.

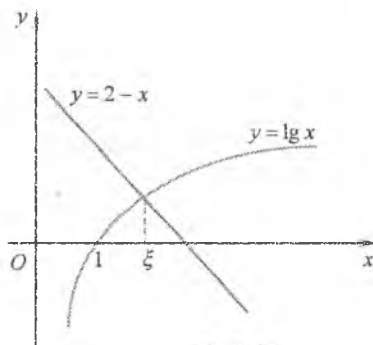
$$f(x) = 2 - x - \lg x \text{ funksiya uchun:}$$

$$f(1,6) = 2 - 1,6 - 0,2041 = 0,1959 > 0;$$

$$f(1,8) = 2 - 1,8 - 0,2553 = -0,0553 < 0;$$

$$f'(x) = -1 - \frac{1}{x} \lg e; \quad f''(x) = \frac{1}{x^2} \lg e;$$

$[1,6;1,8]$  kesmada  $f'(x) < 0$ ,  $f''(x) > 0$ .



33-shakl.

Demak,  $[1,6;1,8]$  kesmada

1) - 3) shartlar bajariladi. Shu sababli  $a = 1,6$  va  $b = 1,8$  deb olamiz va aniqlashtirish formulalarini qo'llaymiz:

$$a_1 = 1,6 - \frac{(1,8 - 1,6)f(1,6)}{f(1,8) - f(1,6)} = 1,6 + 0,1559 = 1,7559;$$

$$b_1 = 1,6 - \frac{f(1,6)}{f'(1,6)} = 1,6 + 0,1540 = 1,7540;$$

$$a_2 = 1,75558; \quad b_2 = 1,75557; \quad a_3 = 1,7555816; \quad b_3 = 1,7555807.$$

$$|a_3 - b_3| = |1,7555816 - 1,7555807| = 0,0000009 < \varepsilon.$$

Demak, izlanayotgan yechim  $\xi \approx a_3 = 1,7555816$ .

### 5.6.3. Mashqlar

1. Tenglamalarning haqiqiy ildizini vatarlar va urinmalar usullarini birgalikda qo'llab,  $\varepsilon = 0,0001$  aniqlikda toping:

1)  $x^5 - 2x + 7 = 0$ ;

2)  $x^3 + x - 1 = 0$ ;

3)  $x \lg x - 1 = 0$ ;

4)  $2x + i - \sin x = 0$ .

# OLIV ALGEBRA ELEMENTLARI

6

- Kompleks sonlar
- Ko'pladlar



**Leonard Eylar**  
(1707–1783) –  
shveysar, nemis va rus  
matematigi hamda  
mexanigi.

*Eylar matematik analiz, differensial geometriya, sonlar nazariyasi, taqribiy hisoblashlar, osmon mexanikasi, matematik fizika, hallistika, kemalar qurilishi, musiqqa nazariyasi va fanning boshqa sohalariga oid 850 dan ortiq ishlar muallifi.*

*Eylar matematikaga qatorlar nazariyasini, kompleks sonlar nazariyasida Eylar formulasi-ni,  $e$  sonini, mavhum birlik uchun  $i$  belgini va boshqa ko'plab maxsus funksiyalarni kiritgan.*

## 6.1. KOMPLEKS SONLAR

Kvadrat ildiz chiqarish amali barcha haqiqiy sonlar uchun aniqlanmaganligi sababli har qanday kvadrat tenglama ham haqiqiy sonlar to'plamida yechimga ega bo'lmaydi. Bu masala hamda uchinchi va to'rtinchi darajali tenglamalarni yechishda yuzaga kelgan ko'plab masalalar haqiqiy sonlar to'plamini kompleks sonlar to'plamigacha kengaytirishni taqozo etdi.

Keyinchalik kompleks sonlar matematika va amaliy matematikaning ko'pchilik muhim masalalarini yechishga tatbiq qilindi va hozirgi matematikani kompleks sonlar tushunchasiz tasavvur qilib bo'lmaydi.

### 6.1.1. Kompleks son tushunchasi va tasviri

#### *Kompleks son tushunchasi*

**1-ta'rif.**  $z$  kompleks son deb ma'lum tartibda berilgan  $x$  va  $y$  haqiqiy sonlar juftiga aytiladi va  $z = (x, y)$  deb yoziladi.

$x$  va  $y$  sonlarga mos ravishda  $z$  kompleks sonning haqiqiy va mavhum qismlari deyilib,  $x = \operatorname{Re} z$ ,  $y = \operatorname{Im} z$  kabi belgilanadi.

$z_1 = (x_1, y_1)$  va  $z_2 = (x_2, y_2)$  kompleks sonlarida  $x_1 = x_2$  va  $y_1 = y_2$  bo'lganida ular teng, ya'ni  $z_1 = z_2$  deyiladi.

$(0,1)$  son *mavhum birlik* deb ataladi va  $i$  bilan belgilanadi:

$$i = (0,1). \quad (1.1)$$

**2-ta'rif.**  $z_1 = (x_1, y_1)$  va  $z_2 = (x_2, y_2)$  kompleks sonlarining yig'indisi deb

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad (1.2)$$

songa aytiladi.

**3-ta'rif.**  $z_1 = (x_1, y_1)$  va  $z_2 = (x_2, y_2)$  kompleks sonlarining ko'paytmasi deb

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2) \quad (1.3)$$

songa aytiladi.

Keltirilgan ta'riflardan haqiqiy sonlardagi kabi

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

va

$$(x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0)$$

kelib chiqadi.

Shunday qilib, kompleks sonlar haqiqiy sonlarni «to'ldiradi».

(1.1) va (1.3) formulalar asosida topamiz:

$$x = (x, 0), \quad iy = (0, 1)y = (0, 1) \cdot (y, 0) = (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y) = (0, y).$$

Bu ifodalardan  $z = (x, y)$  kompleks son yozilishining boshqa bir ko'rinishi kelib chiqadi:

$$z = (x, y) = (x, 0) + (0, y) = x + iy.$$

Kompleks sonlarni ko'paytirish ta'rifidan topamiz:

$$i^2 = \bar{i}i = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

Demak,  $i = (0, 1)$ – kvadrati minus birga teng bo'lgan son.

Shunday qilib,  $z = x + iy$  ifodaga kompleks son deyiladi, bu yerda  $x, y$ – haqiqiy sonlar,  $i$ – mavhum birlik.

Agar  $z = x + iy$  ifodada  $y = 0$  bo'lsa,  $z = x$  haqiqiy son, agar  $x = 0$  bo'lsa,  $z = iy$  sof mavhum son hosil bo'ladi.

Faqat  $x = y = 0$  bo'lganida  $z = x + iy$  kompleks son nolga teng

bo'ladi. Kompleks sonlar uchun «katta» va «kichik» tushunchalari kiritilmaydi.

Mavhum qismlarining ishorasi bilan farq qiluvchi  $z = x + iy$  va  $z = x - iy$  sonlariga *qo'shma kompleks sonlar* deyiladi.

Haqiqiy va mavhum qismlarining ishorasi bilan farq qiluvchi  $z_1 = x + iy$  va  $z_2 = -x - iy$  sonlariga *qarama-qarshi kompleks sonlar* deyiladi.

### *Kompleks sonlarning geometrik tasviri*

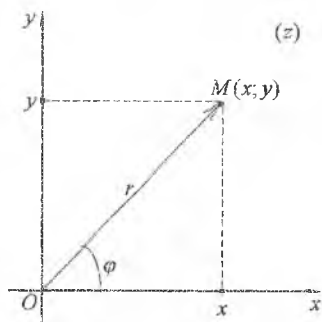
Har bir  $z = x + iy$  kompleks sonni  $Oxy$  koordinatalar tekisligining  $M(x; y)$  nuqtasi bilan ifodalash mumkin (bu yerda  $x = \operatorname{Re} z$ ,  $y = \operatorname{Im} z$ ) va aksincha, koordinatalar tekisligining har bir  $M(x; y)$  nuqtasini  $z = x + iy$  kompleks sonning geometrik tasviri deb qarash mumkin (1-shakl).

$Oxy$  tekislikka *kompleks tekislik* deyiladi va  $(z)$  kabi belgilanadi. Bunda,  $z = x$  haqiqiy sonlar *haqiqiy o'q* deb ataluvchi  $Ox$  o'qning nuqtalari bilan aniqlanadi;  $z = iy$  mavhum sonlar *mavhum o'q* deb ataluvchi  $Oy$  o'qning nuqtalari bilan aniqlanadi.

Shuningdek,  $z = x + iy$  kompleks sonni  $M(x; y)$  nuqtaning radius vektori  $\vec{r} = \overline{OM}$  orqali ifodalash mumkin (1-shakl). Bunda:

$\vec{r}$  vektorning uzunligiga *kompleks sonning moduli* deyiladi va  $|z|$  yoki  $r$  bilan belgilanadi;  $\vec{r}$  vektorning  $Ox$  o'qning musbat yo'nalishi bilan hosil qilgan burchagiga *kompleks sonning argumenti* deyiladi va  $\operatorname{Arg} z$  bilan belgilanadi.

$z = 0$  kompleks sonning argumenti aniqlanmagan.  $z \neq 0$  kompleks sonning argumenti ko'p qiymatli bo'lib,  $2\pi k$  ( $k = 0, -1, 1, -2, 2, \dots$ ) qo'shiluvchigacha aniqlikda topiladi:  $\operatorname{Arg} z = \arg z + 2\pi k$ ,  $k \in \mathbb{Z}$ , bu yerda  $\arg z - \text{argumentning } (-\pi; \pi]$  oraliqda yotuvchi *bosh qiymati*, ya'ni  $-\pi < \arg z \leq \pi$  (ayrim hollarda argumentning bosh qiymati sifatida  $[0; 2\pi)$  oraliqqa tegishli qiymat olinadi).



1-shakl.

## 6.1.2. Kompleks sonlarning yozilish shakllari

Ushbu

$$z = x + iy$$

ifodaga *kompleks sonning algebraik shakli* deyiladi.

Kompleks sonning  $r$  moduli va  $\varphi = \text{Arg}z$  argumentini  $z = x + iy$  kompleks sonni ifodalovchi  $\vec{r} = \overline{OM}$  vektorning qutb koordinatalari deb qarash mumkin. Bunda  $x = r \cos \varphi$ ,  $y = r \sin \varphi$  bo'ladi (1-shakl).

U holda  $z = x + iy$  kompleks sonni  $z = r \cos \varphi + ir \sin \varphi$  yoki

$$z = r(\cos \varphi + i \sin \varphi)$$

ko'rinishda yozish mumkin. Bu ifodaga *kompleks sonning trigonometrik shakli* deyiladi.

Bunda  $r = |z|$  modul quyidagi formula bilan aniqlanadi:

$$r = |z| = \sqrt{x^2 + y^2}.$$

$\varphi$  argument esa

$$\cos \varphi = \frac{x}{r}, \quad \sin \varphi = \frac{y}{r}, \quad \operatorname{tg} \varphi = \frac{y}{x}$$

formulalardan topiladi.

$\varphi = \text{Arg}z = \arg z + 2\pi k$ ,  $k \in z$  ekanidan

$$\cos \varphi = \cos(\arg z + 2\pi k) = \cos(\arg z), \quad \sin \varphi = \sin(\arg z)$$

kelib chiqadi. Shu sababli kompleks sonning algebraik shaklidan trigonometrik shakliga o'tishda kompleks son argumentining bosh qiymatini aniqlash yetarli bo'ladi.

$-\pi < \arg z \leq \pi$  bo'lgani uchun  $\operatorname{tg} \varphi = \frac{y}{x}$  tenglikdan topamiz:

$$\arg z = \begin{cases} \operatorname{arctg} \frac{y}{x}, & I, IV \text{ choraklarning ichki nuqtalarida,} \\ \operatorname{arctg} \frac{y}{x} + \pi, & II \text{ chorakning ichki nuqtalarida,} \\ \operatorname{arctg} \frac{y}{x} - \pi, & III \text{ chorakning ichki nuqtalarida.} \end{cases}$$

Agar nuqta haqiqiy yoki mavhum o'qlarda yotsa,  $\arg z$  ni bevosita topish mumkin.

Masalan,  $z_1 = 2$  uchun  $\arg z_1 = 0$ ;  $z_2 = -3$  uchun  $\arg z_2 = \pi$ ;  $z_3 = i$  uchun  $\arg z_3 = \frac{\pi}{2}$ ;  $z_4 = -4i$  uchun  $\arg z_4 = -\frac{\pi}{2}$  (2-shakl).

Eyler formulasi deb ataluvchi

$$\cos \varphi + i \sin \varphi = e^{i\varphi}$$

ifoda yordamida  $z = r(\cos \varphi + i \sin \varphi)$  tenglikdan  $z = re^{i\varphi}$  ifoda keltirib chiqariladi. Ushbu

$$z = re^{i\varphi}$$

ifodaga  $z = r(\cos \varphi + i \sin \varphi)$  kompleks sonning ko'rsatkichli (yoki eksponensial) shakli deyiladi, bu yerda  $r = |z|$  — kompleks sonning moduli;  $\varphi = \operatorname{Arg} z$  — kompleks sonning argumenti.

Eyler formulasiga ko'ra  $e^{i\varphi}$  funksiya  $2\pi$  davrli davriy funksiya bo'ladi. Shu sababli  $z$  kompleks sonni ko'rsatkichli shaklda yozish uchun kompleks son argumentining bosh qiymatini, ya'ni  $\varphi = \arg z$  ni aniqlash yetarli bo'ladi.

1-misol.  $z = -\sin \frac{\pi}{8} + i \cos \frac{\pi}{8}$  kompleks sonni turli (algebraik, trigonometrik va ko'rsatkichli) shakllarda yozing.

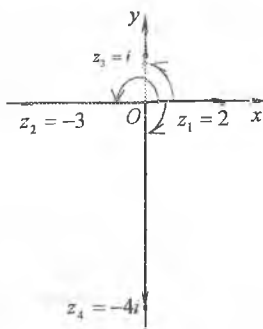
*Yechish.*  $z = -\sin \frac{\pi}{8} + i \cos \frac{\pi}{8}$  kompleks son trigonometrik shaklda berilgan emas. Shu sababli shunday  $\varphi$  burchakni topamizki,  $\cos \varphi = -\sin \frac{\pi}{8}$  va  $\sin \varphi = \cos \frac{\pi}{8}$  bo'lsin.

Bunday burchak  $\varphi = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}$  bo'ladi.

$\cos \frac{5\pi}{8} = -\frac{\sqrt{2-\sqrt{2}}}{2}$  va  $\sin \frac{5\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$  ni hisobga olib, kompleks

sonining turli shakllarini yozamiz:

$$z = -\frac{\sqrt{2-\sqrt{2}}}{2} + i \frac{\sqrt{2+\sqrt{2}}}{2} = \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8} = e^{\frac{5\pi}{8}i}$$



2-shakl.

### 6.1.3. Kompleks sonlar ustida amallar

#### *Kompleks sonlarni qo'shish*

Agar  $z_1 = x_1 + iy_1$  va  $z_2 = x_2 + iy_2$  bo'lsa, yuqorida keltirilgan kompleks sonlarni qo'shish ta'rifiga ko'ra,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \quad (1.4)$$

Kompleks sonlarni qo'shish kommutativlik va assosiativlik xossalariga ega:

$$z_1 + z_2 = z_2 + z_1, \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(1.4) tenglikdan kompleks sonlar geometrik jihatdan vektorlar kabi qo'shilishi kelib chiqadi (3-shakl).

3-shakldan bevosita ko'rinadiki,  $|z_1 + z_2| \leq |z_1| + |z_2|$ . Bu tengsizlikka *uchburchak tengsizligi* deyiladi.

#### *Kompleks sonlarni ayirish*

Kompleks sonlarni ayirish amali qo'shishga teskari amal sifatida aniqlanadi.

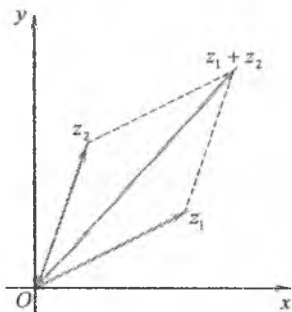
$z_1$  va  $z_2$  kompleks sonlarning ayirmasi deb,  $z_2$  ga qo'shilganida  $z_1$  ni hosil qiluvchi  $z$  kompleks soniga aytiladi va  $z = z_1 - z_2$  tarzda yoziladi.

Agar  $z_1 = x_1 + iy_1$  va  $z_2 = x_2 + iy_2$  bo'lsa, ta'rifga ko'ra,

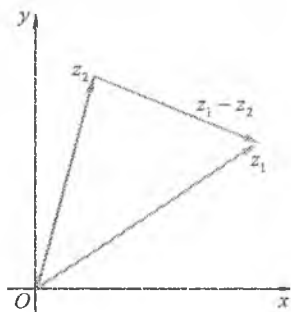
$$z = z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \quad (1.5)$$

(1.5) tenglikdan kompleks sonlar geometrik jihatdan vektorlar kabi ayrilishi kelib chiqadi. (4-shakl).

4-shakldan ko'rinadiki,  $|z_1 - z_2| \geq |z_1| - |z_2|$ .



3-shakl.



4-shakl.

## Kompleks sonlar uchun

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

bo'ladi, ya'ni ikkita kompleks sonlar ayirmasining moduli tekislikda bu sonlarni ifodalovchi nuqtalar orasidagi masofaga teng.

Shu sababli, masalan  $|z - 2i| = 1$  tenglik kompleks tekisligida  $z_0 = 2i$  nuqtadan birga teng masofada yotuvchi  $z$  nuqtalar to'plamini, ya'ni markazi  $z_0 = 2i$  nuqtada joylashgan va radiusi birga teng aylanani aniqlaydi.

### *Kompleks sonlarni ko'paytirish*

Agar  $z_1 = x_1 + iy_1$  va  $z_2 = x_2 + iy_2$  bo'lsa, yuqorida keltirilgan kompleks sonlarni ko'paytirish ta'rifiga ko'ra,

$$z = z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \quad (1.6)$$

Kompleks sonlarni ko'paytirish kommutativlik, assosiativlik va qo'shishga nisbatan distributivlik xossalari ega:

$$\begin{aligned} z_1 z_2 &= z_2 z_1; \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3); \\ z_1 (z_2 + z_3) &= z_1 z_2 + z_1 z_3. \end{aligned}$$

*2-misol.*  $z_1 = 1 + 3i$ ,  $z_2 = -3 + i$  bo'lsa,  $z_1 + z_2$ ,  $2z_1 - z_2$ ,  $z_1 \cdot z_2$  larni hisoblang.

*Yechish.*  $z_1 + z_2 = (1 + 3i) + (-3 + i) = -2 + 4i;$

$$2z_1 - z_2 = (2 + 6i) - (-3 + i) = 5 + 5i;$$

$$z_1 \cdot z_2 = (1 + 3i) \cdot (-3 + i) = (-3 - 3) + i(1 - 9) = -6 - 8i.$$

Trigonometrik shaklda berilgan

$$z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1), \quad z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$$

kompleks sonlarni ko'paytiramiz:

$$\begin{aligned} z_1 z_2 &= r_1(\cos \varphi_1 + i \sin \varphi_1) r_2(\cos \varphi_2 + i \sin \varphi_2) = \\ &= r_1 r_2 (\cos \varphi_1 \cos \varphi_2 + i \sin \varphi_1 \cos \varphi_2 + i \cos \varphi_1 \sin \varphi_2 - \sin \varphi_1 \sin \varphi_2) = \\ &= r_1 r_2 (\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + i(\sin \varphi_1 \cos \varphi_2 + i \cos \varphi_1 \sin \varphi_2) = \\ &= r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)), \end{aligned}$$



ya'ni

$$z_1 z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)). \quad (1.7)$$

Demak, kompleks sonlar ko'paytirilganda ularning modullari ko'paytiriladi va argumentlari qo'shiladi.

Bu qoida istalgan sondagi ko'paytuvchilar uchun ham o'rinli bo'ladi. Xususan,  $n$  ta bir xil ko'paytuvchilar uchun

$$z^n = (r(\cos \varphi + i \sin \varphi))^n = r^n (\cos n\varphi + i \sin n\varphi) \quad (1.8)$$

bo'ladi.

(1.8) formulaga *Muavr formulasi* deyiladi.

3- misol.  $z = \sqrt{3} + i$  bo'lsa,  $z^6$  ni hisoblang.

*Yechish.* Avval kompleks sonni trigonometrik shaklga keltiramiz.

$$x = \sqrt{3}, \quad y = 1 \text{ bo'lgani uchun } r = \sqrt{\sqrt{3}^2 + 1^2} = 2, \quad \arg z = \operatorname{arctg} \frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

Bundan

$$z = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right).$$

Muavr formulasiga ko'ra:

$$z^6 = 2^6 \left( \cos \frac{\pi}{6} \cdot 6 + i \sin \frac{\pi}{6} \cdot 6 \right) = 64 (\cos \pi + i \sin \pi) = -64.$$

### *Kompleks sonlarni bo'lish*

Kompleks sonlarni bo'lish amali ko'paytirishga teskari amal sifatida aniqlanadi.

$z_1$  va  $z_2 \neq 0$  kompleks sonlarning bo'linmasi deb,  $z_2$  ga ko'paytirilganida  $z_1$  ni hosil qiluvchi  $z$  kompleks soniga aytiladi va

$$z = \frac{z_1}{z_2} \text{ kabi yoziladi.}$$

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2 \neq 0 \text{ va } z = x + iy \text{ bo'lsin.}$$

U holda  $(x_2 + iy_2)(x + iy) = x_1 + iy_1$  tenglikdan

$$\begin{cases} xx_2 - yy_2 = x_1, \\ xy_2 + yx_2 = y_1 \end{cases}$$

tenglamalar sistemasi kelib chiqadi.

Sistemadan  $x$  va  $y$  ni topamiz:

$$x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \quad y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}.$$

Shunday qilib,

$$z = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}. \quad (1.9)$$

Amalda ikki kompleks sonning bo'linmasi uning surat va maxrajini maxrajning qo'shmasiga ko'paytirish orqali topiladi (maxraj mavhumlikdan qutqariladi).

4- misol.  $z_1 = 1 + 2i$ ,  $z_2 = 3 + i$  bo'lsa,  $\frac{z_1}{z_2}$  ni toping.

*Yechish.*  $\frac{z_1}{z_2}$  kasrning surat va maxrajini  $\bar{z}_2$  ga ko'paytirib, topamiz:

$$\frac{z_1}{z_2} = \frac{1 + 2i}{3 + i} = \frac{(1 + 2i)(3 - i)}{(3 + i)(3 - i)} = \frac{3 + 6i - i + 2}{9 + 1} = \frac{5 + 5i}{10} = \frac{1}{2} + \frac{1}{2}i.$$

Trigonometrik shaklda berilgan  $z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1)$  kompleks sonini  $z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$  kompleks soniga bo'lamiz:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos \varphi_1 + i \sin \varphi_1)}{r_2(\cos \varphi_2 + i \sin \varphi_2)} = \frac{r_1(\cos \varphi_1 + i \sin \varphi_1) \cdot (\cos \varphi_2 - i \sin \varphi_2)}{r_2(\cos \varphi_2 + i \sin \varphi_2) \cdot (\cos \varphi_2 - i \sin \varphi_2)} \\ &= \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)). \end{aligned}$$

Demak,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)). \quad (1.10)$$

Shunday qilib, bir kompleks sonni ikkinchisiga bo'lganda ularning modullari bo'linadi va argumentlari ayriladi.

### *Kompleks sonlardan ildiz chiqarish*

Kompleks sondan  $n$ -darajali ildiz chiqarish amali  $n$ -natural darajaga oshirish analiga teskari amal sifatida aniqlanadi.

$z$  kompleks sonining  $n$ -darajali ildizi deb,  $w^n = z$  tenglikni qanoatlantiruvchi  $w$  kompleks soniga aytiladi.

$z = r(\cos \varphi + i \sin \varphi)$  va  $w = \rho(\cos \theta + i \sin \theta)$  bo'lsin.

Ildizning ta'rifi va Muavr formulasidan foydalanib topamiz:

$$z = w^n = \rho^n (\cos n\theta + i \sin n\theta) = r(\cos \varphi + i \sin \varphi).$$

Bundan

$$\rho^n = r, \quad n\theta = \varphi + 2\pi k, \quad k = 0, -1, 1, -2, 2, \dots$$

yoki

$$\theta = \frac{\varphi + 2\pi k}{n}, \quad \rho = \sqrt[n]{r}$$

kelib chiqadi.

U holda  $w = \sqrt[n]{z}$  tenglik quyidagi ko'rinishga keladi:

$$w_k = \sqrt[n]{r(\cos \varphi + i \sin \varphi)} = \sqrt[n]{r} \left( \cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right), \quad k = 0, 1, \dots, n-1. \quad (1.11)$$

Sinus va kosinus funksiyala-rning davriyligi sababli  $z$  kompleks

sonining  $n$ - darajali ildizlari soni

$n$  ga teng bo'ladi va ular  $k$  ning

$n$  ta  $k = 0, 1, \dots, n-1$  qiymatlarida

aniqlanadi. Ildizlarning moduli

$r$  haqiqiy sonining  $n$ -darajali

algebraik ildizidan iborat bo'ladi,

argumentlari esa bir-biridan  $\frac{2\pi}{n}$  ga

karrali songa farq qiladi. Bunda

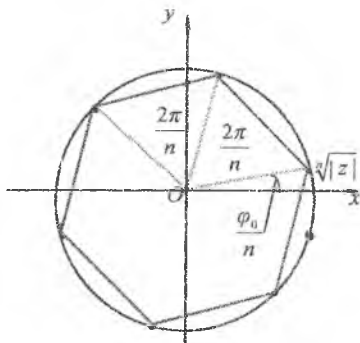
barcha ildizlar kompleks tekisligida

markazi  $z = 0$  nuqtada bo'lgan va

radiusi  $\sqrt[n]{|z|}$  ga teng aylanaga

ichki chizilgan  $n$  burchakli

muntazam ko'pburchakning uchlarini tasvirlaydi (5-shakl).



5-shakl.

5-misol.  $\sqrt[4]{-1}$  ning barcha ildizlarini toping.

*Yechish.* Ildiz ostidagi ifodani trigonometrik shaklda yozamiz:

$$-1 = \cos \pi + i \sin \pi.$$

Bundan

$$\sqrt[4]{-1} = \cos \frac{\pi + 2\pi k}{4} + i \sin \frac{\pi + 2\pi k}{4}, \quad k = 0, 1, 2, 3.$$

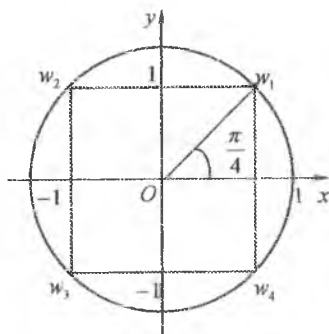
$k$  ga 0, 1, 3 va 4 qiymatlar berib, topamiz:

$$w_0 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i),$$

$$w_1 = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}(-1+i),$$

$$w_2 = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = \frac{\sqrt{2}}{2}(-1-i),$$

$$w_3 = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \frac{\sqrt{2}}{2}(1-i).$$



6-shakl.

Bu ildizlar ( $z$ ) kompleks tekisligida birlik aylanaga ichki chizilgan muntazam to'rtburchakning (kvadratning) uchlarida yotadi (6-shakl).

#### 6.1.4. Mashqlar

1.  $x, y$  ning qanday haqiqiy qiymatlarida  $z_1 = x - 3yi - y - \frac{2x}{i}$  va

$z_2 = 2x + i^2 - 3xi - yi^3$  kompleks sonlar qo'shma bo'ladi?

2.  $x, y$  ning qanday haqiqiy qiymatlarida  $z_1 = y + 2i^3 + 3 - 2xi$  va

$z_2 = 3x + 8i + \frac{2y}{i} + 2i^2$  kompleks sonlar teng bo'ladi?

3.  $x, y$  ning qanday haqiqiy qiymatlarida  $z_1 = 3x - 2yi + 5i^3 - 1$  va

$z_2 = 3y - i^3 + \frac{8x}{i} + 2i^2$  kompleks sonlar qarama-qarshi bo'ladi?

4.  $x, y$  ning qanday haqiqiy qiymatlarida  $z_1 = 5x + \frac{3y}{i^2} + 3yi + i^3$  va

$z_2 = 3y(1+i) + \frac{5x}{i} - i^4$  kompleks sonlar nolga teng bo'ladi?

5. ( $z$ ) tekislikda berilgan tenglamalarni yeching:

1)  $z^2 + 6z + 25 = 0;$

2)  $2z^2 + iz + 1 = 0;$

3)  $iz^2 - 2z + 3i = 0;$

4)  $z^2 - 6iz - 5 = 0.$

6. (z) tekislikda berilgan shartlar bilan qanday nuqtalar to'plami aniqlanadi?

1)  $\operatorname{Re} z = a$ ;

2)  $\operatorname{Im} z = b$ ;

3)  $r < |z| < R$ ;

4)  $\varphi < \arg z < \psi$ ;

5)  $r < |z| < R$ ,  $\varphi < \arg z < \psi$ , bu yerda  $a, b, r, R, \varphi, \psi$  — haqiqiy sonlar.

7. Berilgan kompleks sonlarni turli (algebraik, trigonometrik va ko'rsatkichli) shakllarda yozing:

1)  $z = -2 + 2\sqrt{3}i$ ;

2)  $z = \sqrt{3} - i$ ;

3)  $z = \sqrt{3} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$ ;

4)  $z = 2\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$ ;

5)  $z = \sqrt{2} e^{-\frac{\pi}{4}}$ ;

6)  $z = \sqrt{13} e^{i \left( \arctan \frac{2}{3} - \pi \right)}$ ;

7)  $z = 2 \cos 60^\circ - 2i \sin 60^\circ$ ;

8)  $z = -2 \cos 45^\circ - 2i \sin 45^\circ$ .

8. Berilgan kompleks sonlarning yig'indisi, ayirmasi, ko'paytmasi va bo'linmasini toping:

1)  $z_1 = -5 + 3i$  va  $z_2 = 2 - 4i$ ;

2)  $z_1 = -3 - 4i$  va  $z_2 = 2 + 3i$ .

9. (z) tekislikda berilgan nuqtalar orasidagi masofani toping:

1)  $1 - 3i$  va  $4i$ ;

2)  $1 - 5i$  va  $-4$ ;

3)  $1 + 3i$  va  $3 + 2i$ ;

4)  $8 - 3i$  va  $2 + 5i$ .

10. Hisoblang:

1)  $\frac{2-3i}{1+2i} + (1-i)^2(1+i)$ ;

2)  $\frac{1+3i}{-2+i} \cdot (-2i) + 1$ ;

3)  $(2+3i)^3 - (2-3i)^3$ ;

4)  $(-1+2i)^4 - (1+2i)^4$ .

11. Kompleks sonlarning haqiqiy va mavhum qismlarini toping:

1)  $\frac{3\sqrt{3}-i^7}{2(\sqrt{3}+2i^3)}$ ;

2)  $\frac{-1+i^5}{2+i} + \frac{8+19i^3}{40}$ ;

3)  $\frac{3+i^5}{(1-i^3)(1+2i^7)}$ ;

4)  $(i^3-5) \left( 2i + \frac{3}{2-i^3} \right)$ .

12. Berilgan kompleks sonlarning ko'paytmasi va bo'linmasini toping:

1)  $z_1 = 4 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$  va  $z_2 = 2 \left( \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$ ;

2)  $z_1 = 6 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$  va  $z_2 = \cos \left( -\frac{2\pi}{3} \right) + i \sin \left( -\frac{2\pi}{3} \right)$ ;

$$3) z_1 = 8(\cos 135^\circ + i \sin 135^\circ) \text{ va } z_2 = 2(\cos 45^\circ + i \sin 45^\circ);$$

$$4) z_1 = 8(\cos 90^\circ + i \sin 90^\circ) \text{ va } z_2 = \cos 30^\circ + i \sin 30^\circ.$$

13. Darajalarni hisoblang:

$$1) \left( \frac{\sqrt{2}}{2} \left( \cos \frac{7\pi}{36} + i \sin \frac{7\pi}{36} \right) \right)^{-9};$$

$$2) \left( \frac{1+i}{1-i} \right)^{10};$$

$$3) (2-2i)^9;$$

$$4) (\sqrt{2}(\cos 20^\circ + i \sin 20^\circ))^{12}.$$

14. Berilgan sonlarning barcha ildizlarini toping:

$$1) \sqrt{2-2\sqrt{3}i};$$

$$2) \sqrt[3]{-i};$$

$$3) \sqrt[3]{-8+8\sqrt{3}i};$$

$$4) \sqrt[3]{1+i}.$$

## 6.2. KO'PHADLAR

### 6.2.1. Ko'phadlar ustida amallar

Ushbu

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n \quad (2.1)$$

funksiyaga *n*-darajali ko'phad (yoki *butun ratsional funksiya*) deyiladi. Bunda  $a_0 \neq 0$ ,  $a_1, \dots, a_n$  sonlari ko'phadning koeffitsiyentlari deb ataladi, manfiy bo'lmagan butun *n* soni ko'phadning daraja ko'rsatkichini bildiradi.

Xususan,  $n=0$  da  $P_0(x) = a_0$  ( $a_0 \neq 0$ ) nolinchi darajali ko'phad hosil bo'ladi.

Agar  $P_n(x)$  va  $Q_n(x)$  ko'phadlar  $x$  ning barcha qiymatlarida bir xil qiymatlar qabul qilsa, u holda  $P_n(x)$  va  $Q_n(x)$  ko'phadlarda  $x$  ning bir xil darajalari oldida turgan koeffitsiyentlar teng bo'ladi va aksincha. Bu ko'phadlarga teng ko'phadlar deyiladi va  $P_n(x) = Q_n(x)$  deb yoziladi.

Ko'phadlar ustida qo'shish, ayirish va ko'paytirish amallarini bajarish mumkin.

Ikki ko'phadning yig'indisi, ayirmasi va ko'paytmasi yana ko'phad bo'ladi.

Ko'phadlarni qo'shish, ayirish va ko'paytirish amallari algebraik ifodalardagi kabi bajarilgani uchun arifmetik amallarning asosiy

xossalariga ega bo'ladi.

Ko'phadlarni bo'lish qoldikli va qoldiqsiz bo'lishi mumkin.

$P_n(x)$  va  $Q_m(x)$  ko'phadlar berilgan bo'lsin, bunda  $n \geq m > 0$ .

U holda

$$P_n(x) = Q_m(x) \cdot R_{n-m}(x) + r_k(x), \quad 0 \leq k < m \quad (2.2)$$

tenglikni qanoatlantiruvchi  $R_{n-m}(x)$  va  $r_k(x)$  ko'phadlar mavjud va yagona bo'ladi. Bunda  $P_n(x)$  bo'linuvchi,  $Q_m(x)$  bo'luvchi,  $R_{n-m}(x)$  bo'linma,  $r_k(x)$  qoldiq deb ataladi.

(2.2) tenglikda  $r_k(x) \equiv 0$  bo'lishi ham mumkin. U holda  $P_n(x)$  ko'phad  $R_{n-m}(x)$  ko'phadga qoldiqsiz bo'linadi deyiladi.

Ko'phadlarni bo'lishdan hosil bo'ladigan bo'linma va qoldiqni topishning har xil usullari mavjud. Ko'p hollarda «burchakli usulida bo'lish» qoidasidan foydalaniladi.

### 6.2.2. Ko'phadlarning ildizi

$P_n(x)$  ko'phadning ildizi deb  $x$  o'zgaruvchining bu ko'phadning qiymatini nolga aylantiradigan  $x_0$  (haqiqiy yoki kompleks) qiymatiga aytiladi, ya'ni bunda  $P_n(x_0) = 0$  bo'ladi.

$P_n(x)$  ko'phadni  $x - \alpha$  ga bo'lishdan hosil bo'ladigan qoldiqni bo'lish jarayonini bajarmasdan topish imkonini beradigan teoremani isbotlaymiz.

**1-teorema.**  $P_n(x)$  ko'phadni  $x - \alpha$  ikkihadga bo'lishdan hosil bo'ladigan qoldiq  $P_n(\alpha)$  ga teng bo'ladi.

*Isboti.*  $P_n(x)$  ko'phadni  $x - \alpha$  ikkihadga bo'lish natijasi

$$P_n(x) = (x - \alpha) \cdot R_{n-1}(x) + r,$$

bo'lsin, bu yerda  $r$  biror o'zgarimas son (0- darajali ko'phad) bo'ladi.

Bu tenglikda  $x$  o'zgaruvchiga  $\alpha$  qiymat berib, topamiz:

$$r = P_n(\alpha).$$

Teorema isbotlandi.

Bu teoremadan quyidagi teorema kelib chiqadi.

**2-teorema (Bezu teoremasi).**  $\alpha$  son  $P_n(x)$  ko'phadning ildizi bo'lishi uchun  $P_n(x)$  ko'phad  $x - \alpha$  ikkihadga qoldiqsiz bo'linishi zarur va yetarli.

ildiz  $k$  marta uchrashi mumkin. U holda bu ildizga  $k$  karrali ildiz deyiladi.  $k=1$  bo'lganida ildiz *oddiy ildiz* deb ataladi.

Agar (2.1) ko'phad  $k_1$  karrali  $\alpha_1$  ildizga,  $k_2$  karrali  $\alpha_2$  ildizga va umuman  $k_r$  karrali  $\alpha_r$  ildizga ega bo'lsa, (2.3) formula

$$P_n(x) = a_0(x - \alpha_1)^{k_1}(x - \alpha_2)^{k_2} \dots (x - \alpha_r)^{k_r} \quad (2.4)$$

ko'rinishni oladi, bu yerda  $k_1 + k_2 + \dots + k_r = n$ .

(2.3) tenglikda  $\alpha_1, \alpha_2, \dots, \alpha_n$  ildizlar orasida komplekslari bo'lishi ham mumkin. Kompleks ildizlar uchun o'rinli bo'ladigan teoremani isbotsiz keltiramiz.

**5-teorema.** Agar haqiqiy koeffitsiyentli  $P_n(x)$  ko'phad  $a + ib$  kompleks ildizga ega bo'lsa, u holda u  $a - ib$  qo'shma ildizga ham ega bo'ladi.

Bu teoremaga ko'ra, ko'phadning (2.3) yoyilmasida kompleks ildizlar o'z qo'shma juftlari bilan qatnashadi. Bu juftga mos chiziqli ko'paytuvchilarni ko'paytiramiz:

$$\begin{aligned} (x - (a + ib))(x - (a - ib)) &= ((x - a) - ib)((x - a) + ib) = (x - a)^2 + b^2 = \\ &= x^2 - 2ax + a^2 + b^2 = x^2 + px + q, \end{aligned}$$

bu yerda  $p = -2a$ ,  $q = a^2 + b^2$ .

Demak, qo'shma ildizlarga mos chiziqli ko'paytuvchilar ko'paytmasini haqiqiy koeffitsiyentli, diskriminanti manfiy bo'lgan kvadrat uchhad bilan almashtirish mumkin.

Shu kabi

$$(x - (a + ib))^k (x - (a - ib))^k = ((x - (a + ib))(x - (a - ib)))^k = (x^2 + px + q)^k$$

almashtirish bajarish mumkin.

Shunday qilib, yuqorida keltirilganlar asosida quyidagi tasdiqni hosil qilamiz.

**6-teorema.** Har qanday haqiqiy koeffitsiyentli ko'phad haqiqiy koeffitsiyentli chiziqli va kvadrat uchhadlardan iborat ko'paytuvchilarga ajratiladi, ya'ni  $P_n(x)$  ko'phadni

$$\begin{aligned} P_n(x) &= a_0(x - \alpha_1)^{k_1}(x - \alpha_2)^{k_2} \dots (x - \alpha_r)^{k_r} \times \\ &\times (x^2 + p_1x + q_1)^{s_1}(x^2 + p_2x + q_2)^{s_2} \dots (x^2 + p_lx + q_l)^{s_l} \end{aligned} \quad (2.5)$$

ko'rinishda ifodalash mumkin. Bunda  $\alpha_0$  - ko'phadning bosh



ko'effitsiyenti,  $\alpha_1, \alpha_2, \dots, \alpha_n$  – ko'phadning mos ravishda  $k_1, k_2, \dots, k_l$  karrali ildizlari,  $x^2 + p_i x + q_i$  ( $i = \overline{1, l}$ ) kvadrat uchhadlar uchun diskriminant  $D = p^2 - 4q < 0$ ,  $k_1 + k_2 + \dots + k_r + 2s_1 + 2s_2 + \dots + 2s_l = n$ ;  $r, l, k_1, k_2, \dots, k_r, s_1, s_2, \dots, s_l$  – natural sonlar.

2-misol.  $P_4(x) = x^4 + 3x^3 - x - 3$  ko'phadni ko'paytuvchilarga ajrating.

*Yechish.*  $P_4(x) = x^4 + 3x^3 - x - 3 = x^3(x+3) - (x+3) =$   
 $= (x+3)(x^3 - 1) = (x+3)(x-1)(x^2 + x + 1).$

### 6.2.4. Ratsional kasrlarni sodda kasrlarga yoyish

Ikkita  $Q_m(x)$  va  $P_n(x)$  ko'phadning nisbatiga

$$R(x) = \frac{Q_m(x)}{P_n(x)} = \frac{b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m}{a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n}$$

ratsional funksiya (ratsional kasr) deyiladi.

$m < n$  bo'lganda ratsional kasr to'g'ri kasr,  $m \geq n$  bo'lganda noto'g'ri kasr deyiladi.

Noto'g'ri kasrda uning  $Q_m(x)$  suratini  $P_n(x)$  maxrajiga odatdagidek bo'lish yo'li bilan kasrdan butun qismi  $q(x)$  ajratiladi, ya'ni

$$R(x) = \frac{Q_m(x)}{P_n(x)} = q(x) + \frac{r(x)}{P_n(x)}$$

tenglik hosil qilinadi, bu yerda  $q(x)$  – butun qism deb ataluvchi ko'phad,  $\frac{r(x)}{P_n(x)}$  – to'g'ri kasr, chunki  $r(x)$  qoldiqning darajasi  $P_n(x)$  ning darajasidan kichik.

3- misol.  $R(x) = \frac{3x^4 - 2x^3 + 1}{x^3 + 2x + 2}$  ratsional kasrdan butun qismini ajrating.

*Yechish.* Ko'phadlarni bo'lish qoidasi bo'yicha kasrning suratini maxrajiga bo'lamiz:

$$\begin{array}{r} 3x^4 - 2x^3 + 1 \quad \Big| \quad x^3 + 2x + 2 \\ 3x^4 + 6x^3 + 6x^2 \quad \Big| \quad 3x^2 - 8x + 10 \\ \hline -8x^3 - 6x^2 + 1 \\ -8x^3 - 16x^2 - 16x \\ \hline 10x^2 + 16x + 1 \\ -10x^2 + 20x + 20 \\ \hline -4x - 19 \end{array}$$

Demak,

$$R(x) = 3x^2 - 8x + 10 + \frac{-4x - 19}{x^2 + 2x + 2}.$$

Quyidagi to'g'ri kasrlarga *sodda (elementar) kasrlar* deyiladi:

- I.  $\frac{A}{x - \alpha}$ ;
- II.  $\frac{A}{(x - \alpha)^k}$ , ( $k \geq 2$ ,  $k \in \mathbb{N}$ );
- III.  $\frac{Mx + N}{x^2 + px + q}$ , ( $p^2 - 4q < 0$ );
- IV.  $\frac{Mx + N}{(x^2 + px + q)^s}$ , ( $s \geq 2$ ,  $s \in \mathbb{N}$ ,  $p^2 - 4q < 0$ ),

bu yerda  $A$ ,  $M$ ,  $N$ ,  $\alpha$ ,  $p$ ,  $q$  - haqiqiy sonlar.

**7-teorema.** Maxraji (2.5) ko'rinishda ko'paytuvchilarga ajratilgan har qanday  $\frac{Q_n(x)}{P_n(x)}$  to'g'ri kasrni sodda kasrlar yig'indisiga yagona tarzda yoyish mumkin.

Bunda:

1) (2.5) ifodaning  $(x - \alpha)$  ko'rinishdagi ko'paytuvchisiga I turdagi  $\frac{A}{x - \alpha}$  kasr mos keladi;

2) (2.5) ifodaning  $(x - \alpha)^k$  ko'rinishdagi ko'paytuvchisiga I va II turdagi  $k$  ta kasrlar yig'indisi  $\frac{A_1}{x - \alpha} + \frac{A_2}{(x - \alpha)^2} + \dots + \frac{A_k}{(x - \alpha)^k}$  mos keladi;

3) (2.5) ifodaning  $x^2 + px + q$  ko'rinishdagi ko'paytuvchisiga

III turdagi  $\frac{Mx + N}{x^2 + px + q}$  kasr mos keladi;

4) (2.5) ifodaning  $(x^2 + px + q)^s$  ko'rinishdagi ko'paytuvchisiga III va IV turdagi  $s$  ta kasrlar yig'indisi

$$\frac{M_1x + N_1}{x^2 + px + q} + \frac{M_2x + N_2}{(x^2 + px + q)^2} + \dots + \frac{M_sx + N_s}{(x^2 + px + q)^s} \quad \text{mos keladi.}$$

Shunday qilib, teoreмага ko'ra,

$$\frac{Q_n(x)}{P_n(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \dots + \frac{A_k}{(x-\alpha)^k} + \dots + \frac{M_1x + N_1}{x^2 + px + q} + \frac{M_2x + N_2}{(x^2 + px + q)^2} + \dots + \frac{M_sx + N_s}{(x^2 + px + q)^s}, \quad (2.6)$$

bu yerda  $A_1, A_2, \dots, A_k, M_1, N_1, M_2, N_2, \dots, M_s, N_s$  – koeffitsiyentlar.

(2.6) tenglikdagi noma'lum koeffitsiyentlarini topishning turli usullari mavjud. *Masalan*, noma'lum koeffitsiyentlarni topishda *koeffitsiyentlarni tenglashtirish* usulini qo'llash mumkin.

Bu usul quyidagi tartibda amalga oshiriladi:

1. (2.6) yoyilmaning o'ng tomoni umumiy maxraj  $P_n(x)$  ga keltiriladi. Natijada  $\frac{Q_n(x)}{P_n(x)} = \frac{S_m(x)}{P_n(x)}$  ayniyat hosil bo'ladi, bu yerda

$S_m(x)$  – koeffitsiyentlari

noma'lum bo'lgan ko'phad.

2. Hosil bo'lgan ayniyatda maxrajlar teng bo'lgani uchun, suratlar ham aynan teng bo'ladi, ya'ni  $Q_n(x) = S_m(x)$ .

3.  $Q_n(x) = S_m(x)$  tenglikda  $x$  ning bir xil darajalari oldidagi koeffitsiyentlar tenglashtiriladi (ko'phadlarning aynan tengligi haqidagi teoreмага ko'ra);

Natijada tenglamalar sistemasi hosil bo'ladi va bu sistemadan izlanayotgan  $A_1, A_2, \dots, A_k, M_1, N_1, M_2, N_2, \dots, M_s, N_s$  koeffitsiyentlar topiladi.

4- misol.  $R(x) = \frac{x^4 - 2x + 1}{x^2(x^3 + 1)}$  to'g'ri kasrni oddiy kasrlar yig'indisiga yoying.

*Yechish.*  $R(x)$  ning maxrajini ko'paytuvchilarga ajratamiz:

$$x^2(x^3 + 1) = x^2(x + 1)(x^2 - x + 1)$$

$R(x)$ ni 1-teoreмага asosan, sodda kasrlar yig'indisiga yoyamiz:

$$R(x) = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A}{x+1} + \frac{Mx + N}{x^2 - x + 1}.$$

Noma'lum koeffitsiyentlarini koeffitsiyentlarni tenglashtirish usuli bilan topamiz. Buning uchun tenglikning o'ng qismini umumiy

maxrajga keltiramiz, hosil bo'lgan tenglikning har ikkala tomonidagi maxrajlarni tashlab yuboramiz va quyidagi tenglikni hosil qilamiz:

$$x^4 - 2x + 1 = A_1x(x^3+1) + A_2(x^3+1) + Ax^2(x^2-x+1) + Mx^3(x+1) + Nx^2(x+1).$$

$x$  ning bir xil darajalari oldidagi koeffitsiyentlarni tenglashtiramiz:

$$\begin{cases} x^4: A + A_1 + M = 1, \\ x^3: -A + A_2 + M + N = 0, \\ x^2: A + N = 0, \\ x^1: A_1 = -2, \\ x^0: A_2 = 1. \end{cases}$$

Bu tenglamalar sistemasini yechamiz:

$$A = \frac{4}{3}, \quad A_1 = -2, \quad A_2 = 1, \quad M = \frac{5}{3}, \quad N = -\frac{4}{3}.$$

Demak,

$$R(x) = -\frac{2}{x} + \frac{1}{x^2} + \frac{4}{3(x+1)} + \frac{5x-4}{3(x^2-x+1)}.$$

## 6.2.5. Mashqlar

1. Berilgan  $P(x)$  va  $Q(x)$  ko'phadlarning yig'indisini, ayirmasini va ko'paytmasini toping:

1)  $P(x) = 2x^3 - x^2 + 4, \quad Q(x) = x^5 + 3x^3 - x;$

2)  $P(x) = x^4 + x^2 - 5, \quad Q(x) = 3x^4 + x^3 - x^2.$

2.  $P(x)$  ko'phadni  $Q(x)$  ko'phadga bo'lishda hosil bo'ladigan bo'linma va qoldiqni toping:

1)  $P(x) = x^3 + 2x^2 - x + 1, \quad Q(x) = x^2 + x - 1;$

2)  $P(x) = x^4 + 5x^3 - 6x + 5, \quad Q(x) = x^3 + 2x^2 - 1;$

3)  $P(x) = 3x^5 + 4x^3 + 2x - 1, \quad Q(x) = x^3 + 3x + 7;$

4)  $P(x) = 2x^4 - 13x^3 + 32x^2 - 24x + 1, \quad Q(x) = x^2 - 5x + 6.$

3.  $P(x)$  ko'phadni  $x - \alpha$  ikkihadga bo'lganda hosil bo'ladigan qoldiqni toping:

1)  $P(x) = x^7 - 3x^4 - x^3 + 1, \quad x - 2;$

2)  $P(x) = x^{11} - 6x^7 + x^5 - 8, \quad x + 1;$

3)  $P(x) = 3x^6 + x^5 - 64, \quad x + 2;$

4)  $P(x) = x^5 - 6x^3 + x, \quad x - 3.$

4.  $P(x) = ax^3 + bx^2 + x - 1$  va  $Q(x) = 3x^3 - x^2 + x - c$  ko'phadlar bit-biriga aynan teng.  $a$ ,  $b$ ,  $c$  larni toping.

5.  $ax^4 + x^3 - 3x + b = 2x^4 + cx^3 - 3x + 1$ .  $a$ ,  $b$ ,  $c$  larni toping.

6. Berilgan ko'phadlarni ko'paytuvchilarga ajrating:

1)  $x^4 - 16$ ;

2)  $x^3 - 81x$ ;

3)  $5x^4 - 40x^3 + 115x^2 - 140x + 60$ ;

4)  $x^5 - 6x^4 + 9x^3 - x^2 + 6x - 9$ .

7. Berilgan to'g'ri kasrlarni sodda kasrlar yig'indisiga yoying:

1)  $\frac{x^2 + 4x + 1}{x^3 + x^2}$ ;

2)  $\frac{3x^3 - 5x^2 + 8x - 4}{x^4 + 4x^2}$ ;

3)  $\frac{3x - 2}{x^3 + x^2 - 2x}$ ;

4)  $\frac{x^2 + 5x + 1}{x^4 + x^2 + 1}$ .

8. Berilgan to'g'ri kasrlarni sodda kasrlar yig'indisiga yoying va koeffitsiyentlarni noma'lum koeffitsiyentlarni tenglashtirish usuli bilan toping:

1)  $\frac{x^2 + 2x + 3}{x^4 + x^3}$ ;

2)  $\frac{2x^2 - 11x - 6}{x^3 + x^2 - 6x}$ ;

3)  $\frac{3x^3 - 2x^2 - 2x + 7}{x^4 - x^2}$ ;

4)  $\frac{2x - 1}{x^4 + x}$ .

- Aniqmas integral
- Ratsional funksiyalarni integrallash
- Trigonometrik funksiyalarni integrallash
- Irratsional ifodalarni integrallash
- Aniq integral
- Xosmas integrallar
- Aniq integralning tatbiqlari



*Isaak Nyuton*  
(1642–1727) –

ingliz matematigi, fizigi,  
mexanigi va astronomi

*Nyuton "Natural falsafa-  
ning matematik asos-  
lari" asarida dunyo olam  
tortishish qonunini va  
mexanikaning uch qonu-  
nini hayon qilgan.*

*U matematik analiz  
asoslarini, Nyuton binomi  
formulasini, tenglamalar-  
ni yechishning Nyuton  
usulini yaratgan, mate-  
matik hisoblashlarda qo-  
torlardan foydalanishni  
ko'rsatgan.*

## BIR O'ZGARUVCHI FUNKSIYASINING INTEGRAL HISOBI

*Integral hisob* – bu matematikaning integral tushunchasi, uning xossalari va hisoblash usullari o'rganiladigan bo'limidir. Integral hisob differensial hisob bilan uzviy bog'liq va ular birgalikda matematik analizning asosini tashkil qiladi.

Differensial va integral hisobning asosiy tushunchalari, jumladan, differensiallash va integrallash amallari o'rtasidagi bog'lanish, ularning amaliy masalalarni yechishga tatbiqi XVII asrning ikkinchi yarmida *I.Nyuton* va *G.Leybnis* tomonidan ishlab chiqilgan. Ularning izlanishlari matematik analizning gurkirab rivojlanish davrini boshlab bergan.

Integral hisob yordamida ko'plab yangi masalalar bilan bir qatordam ilgari yechilmagan bir qancha nazariy va amaliy masalalarni yechish imkoni yaratilgan.

Integral hisobning asosiy tushunchalari bir-biri bilan uzviy bog'liq aniqmas va aniq integrallardir.

### 7.1. ANIQMAS INTEGRAL

#### 7.1.1. Bosqlang'ich funksiya va aniqmas integral

Berilgan funksiyaning hosilasini topish differensial hisobning asosiy masalalaridan biri hisoblanadi. Matematik analiz masalalarining turliligi, uning geometriya,

mexanika, fizika va texnikadagi keng miqyosdagi tatbiqi berilgan  $f(x)$  funksiya uchun hosilasi shu funksiyaga teng bo'lgan  $F(x)$  funksiyani topishga olib keladi.

Funksiyaning berilgan hosilasiga ko'ra, uning o'zini topish masalasi integral hisobning asosiy masalaridan biri hisoblanadi.

$y = f(x)$  funksiya  $(a;b)$  intervalda aniqlangan bo'lsin.

**1-ta'rif.** Agar  $(a;b)$  intervalda differensiallanuvchi  $F(x)$  funksiyaning hosilasi berilgan  $f(x)$  funksiyaga teng, ya'ni

$$F'(x) = f(x) \quad (\text{yoki } dF(x) = f(x)dx), \quad x \in (a;b)$$

bo'lsa,  $F(x)$  funksiyaga  $(a;b)$  intervalda  $f(x)$  funksiyaning *boshlang'ich funksiyasi* deyiladi.

Masalan: 1)  $F(x) = x^3$  funksiya sonlar o'qida  $f(x) = 3x^2$  funksiyaning boshlang'ich funksiyasi bo'ladi, chunki  $x \in R$  da  $(x^3)' = 3x^2$ ;

2)  $F(x) = \sqrt{1-x^2}$  funksiya  $(-1;1)$  intervalda  $f(x) = -\frac{x}{\sqrt{1-x^2}}$  funksiyaning

boshlang'ich funksiyasi bo'ladi, chunki  $x \in (-1;1)$  da  $(\sqrt{1-x^2})' = -\frac{x}{\sqrt{1-x^2}}$ .

**Lemma.** Agar  $F(x)$  va  $\Phi(x)$  funksiyalar  $(a;b)$  intervalda  $f(x)$  funksiyaning boshlang'ich funksiyalari bo'lsa, u holda  $F(x)$  va  $\Phi(x)$  bir-biridan o'zgarmas songa farq qiladi.

*Isboti.*  $F(x)$  va  $\Phi(x)$  funksiyalar  $(a;b)$  intervalda  $f(x)$  funksiyaning boshlang'ich funksiyalari bo'lsin:  $F'(x) = f(x)$ ,  $\Phi'(x) = f(x)$ .

U holda istalgan  $x \in (a;b)$  da

$$(\Phi(x) - F(x))' = \Phi'(x) - F'(x) = f(x) - f(x) = 0$$

bo'ladi.

Bundan  $\Phi(x) - F(x) = C$  yoki  $\Phi(x) = F(x) + C$  kelib chiqadi, bu yerda  $C$  - ixtiyoriy o'zgarmas son.

Shunday qilib,  $f(x)$  funksiya  $(a;b)$  intervalda biror  $F(x)$  boshlang'ich funksiyaga ega bo'lsa, uning qolgan barcha boshlang'ich funksiyalari  $\{F(x) + C \mid C \in R\}$  to'plamni tashkil qiladi.

**2-ta'rif.**  $f(x)$  funksiyaning  $(a;b)$  intervaldagi boshlang'ich funksiyalari to'plamiga  $f(x)$  funksiyaning *aniqmas integrali* deyiladi va  $\int f(x)dx$  kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\int f(x)dx = F(x) + C, \quad (1.1)$$

bu yerda  $f(x)$  — integral ostidagi funksiya,  $f(x)dx$  — integral ostidagi ifoda;  $x$  — integrallash o'zgaruvchisi,  $\int$  — integrallash belgisi deyiladi.

Aniqmas integralni topish, ya'ni berilgan funksiyaning boshlang'ich funksiyalari to'plamini aniqlash masalasi *funksiyani integrallash* deb yuritiladi.

Demak, funksiyaning integrallash amali funksiyaning differensiallashga teskari amal bo'ladi.

Berilgan  $f(x)$  funksiya qachon boshlang'ich funksiya ega bo'ladi degan savolga quyidagi teorema javob beradi (teoremani isbotsiz keltiramiz).

**1-teorema.** Agar  $f(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lsa u holda u bu kesmada uzluksiz bo'lgan boshlang'ich funksiya ega bo'ladi.

Ko'p hollarda  $F(x)$  funksiya  $f(x)$  funksiyaning boshlang'ich funksiyasi bo'ladigan  $(a; b)$  interval ko'rsatilmaydi. Bunday holda  $(a; b)$  interval sifatida  $f(x)$  funksiyaning aniqlanish sohasi tushuniladi. Shu sababli bundan keyin integral ostidagi funksiyalar uzluksiz va (1.1) formula ma'noga ega deb hisoblaymiz.

Masalan,  $f(x) = \frac{1}{x}$  funksiya  $(-\infty; 0)$  va  $(0; \infty)$  intervalda uzluksiz.

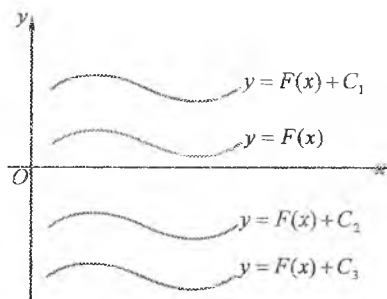
Shu sababli uning aniqmas integrali deb

$$\int \frac{dx}{x} = \begin{cases} \ln x + C, & x > 0, \\ \ln(-x) + C, & x < 0 \end{cases} = \ln|x| + C \quad (x \neq 0)$$

funksiya tushuniladi.

Boshlang'ich funksiyaning grafigi *integral egri chiziq* deb ataladi.

Aniqmas integral *geometrik jihatdan* ixtiyoriy  $C$  o'zgaruvchiga bog'liq bo'lgan barcha integral egri chiziqlar to'plamini ifodalaydi. Agar  $F(x)$  funksiyaning grafigi integral egri chiziq bo'lsa, boshqa



1-shakl.



integral egri chiziqlar uni  $Oy$  o'qi bo'yicha parallel ko'chirish yordamida hosil qilinadi (1-shakl).

### 7.1.2. Aniqmas integralning xossalari

Aniqmas integral quyidagi xossalarga ega.

1°. Aniqmas integralning hosilasi (differensial) integral ostidagi funksiyaga (ifodaga) teng:

$$(\int f(x)dx)' = f(x). \quad (d\int f(x)dx = f(x)dx).$$

*Isboti.*  $F(x)$  funksiya  $f(x)$  funksiyaning boshlang'ich funksiyasi, ya'ni  $F'(x) = f(x)$  bo'lsin.

U holda

$$(\int f(x)dx)' = (F(x) + C)' = F'(x) + 0 = f(x).$$

$$(d\int f(x)dx = d(F(x) + C) = dF(x) + dC = F'(x)dx = f(x)dx).$$

Bu xossa *integral amali to'g'ri bajarilganligini differensiallash orqali tekshirish* imkonini beradi.

Masalan,  $\int(3x^2 + 5)dx = x^3 + 5x + C$  to'g'ri, chunki  $(x^3 + 5x + C)' = 3x^2 + 5$ .

2°. Funksiya differensialining aniqmas integrali shu funksiya bilan o'zgarmas sonning yig'indisiga teng:

$$\int dF(x) = F(x) + C.$$

*Isboti.*  $F'(x) = f(x)$  bo'lsin.

U holda

$$\int dF(x) = \int F'(x)dx = \int f(x)dx = F(x) + C.$$

3°. Ozgarmas ko'paytuvchini aniqmas integral belgisidan tashqariga chiqarish mumkin:

$$\int kf(x)dx = k\int f(x)dx, \quad k = \text{const}, k \neq 0.$$

*Isboti.*  $F'(x) = f(x)$  bo'lsin.

Bundan

$$\int kf(x)dx = \int kF'(x)dx = \int (kF(x))'dx = k(F(x) + C) = kF(x) + C_1 = k\int f(x)dx$$

( $C_1 = kC$  deb olindi).

4°. Chekli sondagi funksiyalar algebraik yig'indisining aniqmas integrali shu funksiyalar aniqmas integrallarining algebraik yig'indisiga teng:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$$

*Isboti.*  $F'(x) = f(x)$ ,  $G'(x) = g(x)$  bo'lsin.

U holda

$$\begin{aligned} \int (f(x) \pm g(x))dx &= \int (F'(x) \pm G'(x))dx = \\ &= \int (F(x) \pm G(x))'dx = \int d(F(x) \pm G(x)) = \\ &= F(x) \pm G(x) + C = (F(x) + C_1) \pm (G(x) + C_2) = \\ &= \int f(x)dx \pm \int g(x)dx, \quad C_1 \pm C_2 = C. \end{aligned}$$

5°. Agar  $\int f(x)dx = F(x) + C$  bo'lsa, u holda  $x$  ning istalgan differensiallanuvchi funksiyasi  $u = u(x)$  uchun  $\int f(u)du = F(u) + C$  bo'ladi.

*Isboti.*  $x$  erkli o'zgaruvchi,  $f(x)$  uzluksiz funksiya,  $F(x)$  funksiya  $f(x)$  funksiyaning boshlang'ich funksiyasi bo'lsin.

U holda  $\int f(x)dx = F(x) + C$  bo'ladi.

$u = \varphi(x)$  bo'lsin, bu yerda  $\varphi(x)$  – uzluksiz hosilaga ega bo'lgan funksiya.

Birinchi differensialning invariantlik xossasiga ko'ra,

$$dF(u) = F'(u)du = f(u)du$$

bo'ladi.

Bundan

$$\int f(u)du = \int d(F(u)) = F(u) + C.$$

Bu xossa integrallash formulasining invariantligi xossasi deyiladi. Demak, aniqmas integral integrallash o'zgaruvchisi erkli o'zgaruvchi yoki erkli o'zgaruvchining uzluksiz hosilaga ega bo'lgan ixtiyoriy funksiyasi bo'lishidan qat'iy nazar bir xil formula bilan topiladi.

### 7.1.3. Asosiy integrallar jadvali

Integrallash amali differensiallash amaliga teskari amal bo'lgani uchun asosiy integrallar jadvalini differensial hisobning mos formulalarini (differensiallar jadvalini) qo'llash va aniqmas integralning xossaligidan foydalanish orqali hosil qilish mumkin.

Masalan,  $d(\sin u) = \cos u du$  ekanidan  $\int \cos u du = \int d(\sin u) = \sin u + C$ .  
 Quyida keltiriladigan integrallar asosiy integrallar jadvali deyiladi.

*Asosiy integrallar jadvali.*

- |  |   |
|--|---|
| 1. $\int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1} + C, (\alpha \neq -1);$         | 2. $\int \frac{du}{u} = \ln  u  + C;$   |
| 3. $\int a^u du = \frac{a^u}{\ln a} + C, (0 < a \neq 1);$                            | 4. $\int e^u du = e^u + C;$   |
| 5. $\int \sin u du = -\cos u + C;$   | 6. $\int \cos u du = \sin u + C;$   |
| 7. $\int \operatorname{tg} u du = -\ln  \cos u  + C;$                                | 8. $\int \operatorname{ctg} u du = \ln  \sin u  + C;$   |
| 9. $\int \frac{du}{\cos^2 u} = \operatorname{tg} u + C;$                             | 10. $\int \frac{du}{\sin^2 u} = -\operatorname{ctg} u + C;$   |
| 11. $\int \frac{du}{\sin u} = \ln \left  \operatorname{tg} \frac{u}{2} \right  + C;$ | 12. $\int \frac{du}{\cos u} = \ln \left  \operatorname{tg} \left( \frac{u}{2} + \frac{\pi}{4} \right) \right  + C;$ |
| 13. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C;$                    | 14. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln \left  u + \sqrt{u^2 \pm a^2} \right  + C.$                           |
| 15. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \operatorname{arctg} \frac{u}{a} + C;$  | 16. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left  \frac{u-a}{u+a} \right  + C;$                              |
| 17. $\int \operatorname{sh} u du = \operatorname{ch} u + C;$                         | 18. $\int \operatorname{ch} u du = \operatorname{sh} u + C;$  |
| 19. $\int \frac{du}{\operatorname{ch}^2 u} = \operatorname{th} u + C;$               | 20. $\int \frac{du}{\operatorname{sh}^2 u} = -\operatorname{cth} u + C.$  |

Asosiy integrallar jadvalida integrallash o'zgaruvchisi  $u$  erkli o'zgaruvchi yoki erkli o'zgaruvchining funksiyasi (5- xossaga ko'ra) bo'lishi mumkin.

Jadvalda keltirilgan formulalarning to'g'riligiga uning o'ng tomonini differensiallash va bu differensialning formula chap tomonidagi integral ostidagi ifodaga teng bo'lishini tekshirish orqali ishonch hosil qilish mumkin.

Bu integrallardan birining, masalan 13-formulaning to'g'riligini ko'rsatamiz:

$$d\left(\arcsin \frac{u}{a} + C\right) = \frac{1}{\sqrt{1 - \left(\frac{u}{a}\right)^2}} \cdot \frac{1}{a} du = \frac{du}{\sqrt{a^2 - u^2}}.$$

## 7.1.4. Integrallash usullari

### Bevosita integrallash usuli

Integral ostidagi funksiyada (yoki ifodada) almashtirishlar bajarish va aniqmas integrallarning xossalarini qo'llash orqali berilgan integralni bir yoki bir nechta jadval integraliga keltirib integrallash usuliga *bevosita integrallash usuli* deyiladi.

*Misollar:*

$$\begin{aligned} 1) \int \left( 5 \sin x - \frac{2}{x^2 + 1} + x^3 \right) dx &= 5 \int \sin x dx - 2 \int \frac{dx}{x^2 + 1} + \int x^3 dx = \\ &= -5 \cos x - 2 \operatorname{arctg} x + \frac{x^4}{4} + C; \end{aligned}$$

$$\begin{aligned} 2) \int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx &= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x} dx = \int \left( \frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx = \\ &= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = -\operatorname{ctg} x - \operatorname{tg} x + C = -\frac{2}{\sin 2x} + C; \end{aligned}$$

$$\begin{aligned} 3) \int \frac{x^4}{1+x^2} dx &= -\int \frac{1-x^4-1}{1+x^2} dx = -\int (1-x^2) dx + \int \frac{dx}{1+x^2} = \\ &= -\int dx + \int x^2 dx + \int \frac{dx}{1+x^2} = -x + \frac{x^3}{3} + \operatorname{arctg} x + C. \end{aligned}$$

Berilgan integralni jadval integrallariga keltirishda differensialning quyidagi almashtirishlari («differensial amali ostiga kiritish» jarayoni) qo'llaniladi:

$$du = d(u+a), \quad a - \text{son}; \quad du = \frac{1}{a} d(au); \quad udu = \frac{1}{2} d(u^2); \quad \cos u du = d(\sin u);$$

$$\sin u du = -d(\cos u); \quad \frac{1}{u} du = d(\ln u); \quad \frac{1}{\cos^2 u} du = d(\operatorname{tg} u).$$

Umuman olganda,  $f'(u)du = d(f(u))$ .

Bu formuladan integrallarni topishda ko'p foydalaniladi.

*Misollar:*

$$1) \int \frac{dx}{16+9x^2} = \frac{1}{3} \int \frac{d(3x)}{16+(3x)^2} = \frac{1}{3} \cdot \frac{1}{4} \operatorname{arctg} \frac{3x}{4} + C = \frac{1}{12} \operatorname{arctg} \frac{3x}{4} + C;$$

$$2) \int \frac{\cos x + \sin x}{\sin x - \cos x} dx = \int \frac{d(\sin x - \cos x)}{\sin x - \cos x} = \ln |\sin x - \cos x| + C.$$

### *O'rniga qo'yish (o'zgaruvchini almashtirish) usuli*

Ko'p hollarda integraldagi o'zgaruvchini almashtirish uni bevosita integrallashga olib keladi. Integrallashning bu usuli *o'rniga qo'yish (o'zgaruvchini almashtirish) usuli* deb yuritiladi. Bu usul quyidagi teorema asoslanadi.

**2-teorema.** Biror  $T$  oraliqda aniqlangan va differensiallanuvchi  $x = \varphi(t)$  funksiyaning qiymatlar sohasi  $X$  oraliqdan iborat bo'lib,  $X$  da  $f(x)$  funksiya aniqlangan va uzluksiz, ya'ni  $T$  oraliqda  $f(\varphi(t))$  murakkab funksiya aniqlangan va uzluksiz bo'lsin. U holda

$$\int f(x) dx = \int f(\varphi(t)) \varphi'(t) dt \quad (1.2)$$

bo'ladi.

*Isboti.*  $X$  oraliqda  $F(x)$  funksiya  $f(x)$  funksiyaning boshlang'ichi bo'lsin. U holda  $F(\varphi(t))$  murakkab funksiya  $T$  oraliqda aniqlangan, differensiallanuvchi hamda

$$F'_x(\varphi(t)) = F'_x(\varphi(t)) \varphi'(t) = f(\varphi(t)) \varphi'(t)$$

bo'ladi.

Bundan

$$\begin{aligned} \int f(\varphi(t)) \varphi'(t) dt &= \int F'_x(\varphi(t)) dt = F(\varphi(t)) + C = \\ &= F(x) + C \Big|_{x=\varphi(t)} = \int f(x) dx \Big|_{x=\varphi(t)}. \end{aligned}$$

hisobga olinsa,

$$\int f(x) dx = \int f(\varphi(t)) \varphi'(t) dt.$$

(1.2) formula *aniqmas integralda o'zgaruvchini almashtirish* formulasi deb yuritiladi.

Ayrim hollarda  $t = \varphi(x)$  o'rniga qo'yish bajarishga to'g'ri keladi. U holda  $\int f(\varphi(x)) \varphi'(x) dx = \int f(t) dt$  bo'ladi. Demak, (1.2) formula o'ngdan chapga qo'llanishi ham mumkin.

*1-misol.*  $\int x \sqrt{x-3} dx$  integralni toping.

*Yechish.*  $\sqrt{x-3} = t$  almashtirish bajaramiz.

U holda  $x = t^2 + 3$ ,  $dx = 2t dt$ .

Shu sababli

$$\begin{aligned} \int x\sqrt{x-3}dx &= \int (t^2+3)t \cdot 2tdt = 2\int (t^4+3t^2)dt = \\ &= 2\int t^4 dt + 6\int t^2 dt = 2 \cdot \frac{t^5}{5} + 6 \cdot \frac{t^3}{3} + C = \frac{2}{5}\sqrt{(x-3)^5} + 2\sqrt{(x-3)^3} + C. \end{aligned}$$

2- misol.  $\int \frac{\sqrt{1+\ln x}}{x \ln x} dx$  integralni toping.

*Yechish.*  $1 + \ln x = t^2$  bo'lsin.

Bundan

$$\ln x = t^2 - 1, \quad \frac{dx}{x} = 2tdt.$$

U holda (1.2) formulaga ko'ra,

$$\begin{aligned} \int \frac{\sqrt{1+\ln x}}{x \ln x} dx &= \int \frac{t \cdot 2tdt}{t^2-1} = 2\int \frac{t^2 dt}{t^2-1} = \\ &= 2\int \left(1 + \frac{1}{t^2-1}\right) dt = 2\left(t + \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| \right) + C = \\ &= 2t + \ln \left| \frac{(t-1)^2}{t^2-1} \right| + C = 2\sqrt{1+\ln x} + \ln \left| \frac{(\sqrt{1+\ln x})^2}{1+\ln x-1} \right| + C = \\ &= 2\sqrt{1+\ln x} + 2\ln |\sqrt{1+\ln x} - 1| - \ln |\ln x| + C. \end{aligned}$$

3- misol.  $\int \sqrt{1+\cos^2 x} \sin 2x dx$  integralni toping.

*Yechish.*  $1 + \cos^2 x = t^2$  deymiz.

Bundan

$$-2 \cos x \sin x dx = 2tdt \text{ yoki } \sin 2x dx = -2tdt.$$

U holda

$$\begin{aligned} \int \sqrt{1+\cos^2 x} \sin 2x dx &= \int t(-2t)dt = \\ &= -2 \cdot \frac{t^3}{3} + C = -\frac{2}{3} \sqrt{(1+\cos^2 x)^3} + C. \end{aligned}$$

*Izoh.* Ayrim hollarda integrallashning o'zgaruvchini almashtirish usuli takroran qo'llaniladi, ya'ni bunda bajarilgan o'rniga qo'yishdan so'ng shunday integral hosil bo'ladi, bu integralni boshqa o'rniga qo'yish orqali soddalashtirish yoki jadval integraliga keltirish mumkin bo'ladi.

### Bo'laklab integrallash usuli

Bo'laklab integrallash usuli ikki funktsiya ko'paytmasining differensial formulasi asoslanadi.

**3-teorema.**  $u(x)$  va  $v(x)$  funksiyalar qandaydir  $X$  oraliqda aniqlangan va differentsiallanuvchi bo'lib,  $u'(x)v(x)$  funktsiya bu oraliqda boshlang'ich funksiyaga ega, ya'ni  $\int u'(x)v(x)dx$  integral mavjud bo'lsin. U holda  $X$  oraliqda  $u(x)v'(x)$  funktsiya boshlang'ich funksiyaga ega va

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \quad (1.3)$$

bo'ladi.

*Isboti.*  $(u(x)v(x))' = u'(x)v(x) + v'(x)u(x)$  tenglikdan

$$u(x)v'(x) = (u(x)v(x))' - v(x)u'(x).$$

$(u(x)v(x))'$  va  $u'(x)v(x)$  funksiyalar  $X$  intervalda boshlang'ich funksiyaga ega bo'lgani uchun  $v'(x)u(x)$  ham  $X$  intervalda boshlang'ich funksiyaga ega bo'ladi. Oxirgi tenglikning chap va o'ng tomonlarini integrallasak, (1.3) formula kelib chiqadi.

(1.3) formulaga *aniqmas integralni bo'laklab integrallash* formulasi deyiladi.

Ma'lumki,

$$v'(x)dx = dv, \quad u'(x)dx = du.$$

Demak, (1.3) formulani

$$\int u dv = uv - \int v du \quad (1.4)$$

ko'rinishda yozish mumkin.

Bo'laklab integrallash usulining mohiyati berilgan integralda integral ostidagi  $f(x)dx$  ifodani  $u dv$  ko'paytma shaklida tasvirlash va (1.4) formulani qo'llagan holda berilgan  $\int u dv$  integralni oson integrallanadigan  $\int v du$  integral bilan almashtirib topishdan iborat.

Bo'laklab integrallash orqali topiladigan integrallarning, asosan, uchta guruhini ajratish mumkin:

1)  $\int P(x)\arctg x dx$ ,  $\int P(x)\arccot x dx$ ,  $\int P(x)\ln x dx$ ,  $\int P(x)\arcsin x dx$ ,  $\int P(x)\arccos x dx$  (bu yerda  $P(x)$  - ko'phad) ko'rinishdagi 1-guruh

integrallar. Bunda  $dv = P(x)dx$  deb, qolgan ko'paytuvchilar esa  $u$  bilan belgilanadi;

2)  $\int P(x)e^{kx} dx$ ,  $\int P(x)\sin kx dx$ ,  $\int P(x)\cos kx dx$  ko'rinishdagi 2-guruh integrallar. Bunda  $u = P(x)$  deb, qolgan ko'paytuvchilar  $dv$  deb olinadi;

3)  $\int e^{kx} \sin kx dx$ ,  $\int e^{kx} \cos kx dx$  ko'rinishdagi 3-guruh integrallar bo'laklab integrallash orqasini takroran qo'llash orqali topiladi.

4- misol.  $\int \arctg x dx$  integralni toping.

$$\begin{aligned} \text{Yechish. } \int \arctg x dx &= \left. \begin{array}{l} u = \arctg x, \quad du = \frac{dx}{1+x^2} \\ dv = dx, \quad v = x \end{array} \right| = x \arctg x - \int \frac{x}{1+x^2} dx = \\ &= x \arctg x - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2} dx = x \arctg x - \frac{1}{2} \ln|1+x^2| + C. \end{aligned}$$

5- misol.  $\int xe^x dx$  integralni toping.

Yechish.

$$\int xe^x dx = \left. \begin{array}{l} u = x, \quad du = dx \\ dv = e^x dx, \quad v = e^x \end{array} \right| = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C.$$

6- misol.  $I = \int \sin xe^{2x} dx$  integralni toping.

Yechish.

$$\begin{aligned} I = \int \sin xe^{2x} dx &= \left. \begin{array}{l} u = e^{2x}, \quad du = 2e^{2x} dx \\ dv = \sin x dx, \quad v = -\cos x \end{array} \right| = -e^{2x} \cos x + 2 \int e^{2x} \cos x dx = \\ &= \left. \begin{array}{l} u = e^{2x}, \quad du = 2e^{2x} dx \\ dv = \cos x dx, \quad v = \sin x \end{array} \right| = -e^{2x} \cos x + 2(2^{\sin x} \sin x - 2 \int e^{2x} \sin x dx) = \\ &= e^{2x} (2 \sin x - \cos x) - 4I. \end{aligned}$$

Bundan

$$I = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C.$$



Ko'rsatilgan uchta guruh bo'laklab integrallanadigan barcha integrallarni o'z ichiga olmaydi. Masalan,  $\int \frac{x dx}{\cos^2 x}$  integral yuqorida keltirilgan integrallar guruhlariga kirmaydi, lekin uni bo'laklab integrallash usuli bilan topish mumkin:

$$\int \frac{x dx}{\cos^2 x} = \left| \begin{array}{l} u = x, \quad du = dx \\ dv = \frac{dx}{\cos^2 x}, \quad v = \operatorname{tg} x \end{array} \right| = x \operatorname{tg} x - \int \operatorname{tg} x dx = x \operatorname{tg} x - \ln |\cos x| + C.$$

### 7.1.5. Mashqlar

1. Berilgan integrallarni aniqmas integralling xossalari va integrallar jadvalini qo'llab toping:

- |  |   |
|--|---|
| 1) $\int \left( 5 \operatorname{tg} x - \frac{2}{x^2 + 1} + x^4 \right) \cdot dx;$ | 2) $\int \frac{x^2 - 7}{x + 3} dx;$   |
| 3) $\int \frac{\sqrt[3]{x} - x^2 e^{-x} - x}{x^2} dx;$                             | 4) $\int \left( \frac{3}{1 + x^2} - \frac{2}{\sqrt{1 - x^2}} \right) \cdot dx;$ |
| 5) $\int \frac{2 \cdot 3^x - 3 \cdot 2^x}{3^x} dx;$                                | 6) $\int \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 dx;$              |
| 7) $\int e^x \left( 1 + \frac{e^{-x}}{\cos^2 x} \right) dx;$                       | 8) $\int \frac{1 - \sin^3 x}{\sin^2 x} dx;$                                     |
| 9) $\int \operatorname{ctg}^2 x dx;$   | 10) $\int \frac{dx}{\cos^2 x - \cos 2x};$                                       |
| 11) $\int \frac{dx}{25 + 4x^2};$   | 12) $\int \frac{dx}{\sqrt{3 + 4x - 2x^2}}.$                                     |

2. Berilgan integrallarni differensial ostiga kiritish usuli bilan toping:

- |  |   |
|--|---|
| 1) $\int \frac{\operatorname{tg} x}{\cos^2 x} dx;$                 | 2) $\int \cos^2 x \sin x dx;$                                   |
| 3) $\int \frac{\sqrt[3]{\operatorname{arctg}^5 2x}}{1 + 4x^2} dx;$ | 4) $\int \frac{\sqrt{\ln^3(x+5)}}{x+5} dx;$                     |
| 5) $\int e^{\sin x} \cos x dx;$                                    | 6) $\int e^{-x^2} x^2 dx;$                                      |
| 7) $\int \frac{\cos x}{\sin^2 x} dx;$                              | 8) $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx;$                    |
| 9) $\int \frac{e^x dx}{\sqrt{1 - e^{2x}}};$                        | 10) $\int \frac{dx}{\sin^2 4x \sqrt{\operatorname{ctg}^2 4x}}.$ |

3. Berilgan integrallarni o'rniga qo'yish usuli bilan toping:

$$1) \int \frac{e^x - 1}{e^x + 1} dx;$$

$$2) \int \frac{x^5 dx}{x^6 + 2}$$

$$3) \int \sqrt{16 - x^2} dx;$$

$$4) \int \frac{x^4 dx}{\sqrt{x^4 + 4}};$$

$$5) \int x^2 \sqrt{x^3 + 3} dx;$$

$$6) \int \frac{\cos 2x dx}{1 + \sin x \cos x};$$

$$7) \int \frac{dx}{(\arcsin x)^3 \sqrt{1 - x^2}};$$

$$8) \int \frac{4x - 5}{x^2 + 5} dx;$$

$$9) \int \frac{dx}{\sqrt{5 - 4x - x^2}};$$

$$10) \int \frac{dx}{\sqrt{3x^2 - 2x - 1}};$$

$$11) \int x(2x + 7)^{10} dx;$$

$$12) \int \frac{dx}{\sqrt{x(1-x)}};$$

$$13) \int \frac{e^{2x} dx}{e^{4x} - 9};$$

$$14) \int \frac{\ln 2x}{\ln 4x} \frac{dx}{x}.$$

4. Integrallarni bo'laklab integrallash usuli bilan toping:

$$1) \int x \arctg x dx$$

$$2) \int \arcsin x dx;$$

$$3) \int x \ln x dx;$$

$$4) \int x^2 e^x dx;$$

$$5) \int x 3^x dx;$$

$$6) \int x \sin 2x dx;$$

$$7) \int x \ln(x+1) dx;$$

$$8) \int \frac{x \sin x dx}{\cos^3 x};$$

$$9) \int \sin \ln x dx;$$

$$10) \int e^{4x} \sin 4x dx.$$

$$11) \int \frac{\ln \lg x dx}{\cos^2 x};$$

$$12) \int \frac{\ln \arctg x dx}{1 + x^2}.$$

5. Integrallarni toping:

$$1) \int x^3 \sqrt{1 + x^2} dx;$$

$$2) \int \sin 3x \sin 5x dx;$$

$$3) \int e^x \cos^2(e^x) dx;$$

$$4) \int \frac{x dx}{e^{3x}};$$

$$5) \int \frac{1 - \lg x}{1 + \lg x} dx;$$

$$6) \int \frac{\ln x dx}{x(1 - \ln^2 x)};$$

$$7) \int \frac{dx}{(x+1)(2x-3)};$$

$$8) \int \frac{dx}{x(4 + \ln^2 x)};$$

9)  $\int \frac{xdx}{\cos^2 x}$ ;

10)  $\int \frac{dx}{x\sqrt{2x-9}}$ ;

11)  $\int \frac{e^{arctg x} dx}{1+x^2}$ ;

12)  $\int \frac{e^{2x} dx}{\sqrt{3+e^{2x}}}$ ;

13)  $\int \sin^2 \frac{3x}{2} dx$ ;

14)  $\int xtg^2 x^2 dx$ ;

15)  $\int x^2 \ln^2 x dx$ ;

16)  $\int \frac{1-2\cos x}{\sin^2 x} dx$

## 7.2. RATSIONAL FUNKSIYALARNI INTEGRALLASH

### 7.2.1. Sodda kasrlarni integrallash

6.2. banddan bilamizki, quyidagi ratsional kasrlarga *sodda kasrlar* deyiladi:

I.  $\frac{A}{x-\alpha}$ ;

II.  $\frac{A}{(x-\alpha)^k}$ , ( $k \geq 2$ ,  $k \in \mathbb{N}$ );

III.  $\frac{Mx+N}{x^2+px+q}$ , ( $p^2-4q < 0$ );

IV.  $\frac{Mx+N}{(x^2+px+q)^r}$ , ( $s \geq 2$ ,  $s \in \mathbb{N}$ ,  $p^2-4q < 0$ ),

bu yerda  $A$ ,  $M$ ,  $N$ ,  $\alpha$ ,  $p$ ,  $q$  – haqiqiy sonlar.

I va II turdagi sodda kasrlar jadval integrallari orqali topiladi:

$$\int \frac{A dx}{x-\alpha} = A \int \frac{d(x-\alpha)}{x-\alpha} = A \ln|x-\alpha| + C; \quad (2.1)$$

$$\int \frac{A dx}{(x-\alpha)^k} = A \int (x-\alpha)^{-k} d(x-\alpha) = A \frac{(x-\alpha)^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-\alpha)^{k-1}} + C. \quad (2.2)$$

III turdagi sodda kasrni qaraymiz.

$$\int \frac{Mx+N}{x^2+px+q} dx \text{ integralining suratida kasrning maxrajidan olingan hosila}$$

$(x^2+px+q)' = 2x+p$  ni ajratamiz va natijani integrallaymiz:

$$\int \frac{Mx + N}{x^2 + px + q} dx = \int \frac{\frac{M}{2}(2x + p) + N - \frac{Mp}{2}}{x^2 + px + q} dx = \frac{M}{2} \int \frac{2x + p}{x^2 + px + q} dx + \left(N - \frac{Mp}{2}\right) \int \frac{dx}{x^2 + px + q} = \frac{M}{2} J_1 + \left(N - \frac{Mp}{2}\right) J_2$$

yoki

$$\int \frac{Mx + N}{x^2 + px + q} dx = \frac{M}{2} J_1 + \left(N - \frac{Mp}{2}\right) J_2.$$

Bu tenglikning o'ng tomonidagi integrallardan birinchisi

$$J_1 = \ln |x^2 + px + q|.$$

Ikkinchi integral maxrajida to'liq kvadrat ajratamiz va integralni quyidagicha hisoblaymiz:

$$J_2 = \int \frac{dx}{x^2 + px + q} = \int \frac{d\left(x + \frac{p}{2}\right)}{\left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}} = \frac{2}{\sqrt{4q - p^2}} \operatorname{arctg} \frac{2x + p}{\sqrt{4q - p^2}},$$

bunda  $4q - p^2 > 0$ , chunki  $D < 0$ .

Natijada quyidagiga ega bo'lamiz:

$$\int \frac{Mx + N}{x^2 + px + q} dx = \frac{M}{2} \ln |x^2 + px + q| + \frac{2N - Mp}{\sqrt{4q - p^2}} \operatorname{arctg} \frac{2x + p}{\sqrt{4q - p^2}} + C. \quad (2.3)$$

1- misol.  $I = \int \frac{5x + 11}{x^2 + 6x + 13} dx$  integralni toping.

$$\begin{aligned} \text{Yechish. } I &= \int \frac{\frac{5}{2}(2x + 6) + 11 - \frac{5}{2} \cdot 6}{x^2 + 6x + 13} dx = \frac{5}{2} \int \frac{(2x + 6) dx}{x^2 + 6x + 13} - 4 \int \frac{dx}{x^2 + 6x + 13} = \\ &= \frac{5}{2} \ln |x^2 + 6x + 13| - 4J. \end{aligned}$$

Bu yerda

$$J = \int \frac{dx}{(x + 3)^2 + 4} = \int \frac{d(x + 3)}{(x + 3)^2 + 2^2} = \frac{1}{2} \operatorname{arctg} \frac{x + 3}{2}.$$

Bundan

$$\int \frac{5x+11}{x^2+6x+13} dx = \frac{5}{2} \ln|x^2+6x+13| - 2 \operatorname{arctg} \frac{x+3}{2} + C.$$

IV turdagi sodda kasrning integralini topamiz:

$$\begin{aligned} \int \frac{Mx+N}{(x^2+px+q)^s} dx &= \frac{M}{2} \int \frac{(2x+p)dx}{(x^2+px+q)^s} + \\ &+ \left(N - \frac{Mp}{2}\right) \int \frac{d\left(x + \frac{p}{2}\right)}{\left(\left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right)^s}. \end{aligned} \quad (2.4)$$

Bu tenglikning o'ng tomonidagi birinchi integral jadvaldagi integralga keltirib, piladi:

$$\begin{aligned} I &= \int \frac{(2x+p)dx}{(x^2+px+q)^s} = \\ &= \int (x^2+px+q)^{-s} d(x^2+px+q) = \frac{1}{(1-s)(x^2+px+q)^{s-1}}. \end{aligned}$$

Ikkinchi integralga (uni  $t$ , bilan belgilaymiz)  $\left(x + \frac{p}{2}\right) = t$  almashtirish

bajaramiz va  $0 < q - \frac{p^2}{4} = a^2$  belgilash kiritamiz.

U holda

$$\begin{aligned} I_s &= \int \frac{d\left(x + \frac{p}{2}\right)}{\left(\left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right)^s} = \int \frac{dt}{(t^2+a^2)^s} = \frac{1}{a^2} \int \frac{(t^2+a^2) - t^2}{(t^2+a^2)^s} dt = \\ &= \frac{1}{a^2} \int \frac{dt}{(t^2+a^2)^{s-1}} - \frac{1}{a^2} \int \frac{t^2 dt}{(t^2+a^2)^s}. \end{aligned}$$

Bu tenglikning o'ng qismidagi birinchi integral  $I_s$  ga o'xshash bo'lib, unda maxrajning darajasi  $s$  dan bir birlikka kichik. Shu sababli, belgilashga ko'ra,  $I_{s-1}$  bo'ladi. Ikkinchi integralni bo'laklab

integrallaymiz:

$$\begin{aligned} \int \frac{t^2 dt}{(t^2 + a^2)^s} &= \frac{1}{2} \int \frac{t \cdot 2t dt}{(t^2 + a^2)^s} = \\ &= \frac{1}{2} \left( \frac{-t}{(s-1)(t^2 + a^2)^{s-1}} + \frac{1}{s-1} \int \frac{dt}{(t^2 + a^2)^{s-1}} \right) = \\ &= -\frac{t}{2(s-1)(t^2 + a^2)^{s-1}} + \frac{1}{2(s-1)} I_{s-1}. \end{aligned}$$

Demak,  $I_s$  integralni hisoblash uchun  $s$  darajani pasaytirish formulasini hosil qilamiz:

$$\begin{aligned} I_s &= \frac{1}{a^2} I_{s-1} + \frac{t}{2a^2(s-1)(t^2 + a^2)^{s-1}} - \frac{1}{2a^2(s-1)} = \\ &= \frac{t}{2a^2(s-1)(t^2 + a^2)^{s-1}} + \frac{2s-3}{2a^2(s-1)} I_{s-1}. \end{aligned} \quad (2.5)$$

Shunday qilib, (2.5) formula bo'yicha  $I_s$  integralni topamiz, keyin  $I_s$  dagi barcha  $t$  ni  $x + \frac{p}{2}$  bilan almashtirib va  $I_s, I_{s-1}$  integrallarni (2.4) tenglikka qo'yib, IV turdagi sodda kasr integralini topish uchun ifoda hosil qilamiz.

(2.5) formula bo'yicha  $I_s$  integralni topish indeksi bittaga kichik bolgan  $I_{s-1}$  integralni topishga,  $I_{s-1}$  integralni topish esa o'z navbatida  $I_{s-2}$  integralni topishga keltiriladi va bu jarayon quyidagi jadval integralni topishgacha davom ettiriladi:

$$I_1 = \int \frac{dt}{t^2 + a^2} = \frac{1}{a} \operatorname{arctg} \frac{t}{a} + C.$$

Demak, (2.5) formula orqali  $I_s$  dan  $I_{s-1}$  ga, so'ngra  $I_{s-2}$  o'tiladi va hokazo. Shu sababli bunday formulalar *keltirish* yoki *rekurrent (qaytuvchan)* formulalar deyiladi.

2- misol.  $\int \frac{2x+5}{(x^2+4x+8)^2} dx$  integralni toping.

*Yechish.* 
$$\int \frac{2x+4+1}{(x^2+4x+8)^2} dx = \int \frac{2x+4}{(x^2+4x+8)^2} dx + \int \frac{dx}{(x^2+4x+8)^2} =$$

$$= -\frac{1}{x^2 + 4x + 8} + \int \frac{d(x+2)}{[(x+2)^2 + 4]^2} = -\frac{1}{x^2 + 4x + 8} + \int \frac{dt}{t^2 + a^2},$$

bu yerda  $t = x + 2$ ,  $a = 2$ .

(2.5) integraldan foydalanib ayrim hisoblashlardan so'ng

$$I_2 = \frac{t}{2a^2(t^2 + a^2)} + \frac{1}{2a^3} \operatorname{arctg} \frac{t}{a} = \frac{x+2}{8(x^2 + 4x + 8)} + \frac{1}{16} \operatorname{arctg} \frac{x+2}{2}$$

ekanligini topamiz.

Demak,

$$\begin{aligned} \int \frac{2x+5}{(x^2 + 4x + 8)^2} dx &= \\ &= -\frac{1}{x^2 + 4x + 8} + \frac{x+2}{8(x^2 + 4x + 8)} + \frac{1}{16} \operatorname{arctg} \frac{x+2}{2} + C = \\ &= \frac{x-6}{8(x^2 + 4x + 8)} + \frac{1}{16} \operatorname{arctg} \frac{x+2}{2} + C. \end{aligned}$$

### 7.2.2. Ratsional kasrlarni integrallash

6.2 banddan va yuqorida aytilganlardan kelib chiqadiki,

$R(x) = \frac{Q_m(x)}{P_n(x)}$  ratsional kasr funksiyani integrallash quyidagi tartibda amalga oshiriladi:

1) berilgan ratsional kasrning to'g'ri yoki noto'g'ri kasr ekanini tekshirish; agar kasr noto'g'ri bo'lsa, kasrdan butun qismini ajratish;

2) to'g'ri kasrning maxrajini ko'paytuvchilarga ajratish;

3) to'g'ri kasrni sodda kasrlar yig'indisiga yoyish;

4) hosil bo'lgan ko'phad va sodda kasrlar yig'indisini integrallash.

3- misol.  $I = \int \frac{x^4 + 6}{x^3 - 2x^2 + 2x} dx$  integralni toping.

*Yechish.*  $R(x) = \frac{x^4 + 6}{x^3 - 2x^2 + 2x}$  noto'g'ri kasr, chunki

$m = 4$ ,  $n = 3$  ( $m > n$ ).

Bu kasrning suratni maxrajga bo'lish orqali kasrdan butun qismini

ajratamiz:

$$\begin{array}{r} -x^4 + 6 \\ x^4 - 2x^3 + 2x^2 \end{array} \left| \begin{array}{r} x^3 - 2x^2 + 2x \\ x + 2 \end{array} \right.$$
$$\begin{array}{r} -2x^3 - 2x^2 + 6 \\ -2x^3 - 4x^2 + 4x \end{array}$$
$$\hline 2x^2 - 4x + 6$$

Bundan

$$R(x) = x + 2 + \frac{2x^2 - 4x + 6}{x^3 - 2x^2 + 2x}.$$

To'g'ri kasrning maxrajini ko'paytuvchilarga ajratamiz:

$$x^3 - 2x^2 + 2x = x(x^2 - 2x + 2).$$

To'g'ri kasrni sodda kasrlarga yoyilmasi ko'rinishida yozamiz:

$$\frac{2x^2 - 4x + 6}{x(x^2 - 2x + 2)} = \frac{A}{x} + \frac{Mx + N}{x^2 - 2x + 2}.$$

Yoyilmaning noma'lum koeffitsiyentlarini topamiz:

$$2x^2 - 4x + 6 = A(x^2 - 2x + 2) + Mx^2 + Nx,$$

$$\begin{cases} x^2: A + M = 2, \\ x^1: -2A + N = -4, \\ x^0: 2A = 6. \end{cases}$$

Bundan  $A=3$ ,  $M=-1$ ,  $N=2$ .

Shunday qilib,

$$R(x) = x + 2 + \frac{3}{x} + \frac{-x + 2}{x^2 - 2x + 2}.$$

Ko'phad va sodda kasrlar yig'indisini integrallaymiz:

$$\begin{aligned} I &= \int (x + 2) dx + \int \frac{3 dx}{x} + \int \frac{-x + 2}{x^2 - 2x + 2} dx = \frac{x^2}{2} + 2x + 3 \ln |x| - \\ &- \int \frac{\frac{1}{2}(2x - 2) + 1 - 2}{x^2 - 2x + 2} dx = \frac{x^2}{2} + 2x + 3 \ln |x| - \frac{1}{2} \int \frac{2x - 2}{x^2 - 2x + 2} dx + \\ &\frac{x^2}{2} + 2x + 3 \ln |x| - \frac{1}{2} \ln |x^2 - 2x + 2| + \arctg(x - 1) + C. \end{aligned}$$



### 7.2.3. Mashqlar

#### 1. Integrallarni toping:

$$1) \int \frac{2x+3}{(x-2)(x+5)} dx;$$

$$3) \int \frac{x dx}{(x+1)(x+2)(x+3)};$$

$$5) \int \frac{3x^2+2x-3}{x(x-1)(x+1)} dx;$$

$$7) \int \frac{2x^3+2x^2+4x+3}{x^3+x^2} dx;$$

$$9) \int \frac{x^3-3}{x^3-2x^2-x+2} dx;$$

$$11) \int \frac{dx}{x(1+x^2)};$$

$$13) \int \frac{x^4+3x^3+2x^2+x+1}{x^2+x+1} dx;$$

$$15) \int \frac{dx}{x^4-1};$$

$$17) \int \frac{3x+5}{(x^2+2x+2)^2} dx;$$

$$19) \int \frac{dx}{(x^2+4x+5)(x^2+4x+13)};$$

$$21) \int \frac{dx}{(x^2+1)^4};$$

$$23) \int \frac{2x+3}{(x^2-3x+3)^2} dx;$$

$$2) \int \frac{x dx}{(x+1)(2x+1)};$$

$$4) \int \frac{8x dx}{(x+1)(x^2+6x+5)};$$

$$6) \int \frac{x^3-1}{4x^3-x} dx;$$

$$8) \int \frac{2+5x^3}{x(x^2-5x+4)} dx;$$

$$10) \int \frac{dx}{x^2(x^2+1)};$$

$$12) \int \frac{dx}{1+x^3};$$

$$14) \int \frac{x^9 dx}{x^4-1};$$

$$16) \int \frac{dx}{(x^2+9)^3};$$

$$18) \int \frac{x^4+2x^2+x}{(x-1)(x^2+4)^2} dx;$$

$$20) \int \frac{dx}{(x+1)^2(x^2+1)};$$

$$22) \int \frac{2x-1}{(x^2-2x+5)^2} dx;$$

$$24) \int \frac{3x^2-10x+12}{x^4+13x^2+36} dx.$$

### 7.3. TRIGONOMETRIK FUNKSIYALARNI INTEGRALLASH

Trigonometrik funksiyalarni integrallash usullaridan ayrimlari bilan tanishamiz. Faqat trigonometrik funksiyalar ustida ratsional amallar (qo'shish, ayirish, ko'paytirish va bo'lish) bajarilgan ifoda berilgan bo'lsin. Bunday ifodani barcha trigonometrik funksiyalarni  $\sin x$  va  $\cos x$

funksiyalar orqali ratsional ravishda ifodalash va  $R(\sin x, \cos x)$  ko'rinishga keltirish mumkin.

### 7.3.1. $\int R(\sin x, \cos x) dx$ ko'rinishidagi integrallar

$\int R(\sin x, \cos x) dx$  ko'rinishidagi integralni  $\operatorname{tg} \frac{x}{2} = t$  almashtirish orqali hamma vaqt  $t$  o'zgaruvchili ratsional funksiyaning integraliga almashtirish, ya'ni *ratsionallashtirish* mumkin. Shu sababli bu almashtirish *universal trigonometrik almashtirish* deyiladi.

Haqiqatan ham,  $\int R(\sin x, \cos x) dx$  ifodadan

$$\sin x = \frac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}, \quad x = \operatorname{arctg} t, \quad dx = \frac{2dt}{1+t^2}$$

tarzdagi o'rniga qo'yishlar yordamida  $t$  o'zgaruvchili

$$\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \cdot \frac{2dt}{1+t^2} = \int R_1(t) dt$$

ratsional funksiya kelib chiqadi.

*I-misol.*  $I = \int \frac{dx}{3\sin x + 2\cos x + 3}$  integralni toping.

*Yechish.*  $\operatorname{tg} \frac{x}{2} = t$  deymiz. U holda

$$\begin{aligned} I &= \int \frac{\frac{2dt}{1+t^2}}{3 \cdot \frac{2t}{1+t^2} + 2 \cdot \frac{1-t^2}{1+t^2} + 3} = 2 \int \frac{dt}{t^2 + 6t + 5} = 2 \int \frac{dt}{(t+1)(t+5)} = \\ &= \int \left( \frac{A}{t+1} + \frac{B}{t+5} \right) dt = A \ln |t+1| + B \ln |t+5| + C. \end{aligned}$$

Noma'lum koeffitsiyentlarni aniqlaymiz:  $A = \frac{1}{2}$ ,  $B = -\frac{1}{2}$ .

Demak,

$$I = \frac{1}{2} (\ln |t+1| - \ln |t+5|) + C = \frac{1}{2} \ln \left| \frac{t+1}{t+5} \right| = \frac{1}{2} \ln \left| \frac{\operatorname{tg} \frac{x}{2} + 1}{\operatorname{tg} \frac{x}{2} + 5} \right| + C.$$

Universal trigonometrik o'rniga qo'yish natijasida amalda ko'pincha ancha murakkab ratsional funksiyalar hosil bo'lishi mumkin. Bunday hollarda yuqorida keltirilgan integralni topishda quyidagi sodda almashtirishlardan foydalangan ma'qul:

a) agar  $R(\sin x, \cos x)$  ifoda  $\sin x$  ga nisbatan toq, ya'ni

$$R(-\sin x, \cos x) = -R(\sin x, \cos x)$$

bo'lsa, u holda  $\cos x = t$  o'rniga qo'yish bu funksiyani ratsionallashtiradi;

b) agar  $R(\sin x, \cos x)$  ifoda  $\cos x$  ga nisbatan toq, ya'ni

$$R(\sin x, -\cos x) = -R(\sin x, \cos x)$$

bo'lsa, u holda  $\sin x = t$  o'rniga qo'yish orqali bu funksiya ratsionallashtiriladi;

c) agar  $R(\sin x, \cos x)$  ifoda  $\sin x$  va  $\cos x$  larga nisbatan juft, ya'ni

$$R(-\sin x, -\cos x) = R(\sin x, \cos x)$$

bo'lsa, u holda  $\operatorname{tg} x = t$  o'rniga qo'yish bu funksiyani ratsionallashtiradi.

Bunda quyidagi almashtirishlardan foydalaniladi:

$$\sin^2 x = \frac{\operatorname{tg}^2 x}{1 + \operatorname{tg}^2 x} = \frac{t^2}{1 + t^2}, \quad \cos^2 x = \frac{1}{1 + \operatorname{tg}^2 x} = \frac{1}{1 + t^2},$$

$$x = \operatorname{arctgt}, \quad dx = \frac{dt}{1 + t^2}.$$

2-misol.  $I = \int \frac{\cos x dx}{\sin^2 x - 4 \sin x + 5}$  integralni toping.

*Yechish.* Integral ostidagi funksiya  $\cos x$  ga nisbatan toq funksiya. Shu sababli  $\sin x = t$  deb olamiz.

U holda

$$I = \int \frac{dt}{t^2 - 4t + 5} = \int \frac{d(t-2)}{(t-2)^2 + 1} = \operatorname{arctg}(t-2) + C = \operatorname{arctg}(\sin x - 2) + C.$$

3-misol.  $I = \int \frac{dx}{1 - 2 \sin^2 x}$  integralni toping.

*Yechish.* Integral ostidagi funksiya  $\sin x$  ga nisbatan juft funksiya. Shu sababli  $\operatorname{tg} x = t$  o'rniga qo'yishdan foydalanamiz.

U holda

$$I = \int \frac{\frac{dt}{1+t^2}}{1 - \frac{2t^2}{1+t^2}} = \int \frac{dt}{1-t^2} = \frac{1}{2} \ln \left| \frac{t+1}{t-1} \right| = \frac{1}{2} \ln \left| \frac{\operatorname{tg} x + 1}{\operatorname{tg} x - 1} \right| + C.$$

### 7.3.2. $\int \sin^n x \cos^m x dx$ ko'rinishidagi integrallar

$\int \sin^n x \cos^m x dx$  ko'rinishidagi integrallar  $m$  va  $n$  butun sonlarga bog'liq holda quyidagicha topiladi:

a)  $n > 0$  va toq bo'lganida  $\cos x = t$  o'rniga qo'yish integralni ratsionallashtiradi;

a)  $m > 0$  va toq bo'lganida  $\sin x = t$  o'rniga qo'yish orqali integral ratsionallashtiriladi;

c)  $m$  va  $n$  sonlar juft va nomanfiy bo'lganida

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

formularidan foydalanib, darajalar pasaytiriladi;

d)  $m + n < 0$  hamda  $m$  va  $n$  juft bo'lganida  $\operatorname{tg} x = t$  yoki  $\operatorname{ctg} x = t$  o'rniga qo'yishdan foydalaniladi. Bunda  $m < 0$  va  $n < 0$  bo'lsa, suratda  $1 = (\sin^2 x + \cos^2 x)^k$ , bu yerda  $k = \frac{|m+n|}{2} - 1$ , almashtirishdan iborat usul qo'llab, ratsional funksiyalarni integrallashga keltiriladi;

e)  $m, n \leq 0$  va ulardan biri toq bo'lganida  $\sin x$  va  $\cos x$  lardan qaysi birining darajasi toqligiga qarab, surat va maxrajni shu funksiyaga qo'shimcha ko'paytirishdan foydalaniladi.

4-misol.  $\int \sin^5 x \cos^2 x dx$  integralni toping.

*Yechish.*  $\int \sin^5 x \cos^2 x dx$  ( $n > 0$  va toq,  $\cos x = t$ ) =  $\int \sin^4 x \cos^2 x \sin x dx =$

$$= -\int (1-t^2)^2 t^2 dt = -\int t^2 dt + 2\int t^4 dt - \int t^6 dt = -\frac{t^3}{3} + \frac{2t^5}{5} - \frac{t^7}{7} + C =$$

$$= -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C.$$

5-misol.  $\int \sin^4 x \cos^2 x dx$  integralni toping.

*Yechish.*  $\int \sin^4 x \cos^2 x dx$  ( $n, m \geq 0$  va  $n, m$  - juft) =  $\int (\sin x \cos x)^2 \sin^2 x dx =$

$$= \int \left( \frac{\sin^2 2x}{4} \right) \cdot \left( \frac{1 - \cos 2x}{2} \right) dx = \frac{1}{8} \int (\sin^2 2x - \sin^2 2x \cos 2x) dx =$$

$$= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx - \frac{1}{16} \int \sin^2 2x d(\sin 2x) =$$

$$= \frac{1}{16} \left( x - \frac{\sin 4x}{4} \right) - \frac{\sin^3 2x}{48} + C = \frac{1}{16} \left( x - \frac{\sin 4x}{4} - \frac{\sin^3 2x}{3} \right) + C.$$

6-misol.  $I = \int \frac{dx}{\sin^4 x \cos^2 x}$  integralni toping.

*Yechish.* Bunda  $n = -4$ ,  $m = -2$ ,  $n + m = -6 < 0$ ,  $k = \frac{|m+n|}{2} - 1 = 2$ .

Demak,

$$\begin{aligned} I &= \int \frac{(\sin^2 x + \cos^2 x)^2}{\sin^4 x \cos^2 x} dx = \int \frac{\sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x}{\sin^4 x \cos^2 x} dx = \\ &= \int \frac{dx}{\cos^2 x} + 2 \int \frac{dx}{\sin^2 x} + \int \frac{\cos^2 x}{\sin^4 x} dx = \operatorname{tg} x - 2\operatorname{ctg} x - \int \operatorname{ctg}^2 x d(\operatorname{ctg} x) = \\ &= \operatorname{tg} x - 2\operatorname{ctg} x - \frac{1}{3} \operatorname{ctg}^3 x + C. \end{aligned}$$

### 7.3.3. $\int \operatorname{tg}^n x dx$ va $\int \operatorname{ctg}^n x dx$ ko'rinishidagi integrallar

$\int \operatorname{tg}^n x dx$  va  $\int \operatorname{ctg}^n x dx$  (bu yerda  $n > 0$  butun son) ko'rinishidagi integrallar mos rasvishda  $\operatorname{tg} x = t$  va  $\operatorname{ctg} x = t$  o'rniga qo'yish orqali topiladi.

Bunday integrallarni o'rniga qo'yishlardan foydalanmasdan, bevosita

$$\operatorname{tg}^2 x = \frac{1}{\cos^2 x} - 1, \quad \operatorname{ctg}^2 x = \frac{1}{\sin^2 x} - 1$$

formular yordamida hisoblash ham mumkin.

7-misol.  $\int \operatorname{tg}^5 x dx$  integralni hisoblang.

*Yechish.* 1-usul.  $\int \operatorname{tg}^5 x dx = \left| \operatorname{tg} x = t, dx = \frac{dt}{1+t^2} \right| = \int \frac{t^5 dt}{1+t^2} = \int t^3 dt -$

$$- \int t dt + \int \frac{t dt}{1+t^2} = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \int \frac{d(1+t^2)}{1+t^2} = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \ln |1+t^2| + C =$$

$$= \frac{1}{4} \operatorname{tg}^4 x - \frac{1}{2} \operatorname{tg}^2 x - \frac{1}{2} \ln |\cos^2 x| + C = \frac{1}{4} \operatorname{tg}^4 x - \frac{1}{2} \operatorname{tg}^2 x - \ln |\cos x| + C.$$

$$\begin{aligned}
 2\text{-usul. } \int \operatorname{tg}^3 x dx &= \int \operatorname{tg}^3 x \cdot \operatorname{tg}^2 x dx = \int \operatorname{tg}^3 x \cdot \left( \frac{1}{\cos^2 x} - 1 \right) dx = \\
 &= \int \operatorname{tg}^3 x \cdot \frac{dx}{\cos^2 x} - \int \operatorname{tg}^3 x dx = \int \operatorname{tg}^3 x d(\operatorname{tg} x) - \int \operatorname{tg} x \cdot \left( \frac{1}{\cos^2 x} - 1 \right) dx = \\
 &= \frac{1}{4} \operatorname{tg}^4 x - \int \operatorname{tg} x d(\operatorname{tg} x) - \int \operatorname{tg} x dx = \frac{1}{4} \operatorname{tg}^4 x - \frac{1}{2} \operatorname{tg}^2 x - \ln |\cos x| + C.
 \end{aligned}$$

### 7.3.4. $\int \sin mx \cos nx dx$ , $\int \sin mx \sin nx dx$ , $\int \cos mx \cos nx dx$ ko‘rinishidagi integrallar

Bu ko‘rinishdagi integrallarni hisoblashda

$$\sin mx \cos nx = \frac{1}{2} (\sin(m+n)x + \sin(m-n)x),$$

$$\sin mx \sin nx = \frac{1}{2} (\cos(m-n)x - \cos(m+n)x),$$

$$\cos mx \cos nx = \frac{1}{2} (\cos(m+n)x + \cos(m-n)x)$$

trigonometrik formulalardan foydalaniladi.

8-misol.  $\int \cos 3x \cdot \cos 5x dx$  integralni toping.

$$\begin{aligned}
 \text{Yechish. } \int \cos 3x \cdot \cos 5x dx &= \frac{1}{2} \int (\cos 8x + \cos 2x) dx = \\
 &= \frac{1}{2} \left( \frac{1}{8} \sin 8x + \frac{1}{2} \sin 2x \right) + C = \frac{1}{16} (\sin 8x + 4 \sin 2x) + C.
 \end{aligned}$$

### 7.3.5. Mashqlar

1. Berilgan integrallarni toping:

$$1) \int \frac{dx}{5 + 4 \sin x};$$

$$2) \int \frac{dx}{2 \sin x + \sin 2x};$$

$$3) \int \frac{dx}{3 + 5 \sin x + 3 \cos x};$$

$$4) \int \frac{dx}{4 + 2 \sin x + 3 \cos x};$$

$$5) \int \frac{\sin x dx}{\sqrt{3 - \cos^2 x}};$$

$$6) \int \frac{3 \cos^3 x dx}{\sin^4 x};$$

7)  $\int \frac{\cos^3 x dx}{1 + \sin^2 x}$ ;

8)  $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x} dx$ ;

9)  $\int \sin^2 x \cos^4 x dx$ ;

10)  $\int \frac{dx}{\sin x \cos^3 x}$ ;

11)  $\int \frac{dx}{2 + 3 \sin^2 x - 7 \cos^2 x}$ ;

12)  $\int \operatorname{ctg}^3 2x dx$ ;

13)  $\int \frac{\sin^2 x dx}{1 + \cos^2 x}$ ;

14)  $\int \cos 2x \cos 5x dx$ ;

15)  $\int \sin^2 x \cos 3x dx$ ;

16)  $\int \cos x \cos 2x \cos 3x dx$ .

## 7.4. IRRATSIONAL IFODALARNI INTEGRALLASH

Irratsional ifodalarni o'z ichga olgan ayrim integrallarni ko'rib chiqamiz.

**7.4.1.**  $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{m_1}, \left(\frac{ax+b}{cx+d}\right)^{m_2}, \dots\right) dx$  ko'rinishidagi integrallar

$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{n_1}, \left(\frac{ax+b}{cx+d}\right)^{n_2}, \dots\right) dx$  ( $R$  — ratsional funksiya,

$m_1, n_1, m_2, n_2, \dots$  — butun sonlar) ko'rinishdagi integrallar  $\frac{ax+b}{cx+d} = t'$  o'rniga qo'yish yordamida ratsional funksiyaning integraliga keltiriladi, bunda  $s = EKUK(n_1, n_2, \dots)$ .

Xususan,  $\int R\left(x, (ax+b)^{\frac{m_1}{s}}, (ax+b)^{\frac{m_2}{s}}, \dots\right) dx$  integrallar  $ax+b = t'$  o'rniga

qo'yish yordamida,  $\int R\left(x, x^{\frac{m_1}{s}}, x^{\frac{m_2}{s}}, \dots\right) dx$  integrallar esa  $x = t'$  o'rniga

qo'yish yordamida  $t$  o'zgaruvchili ratsional funksiya keltiriladi.

*1-misol.*  $I = \int \frac{x^2 + \sqrt[3]{1+x}}{\sqrt{1+x}} dx$  integralni toping.

*Yechish.* Bu yerda  $EKUK(2,3) = 6$  bo'lgani uchun  $1+x = t^6$  tarzda belgilash kiritamiz.

U holda

$$\sqrt{1+x} = t^3, \quad \sqrt[3]{1+x} = t^2, \quad dx = 6t^5 dt.$$

Demak,

$$\begin{aligned} I &= \int \frac{(t^6 - 1)^2 + t^2}{t^3} \cdot 6t^5 dt = 6 \int t^2 (t^{12} - 2t^6 + t^2 + 1) dt = \\ &= 6 \left( \frac{t^{15}}{15} - 2 \frac{t^9}{9} + \frac{t^3}{5} + \frac{t^3}{3} \right) + C = \frac{2t^3}{15} (3t^{12} - 10t^9 + 9t^2 + 15) + C = \\ &= \frac{2\sqrt{1+x}}{15} (3(1+x)^2 - 10(1+x) + 9\sqrt{1+x} + 15) + C. \end{aligned}$$

#### 7.4.2. $\int R(x, \sqrt{ax^2 + bx + c}) dx$ ko‘rinishidagi integrallar

$\int R(x, \sqrt{ax^2 + bx + c}) dx$  ko‘rinishidagi integrallar *Eylerning uchta o‘rniga qo‘yish usuli* orqali ratsional funksiyalardan olingan integrallarga keltiriladi:

a)  $a > 0$  bo‘lganida  $\sqrt{ax^2 + bx + c} = t \pm \sqrt{ax}$  almashtirish orqali integral ostidagi funksiya ratsionallashtiriladi (*Eylerning birinchi o‘rniga qo‘yishi*);

b)  $c > 0$  bo‘lganida  $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$  almashtirish yordamida integral ostidagi funksiya ratsionallashtiriladi (*Eylerning ikkinchi o‘rniga qo‘yishi*);

c)  $ax^2 + bx + c$  kvadrat uchhad  $a(x - x_1)(x - x_2)$  ko‘rinishda ko‘paytuvchilarga ajralganida integral ostidagi funksiya  $\sqrt{ax^2 + bx + c} = t(x - x_1)$  almashtirish bilan ratsionallashtiriladi (*Eylerning uchinchi o‘rniga qo‘yishi*).

2-misol.  $I = \int \frac{dx}{1 + \sqrt{x^2 + 2x + 2}}$  integralni toping.

*Yechish.* Bunda  $a > 0$ . Shu sababli  $\sqrt{x^2 + 2x + 2} = t - x$  ko‘rinishdagi o‘rniga qo‘yish bajaramiz.

U holda

$$x^2 + 2x + 2 = t^2 - 2tx + x^2, \quad 2x + 2tx = t^2 - 2.$$



Bundan

$$x = \frac{t^2 - 2}{2(1+t)}, \quad dx = \frac{t^2 + 2t + 2}{2(1+t)^2}, \quad 1 + \sqrt{x^2 + 2x + 2} = 1 + t - \frac{t^2 - 2}{2(1+t)} = \frac{t^2 + 4t + 4}{2(1+t)}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$I = \int \frac{2(1+t)(t^2 + 2t + 2)}{(t^2 + 4t + 4)2(1+t)^2} dt = \int \frac{t^2 + 2t + 2}{(1+t)(2+t)^2} dt.$$

Integral ostidagi to'g'ri kasmi sodda kasrlarga yoyamiz:

$$\frac{t^2 + 2t + 2}{(1+t)(2+t)^2} = \frac{A}{1+t} + \frac{B}{2+t} + \frac{C}{(2+t)^2}.$$

Koeffitsiyentlarni tenglashtirish usulini qo'llaymiz:  $A=1$ ,  $B=0$ ,  $C=-2$ .

Bundan

$$I = \int \frac{dt}{1+t} - 2 \int \frac{dt}{(2+t)^2} = \ln|1+t| + \frac{2}{2+t} + C.$$

$x$  o'zgaruvchiga qaytamiz:

$$I = \ln|1+x+\sqrt{x^2+2x+2}| + \frac{2}{x+2+\sqrt{x^2+2x+2}} + C.$$

3-misol.  $I = \int \frac{dx}{\sqrt{x^3 - 3x + 2}}$  integralni toping.

*Yechish.*  $x^2 - 3x + 2 = (x-1)(x-2)$  bo'lgani uchun

$$\sqrt{(x-1)(x-2)} = (x-1)t$$

shaklda o'rni qo'yish bajaramiz.

U holda

$$(x-1)(x-2) = (x-1)^2 t^2, \quad t = \sqrt{\frac{x-2}{x-1}}.$$

Bundan

$$x = \frac{t^2 - 2}{t^2 - 1}, \quad dx = \frac{2tdt}{(t^2 - 1)^2}, \quad \sqrt{x^2 - 3x + 2} = \left( \frac{t^2 - 2}{t^2 - 1} - 1 \right) t = -\frac{t}{t^2 - 1}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$I = \int \frac{-(t^2 - 1)2t dt}{(t^2 - 1)^2 t} = -2 \int \frac{dt}{t^2 - 1} = -\ln \left| \frac{t-1}{t+1} \right| + C = -\ln \left| \frac{1-2t+t^2}{t^2-1} \right| + C.$$

Dastlabki o'zgaruvchiga qaytamiz:

$$I = -\ln \left| \frac{1 - 2\sqrt{\frac{x-2}{x-1}} + \frac{x-2}{x-1}}{\frac{x-2}{x-1} - 1} \right| + C = -\ln |3 - 2x + 2\sqrt{x^2 - 3x + 2}| + C.$$

Eyler o'rninga qo'yishlari ayrim integrallarda murakkab hisoblashlarga olib kelishi mumkin. Bunday hollarda integrallashning quyidagi usullaridan foydalanilsa bo'ladi.

1.  $\int R(x, \sqrt{ax^2 + bx + c}) dx$  ko'rinishidagi integrallarni hisoblashning kvadrat uchhaddan to'la kvadrat ajratish usulida berilgan integrallar  $ax^2 + bx + c$  kvadrat uchhaddan to'la kvadrat ajratish yo'li bilan ushbu integrallardan biriga keltiriladi:

a) agar  $a > 0$  va  $b^2 - 4ac < 0$  bo'lsa, u holda  $\int R(t, \sqrt{m^2 + n^2 t^2}) dt$ ,  
bu yerda  $n^2 = a$ ,  $m^2 = -\frac{b^2 - 4ac}{4a}$ ,  $t = x + \frac{b}{2a}$ ;

b) agar  $a > 0$  va  $b^2 - 4ac > 0$  bo'lsa, u holda  $\int R(t, \sqrt{n^2 t^2 - m^2}) dt$ ,  
bu yerda  $n^2 = a$ ,  $m^2 = \frac{b^2 - 4ac}{4a}$ ,  $t = x + \frac{b}{2a}$ ;

c) agar  $a < 0$  va  $b^2 - 4ac > 0$  bo'lsa, u holda  $\int R(t, \sqrt{m^2 - n^2 t^2}) dt$ ,  
bu yerda  $n^2 = -a$ ,  $m^2 = -\frac{b^2 - 4ac}{4a}$ ,  $t = x + \frac{b}{2a}$ .

Hosil qilingan integrallar mos ravishda  $t = \frac{m}{n} \operatorname{tg} z$ ,  $t = \frac{m}{n \sin z}$ ,  $t = \frac{m}{n} \sin z$

o'rninga qo'yishlar orqali  $\int R(\sin z, \cos z) dz$  ko'rinishga keltiriladi.

4-misol.  $\int \sqrt{5 + 4x - x^2} dx$  integralni toping.

*Yechish.* Kvadrat uchhaddan to'la kvadrat ajratamiz, yangi  $t$  o'zgaruvchi kiritamiz va trigonometrik o'rninga qo'yishdan foydalanib, topamiz:

$$\int \sqrt{5 + 4x - x^2} dx = \int \sqrt{9 - (x-2)^2} dx = \left| \begin{array}{l} x-2=t, \\ dx=dt \end{array} \right| = \int \sqrt{9-t^2} dt =$$

$$\begin{aligned}
 &= \left| \begin{array}{l} t = 3 \sin z, \\ dt = \cos z dz \end{array} \right| = \int \sqrt{9 - 9 \sin^2 z} 3 \cos z dz = \int 9 \cos^2 z dz = \\
 &= \frac{9}{2} \int (1 + \cos 2z) dz = \frac{9}{2} \left( z + \frac{\sin 2z}{2} \right) + C = \frac{9}{2} (z + \sin z \sqrt{1 - \sin^2 z}) + C = \\
 &= \left| z = \arcsin \frac{t}{3} \right| = \frac{9}{2} \left( \arcsin \frac{t}{3} + \frac{t}{3} \sqrt{1 - \frac{t^2}{9}} \right) + C = \frac{9}{2} \arcsin \frac{t}{3} + \frac{t}{2} \sqrt{9 - t^2} + C = \\
 &= \frac{9}{2} \arcsin \frac{x-2}{3} + \frac{1}{2} (x-2) \sqrt{5+4x-x^2} + C.
 \end{aligned}$$

2.  $\int R(x, \sqrt{ax^2 + bx + c}) dx$  ko'rinishidagi ayrim integrallarni hisoblashning boshqa usullarini keltiramiz.

a)  $\int \frac{P_n(x) dx}{\sqrt{ax^2 + bx + c}}$  ko'rinishidagi integrallar, bu yerda  $P_n(x)$  —  $n$ -darajali ko'phad:

1)  $n=0$  da  $\int \frac{A dx}{\sqrt{ax^2 + bx + c}}$  ko'rinishda bo'ladi; bu integral  $a > 0$  bo'lganda jadvaldagi 14- integralga,  $a < 0$  bo'lganda jadvaldagi 13- integralga keltiriladi;

2)  $n=1$  da  $\int \frac{(Ax + B) dx}{\sqrt{ax^2 + bx + c}}$  ko'rinishda bo'ladi; bu integral suratda kvadrat uchhadning hosilasini ajratish natijasida ikkita, biri jadvaldagi 1- integralga va ikkinchisi 1) banddagi integralga keltiriladi;

3)  $n \geq 2$  bo'lganda berilgan integraldan keltirish formulalari yordamida

$$\int \frac{P_n(x) dx}{\sqrt{ax^2 + bx + c}} = Q_{n-1}(x) \sqrt{ax^2 + bx + c} + M \int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

ko'rinishdagi ifoda hosil qilinadi, bu yerda  $Q_{n-1}(x)$  — koeffitsiyentlari noma'lum bo'lgan  $n-1$ -darajali ko'phad,  $M$  — qandaydir o'zgarmas son. Bunda ko'phadning noma'lum koeffitsiyentlari va  $M$  soni oxirgi tenglikni differensiallash hamda  $x$  ning chap va o'ng tomondagi bir xil darajalari oldidagi koeffitsiyentlarni tenglashtirish orqali topiladi.

b)  $\int \frac{dx}{(ax + \beta) \sqrt{ax^2 + bx + c}}$  ko'rinishdagi integral  $ax + \beta = \frac{1}{t}$

almashtirish yordamida 1) banddagi integralga keltiriladi;

$$c) \int \frac{dx}{(\alpha x + \beta)^n \sqrt{ax^2 + bx + c}} \quad (n \in \mathbb{Z}, n > 1) \text{ ko'rinishdagi integral } \alpha x + \beta = \frac{1}{t}$$

o'rniga qo'yish orqali 3) banddagi integralga keltiriladi.

5- misol.  $\int \frac{dx}{(x-2)^3 \sqrt{x^2 - 4x + 5}}$  integralni toping.

*Yechish.*  $x - 2 = \frac{1}{t}$  deymiz. U holda  $dx = -\frac{dt}{t^2}$ ,  $x^2 - 4x + 5 = \frac{1}{t^2} + 1$ .

Bundan

$$\int \frac{dx}{(x-2)^3 \sqrt{x^2 - 4x + 5}} = -\int \frac{\frac{dt}{t^2}}{\frac{1}{t^3} \sqrt{\frac{1}{t^2} + 1}} = -\int \frac{t^2 dt}{\sqrt{t^2 + 1}}$$

Demak, b) banddagi integral hosil qilindi. Bunda  $n=2$  bo'lgani uchun

$$\int \frac{t^2 dt}{\sqrt{t^2 + 1}} = (At + B)\sqrt{t^2 + 1} + M \int \frac{dt}{\sqrt{t^2 + 1}}$$

Tenglikning har ikkala tomonini differensiallaymiz:

$$\frac{t^2}{\sqrt{t^2 + 1}} = A\sqrt{1+t^2} + \frac{(At+B)t}{\sqrt{t^2+1}} + \frac{M}{\sqrt{t^2+1}}$$

yoki

$$t^2 = A(1+t^2) + (At+B)t + M$$

$t$  ning bir xil darajalari oldidagi koeffitsiyentlarni tenglab, topamiz:

$$A = \frac{1}{2}, \quad b = 0, \quad M = -\frac{1}{2}$$

U holda

$$\int \frac{t^2 dt}{\sqrt{1+t^2}} = \frac{t\sqrt{1+t^2}}{2} - \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}} = \frac{t\sqrt{1+t^2}}{2} - \frac{1}{2} \ln|t + \sqrt{1+t^2}| + C$$

Dastlabki o'zgaruvchiga qaytamiz:

$$\int \frac{dx}{(x-2)^3 \sqrt{x^2 - 4x + 5}} = -\frac{\sqrt{x^2 - 4x + 5}}{2(x-2)^2} + \frac{1}{2} \ln \left| \frac{1 + \sqrt{x^2 - 4x + 5}}{x-2} \right| + C$$

### 7.4.3. $\int x^m (a+bx^n)^p dx$ binominal differensialni integrallash

$\int x^m (a+bx^n)^p dx$  ko‘rinishidagi integral *binominal differensial integrali* deyiladi. Bunda integral ostidagi ifoda  $x^m (a+bx^n)^p$  ga *binominal differensial* deyiladi, bu yerda  $m, n, p$  – ratsional sonlar.

Binominal differensial integrali uchta holdagina ratsional funksiyalarni integrallashga keltiriladi:

a)  $p$  butun son bo‘lganida integral  $x=t^s$  (bu yerda  $s = EKUK(m,n)$ ) o‘rniga qo‘yish orqali ratsionallashtiriladi;

b)  $\frac{m+1}{n}$  butun son bo‘lganida integral  $a+bx^n = t^s$  (bu yerda  $s = p$  sonning maxraji) o‘rniga qo‘yish yordamida ratsionallashtiriladi;

c)  $\frac{m+1}{n} + p$  butun son bo‘lganida integralda  $a+bx^n = t^s x^n$  (bu yerda  $s = p$  sonning maxraji) almashtirish bajariladi.

Agar yuqorida keltirilgan shartlar bajarilmasa binominal differensial elementar funksiyalar orqali ifodalanmaydi, ya’ni integrallanmaydi.

*Masalan*,  $\int \sqrt{1+x^3} dx$  integralning integral osti funksiyasi binominal differensial:  $m=0, n=3, p=\frac{1}{2}$ . Bunda  $p=\frac{1}{2}, \frac{m+1}{n}=\frac{1}{3}, \frac{m+1}{n}+p=\frac{5}{6}$  sonlardan birortasi butun son emas. Shu sababli bu integral elementar funksiyalar orqali ifodalanmaydi.

6-misol.  $I = \int \frac{dx}{x^3 \sqrt{1+x^4}}$  integralni toping.

*Yechish.* Shartga ko‘ra,

$$m = -3, n = 4, p = -\frac{1}{2}, \frac{m+1}{n} + p = \frac{-3+1}{4} - \frac{1}{2} = -1.$$

c) bandda aytilganidek, almashtirish bajaramiz:

$$1+x^4 = t^2 x^4, x = (t^2 - 1)^{\frac{1}{4}}, dx = -\frac{1}{2} t (t^2 - 1)^{\frac{5}{4}}, t = \frac{\sqrt{1+x^4}}{x^2}.$$

U holda

$$\begin{aligned} I &= \int x^{-3} (1+x^4)^{-\frac{1}{2}} dx = -\frac{1}{2} \int (t^2 - 1)^{-\frac{1}{4}(-\frac{5}{4})} (t^2)^{\frac{1}{2}} \left( (t^2 - 1)^{\frac{1}{4}} \right)^{-2} t (t^2 - 1)^{\frac{5}{4}} dt = \\ &= -\frac{1}{2} \int (t^2 - 1)^{\frac{3+5}{4}} \cdot t^{-1+1} dt = -\frac{1}{2} \int dt = -\frac{1}{2} t + C = -\frac{\sqrt{1+x^4}}{2x^2} + C. \end{aligned}$$

#### 7.4.4. Elementar funksiyalarda ifodalanmaydigan integrallar

Biz integrallashning elementar funksiyalarning keng sinfini qamrab olgan muhim usullarini ko'rib chiqdik. Bu usullar ko'pchilik hollarda aniqmas integralni topish, ya'ni boshlang'ich funksiyalarni aniqlash imkonini beradi.

Ma'lumki, har qanday uzluksiz funksiya boshlang'ich funksiyaga ega bo'ladi. Agar biror  $f(x)$  elementar funksiyaning boshlang'ich funksiyasi ham elementar funksiya bo'lsa, u holda  $\int f(x)dx$  integral *elementar funksiyalarda ifodalanadi* deyiladi.

Adabiyotlarda keltirilishicha,  $\int \sqrt{x} \cdot \cos x dx$  integral elementar funksiyalarda ifodalanmaydi, chunki hosilasi  $\sqrt{x} \cdot \cos x$  ga teng bo'lgan elementar funksiya mavjud emas. Amaliy tatbiqda muhim ahamiyatga ega bo'lgan elementar funksiyalarda ifodalanmaydigan integrallarga misollar keltiramiz:

$\int e^{-x^2} dx$  – Puasson integrali (ehtimollar nazariyasi);

$\int \frac{dx}{\ln x}$  – integralli logarifm (sonlar nazariyasi);

$\int \cos x^2 dx$ ,  $\int \sin x^2 dx$  – Frenel integrallari (fizika);

$\int \frac{\sin x}{x} dx$ ,  $\int \frac{\cos x}{x} dx$  – integralli sinus va kosinus;

$\int \frac{e^x}{x} dx$  – integralli ko'rsatkichli funksiya.

Elementar funksiyalarda ifodalanmasa-da,  $e^{-x^2}$ ,  $\frac{1}{\ln x}$ ,  $\cos x^2$ ,  $\sin x^2$ ,  $\frac{\sin x}{x}$ ,  $\frac{\cos x}{x}$ ,  $\frac{e^x}{x}$  funksiyalarning boshlang'ich funksiyalari yetarlicha o'rganilgan,  $x$  argumentning turli qiymatlarida ularning qiymatlari uchun mufassal jadvallar tuzilgan.

#### 7.4.5. Mashqlar

1. Berilgan integrallarni toping:

1)  $\int \frac{dx}{\sqrt{x+\sqrt{x}}}$ ;

2)  $\int \frac{dx}{\sqrt{x(1+\sqrt{x})^2}}$ ;

3)  $\int \frac{x^2 + \sqrt[3]{1+x}}{\sqrt{1+x}} dx$

4)  $\int \frac{x - \sqrt{x+1}}{\sqrt[3]{x+1}} dx$ ;

5)  $\int \frac{dx}{\sqrt{2x-1} + \sqrt[3]{(2x-1)^2}};$

7)  $\int \frac{dx}{\sqrt{x^2-3x+2}};$

9)  $\int \frac{dx}{x\sqrt{x^2+x+1}};$

11)  $\int \frac{dx}{1+\sqrt{1-2x-x^2}};$

13)  $\int \sqrt{5+4x-x^2} dx;$

15)  $\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}};$

17)  $\int \frac{x dx}{\sqrt{3-2x-x^2}};$

19)  $\int \frac{dx}{x(1+\sqrt[3]{x})^2};$

21)  $\int x^3 \sqrt{(1+x^3)^2} dx;$

23)  $\int \frac{dx}{x^3 \sqrt{1+x^4}};$

6)  $\int \left( \sqrt[3]{\left(\frac{x+1}{x-1}\right)^2} - \sqrt{\left(\frac{x+1}{x-1}\right)^5} \right) \frac{dx}{1-x^2};$

8)  $\int \frac{dx}{\sqrt{x^2+2x+5}};$

10)  $\int \frac{dx}{x\sqrt{4-2x-x^2}};$

12)  $\int \frac{dx}{1+\sqrt{x^2+2x+2}};$

14)  $\int \sqrt{x^2-4} dx;$

16)  $\int \frac{dx}{(x-1)\sqrt{x^2-2x}};$

18)  $\int \frac{(2x+3)dx}{\sqrt{6x-x^2-8}};$

20)  $\int \frac{dx}{x^3 \sqrt{2-x^3}};$

22)  $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx;$

24)  $\int \frac{\sqrt{1+\sqrt[3]{x}}}{x\sqrt{x}} dx.$

## 7.5. ANIQ INTEGRAL

### 7.5.1. Aniq integral tushunchasiga olib keluvchi masalalar

Aniq integral tabiat va texnikaning bir qancha masalalarini yechishda, xususan, har xil geometrik va fizik kattaliklarni hisoblashda keng qo'llaniladi.

#### *Egri chiziqli trapetsiyaning yuzasi haqidagi masalasi*

Tekislikda  $Oxy$  to'g'ri burchakli dekart koordinatalar sistemasi kiritilgan va  $[a;b]$  kesmada uzluksiz va manfiy bo'lmagan  $y=f(x)$  funksiya aniqlangan bo'lsin.

Yuqoridan  $y = f(x)$  funksiya grafigi bilan, quyidan  $Ox$  o'qi bilan, yon tomonlaridan  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan figuraga *egri chizikli trapetsiya* deyiladi (2-shaklda bu figura  $-aABb$ ).

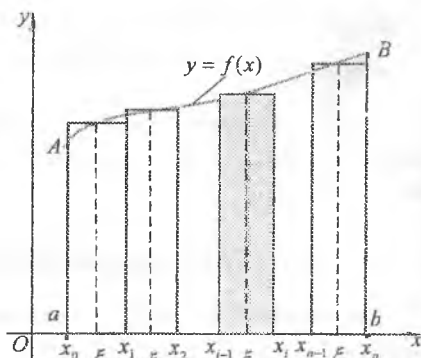
$aABb$  egri chizikli trapetsiyaning  $S$  yuzasiga ta'rif beramiz.

$[a; b]$  kesmani  $n$  ta kichik kesmalarga bo'lamiz: bo'linish nuqtalarining absissalarini  $a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = b$  bilan belgilaymiz.  $\{x_i\} = \{x_0, x_1, \dots, x_n\}$  bo'linish nuqtalari to'plamini  $[a; b]$  kesmaning bo'linishi deymiz.  $x_i$  bo'linish nuqtalari orqali  $Oy$  o'qqa parallel  $x = x_i$  to'g'ri chiziq o'tkazamiz. Bu to'g'ri chiziqlar  $aABb$  trapetsiyaning asoslari  $[x_{i-1}; x_i]$  bo'lgan  $n$  ta bo'lakka bo'ladi.  $aABb$  trapetsiyaning  $S$  yuzasi  $n$  ta tasma yuzalarining yig'indisiga teng bo'ladi.  $n$  yetarlicha katta va barcha  $[x_{i-1}; x_i]$  kesmalar kichik bo'lganida har bir  $n$  ta tasmaning yuzasini hisoblash oson bo'lgan mos to'g'ri to'rtburchakning yuzasi bilan almashtirish mumkin bo'ladi. Har bir  $[x_{i-1}; x_i]$  kesmada biror  $\xi_i$  nuqtani tanlaymiz,  $f(x)$  funksiyaning bu nuqtadagi qiymati  $f(\xi_i)$  ni hisoblaymiz va uni to'g'ri to'rtburchakning balandligi deb qabul qilamiz.  $[x_{i-1}; x_i]$  kesma kichik bo'lganida  $f(x)$  uzluksiz funksiya bu kesmada kichik o'zgarishga ega bo'ladi. Shu sababli bu kesmalarda funksiyaning va taqriban  $f(\xi_i)$  ga teng deyish mumkin. Bitta tasmaning yuzasi  $f(\xi_i)(x_i - x_{i-1})$  ga teng bo'lganidan  $aABb$  egri chizikli trapetsiyaning  $S$  ga yuzasi taqriban  $S_n$  teng bo'ladi:

$$S \approx S_n = \sum_{i=1}^n f(\xi_i) \Delta x_i, \quad \Delta x_i = x_i - x_{i-1} \quad (5.1)$$

(5.1) taqribiy qiymat  $d = \max \Delta x_i (i = \overline{1, n})$  kattalik qancha kichik bo'lsa, shuncha aniq bo'ladi.  $d$  kattalikka  $\{x_i\}$  bo'linishning diametri deyiladi. Bunda  $n \rightarrow \infty$  da  $d \rightarrow 0$ .

Shunday qilib, *egri chizikli trapetsiyaning  $S$  yuzasi* deb,  $S_n$  to'g'ri to'rtburchaklar yuzasining bo'linish diametri nolga intilgandagi limitiga



2-shakl.



aytiladi, ya'ni

$$S = \lim_{d \rightarrow 0} S_n = \lim_{d \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i. \quad (5.2)$$

Demak, egri chizikli trapetsiyaning yuzasini hisoblash masalasi (5.2) ko'rinishdagi limitni hisoblashga keltiriladi.

### *Bosib o'tilgan yo'l masalasi*

Agar moddiy nuqtaning harakat qonuni  $s = f(t)$  (bunda  $t$  – vaqt,  $s$  – bosib o'tilgan yo'l) tenglama bilan berilgan bo'lsa,  $f(t)$  funksiyaning  $f'(t)$  hosilasi moddiy nuqtaning berilgan vaqtdagi harakat tezligi  $v(t)$  ga teng, ya'ni  $v(t) = f'(t)$  bo'ladi. Fizikada quyidagi masalani yechishga to'g'ri keladi. Moddiy nuqta to'g'ri chiziq bo'ylab  $v$  tezlik bilan harakat qilayotgan va  $v$  tezlik  $t$  vaqtning uzluksiz funksiyasi bo'lsin deymiz. Moddiy nuqta vaqtning  $t = a$  dan  $t = b$  gacha bo'lgan biror  $[a; b]$  oraliq'ida bosib o'tgan yo'l  $s$  ni topamiz.  $[a; b]$  kesmani  $a = t_0 < t_1 < \dots < t_{i-1} < t_i < \dots < t_{n-1} < t_n = b$  nuqtalar bilan vaqtning  $n$  ta yetarlicha kichik oraliqlariga bo'lamiz. Vaqtning kichik  $[t_{i-1}; t_i]$  oraliq'ida  $v(t)$  tezlik «deyarli» o'zgarmaydi. Uni bu vaqt oraliq'ida o'zgarmas va taqriban  $v(\xi_i)$  ( $\xi_i \in [t_{i-1}; t_i]$ ) ga teng deyish mumkin. Bunda harakat  $[t_{i-1}; t_i]$  kesmada tekis bo'ladi. U holda bosib o'tilgan yo'l bu vaqt oraliq'ida  $v(\xi_i)(t_i - t_{i-1})$  ga,  $[a; b]$  vaqt oraliq'ida  $s \approx s_n = \sum_{i=1}^n v(\xi_i) \Delta t_i$  ( $\Delta t_i = t_i - t_{i-1}$ ) ga teng bo'ladi. Bu taqribiy qiymat  $d = \max_i \Delta t_i$  ( $i = 1, n$ ) kattalik qancha kichik bo'lsa, shuncha aniq bo'ladi.

Shunday qilib,  $s$  bosib o'tilgan yo'l deb,  $s_n$  yig'indining  $d \rightarrow 0$  dagi limitiga aytiladi, ya'ni

$$s = \lim_{d \rightarrow 0} s_n = \lim_{d \rightarrow 0} \sum_{i=1}^n v(\xi_i) \Delta t_i. \quad (5.3)$$

Demak, bosib o'tilgan yo'l ni hisoblash masalasi (5.3) ko'rinishdagi limitni hisoblashga keltiriladi.

Qaralgan har ikki masalada biror ko'rinishdagi yig'indining limitini topishga olib keluvchi bir xil usul qo'llanildi. Tabiat va texnikaning bir qancha masalalari yuqoridagi kabi yig'indining limitini topishga keltiriladi.

## 7.5.2. Integral yig'indi va aniq integral

$y = f(x)$  funksiya  $[a; b]$  kesmada aniqlangan bo'lsin.

$[a; b]$  kesmani ixtiyoriy ravishda

$$a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = b$$

nuqtalar bilan  $n$  ta qismga bo'lamiz, bunda  $\{x_i\}$  ga  $[a; b]$  kesmaning bo'linishi,  $d = \max_{1 \leq i \leq n} (x_i - x_{i-1})$ ,  $(i = \overline{1, n})$  kattalikka bo'linish diametri deymiz.

Har bir  $[x_{i-1}; x_i]$  kesmada ixtiyoriy  $\xi_i$  nuqtani tanlaymiz. Bunday nuqtalarni *belgilangan nuqtalar* deb ataymiz. Funksiyaning  $f(\xi_i)$  qiymatini mos  $\Delta x_i = x_i - x_{i-1}$  uzunlikka ko'paytirib, bu ko'paytmalardan

$$w_n = \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (5.4)$$

yig'indini tuzamiz. (5.4) yig'indiga  $f(x)$  funksiya uchun  $[a; b]$  kesmaning  $\{x_i\}$  bo'linishidagi *Riman integral yig'indisi* deyiladi.

$w_n$  yig'indining  $d \rightarrow 0$  dagi limiti tushunchasini kiritamiz.

**1-ta'rif.** Agar  $\forall \varepsilon > 0$  son uchun shunday  $\delta > 0$  son topilsa va  $|I - w_n| < \varepsilon$  tengsizlik  $[a; b]$  kesmaning diametri  $d < \delta$  bo'lgan istalgan  $\{x_i\}$  bo'linishida  $\xi_i$  belgilangan nuqtalarning tanlanishiga bog'liq bo'lmagan holda bajarilsa,  $I$  soniga  $w_n$  Riman integral yig'indisining limiti deyiladi va u  $I = \lim_{d \rightarrow 0} w_n$  deb yoziladi.

**2-ta'rif.** Agar (5.4) Riman integral yig'indisi  $d \rightarrow 0$  da chekli limitga ega bo'lsa, u holda bu limitga  $[a; b]$  kesmada  $f(x)$  funksiyadan olingan *aniq* (bir karrali) *integral* (Riman integrali) deyiladi va  $\int_a^b f(x) dx$  kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\int_a^b f(x) dx = \lim_{d \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i, \quad (5.5)$$

bu yerda  $f(x)$  – integral ostidagi funksiya,  $x$  – integrallash o'zgaruvchisi,  $a, b$  – integralning quyi va yuqori chegarasi,  $[a; b]$  – integrallash sohasi (kesmasi) deyiladi.

$[a; b]$  kesmada  $\int_a^b f(x) dx$  aniq integral mavjud bo'lsa,  $y = f(x)$  funksiya shu kesmada *integrallanuvchi* (Riman bo'yicha integrallanuvchi) deyiladi.

**Izoh.** Oliy matematika kursida boshqa aniq integrallar qaralmagani sababli bundan keyin «Riman integrali» va «Riman bo'yicha integrallanuvchi» iboralarini mos ravishda «integral» va «integrallanuvchi» deb ishlatamiz.

Keltirilgan ta'riflarda  $a < b$  bo'lsin deb faraz qilindi. Aniq integral tushunchasini  $a = b$  va  $a > b$  bo'lgan hollar uchun umumlashtiramiz.

$a > b$  bo'lganida 2-ta'rifga ko'ra,

$$\int_a^b f(x) dx = -\int_b^a f(x) dx. \quad (5.6)$$

2-ta'rifga ko'ra,  $a = b$  bo'lganida ((5.5) ga qarang).

$$\int_a^a f(x) dx = 0. \quad (5.7)$$

(5.4) integral yig'indi berilgan funksiyaning argumenti qanday harf bilan belgilanishiga bog'liq bo'lmagani sababli, uning limiti va shuningdek, aniq integral integrallash o'zgaruvchisining belgilanishiga bog'liq bo'lmaydi:

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(z) dz.$$

*1-misol.*  $\int_0^1 x^2 dx$  integralni aniq integralning ta'rifidan foydalanib hisoblang.

*Yechish.*  $[0;1]$  kesmada  $y = x^2$  funksiya uzluksiz.

$[0;1]$  kesmani  $0 = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = 1$  nuqtalar bilan uzunliklari  $\Delta x_i = \frac{1}{n} (i = \overline{1, n})$  bo'lgan  $n$  ta bo'lakka bo'lamiz. Bunda

$d = \max \Delta x_i$ . Demak,  $d \rightarrow 0$  da  $n \rightarrow \infty$ .

$\xi_i$  nuqta sifatida qisman kesmalarning oxirlarini olamiz:

$$\xi_i = x_i = \frac{i}{n}.$$

Tegishli integral yig'indini tuzamiz:

$$\begin{aligned} w_n &= \sum_{i=1}^n f(\xi_i) \Delta x_i = \sum_{i=1}^n \frac{i^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2) = \\ &= \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}. \end{aligned}$$

Bundan

$$\lim_{d \rightarrow 0} w_n = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

Demak, ta'rifga ko'ra,

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

Endi  $\xi_i$  nuqta sifatida qisman kesmalarning boshlarini olamiz:

$$\xi_i = x_{i-1} = \frac{i-1}{n}.$$

Bundan

$$w_n = \sum_{i=1}^n f(\xi_i) \Delta x_i = \sum_{i=1}^n \frac{(i-1)^2}{n^2} \cdot \frac{1}{n} = \frac{(n-1)n(2n-1)}{6n^3} = \frac{(n-1)(2n-1)}{6n^2}$$

yoki

$$\int_0^1 x^2 dx = \lim_{d \rightarrow 0} w_n = \lim_{n \rightarrow \infty} \frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}.$$

Demak, berilgan integralning qiymati  $[0;1]$  kesmani bo'lish usuliga va bu kesmada  $\xi_i$  nuqtani tanlash usuliga bog'liq emas va  $\int_0^1 x^2 dx = \frac{1}{3}$ .

Aniq integral mavjud bo'lishi haqidagi teoremani isbotsiz keltiramiz.

**1-teorema.** (*Koshi teoremasi*). Agar  $y = f(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lsa, u holda  $\int_a^b f(x) dx$  aniq integral mavjud bo'ladi.

Funksiyaning uzluksiz bo'lishi uning integrallanuvchi bo'lishining yetarli sharti bo'ladi. Boshqacha aytganda,  $[a; b]$  kesmada uzilishga ega bo'lgan, ammo bu kesmada integrallanuvchi funksiyalar mavjud bo'lishi ham mumkin.

**2-teorema.**  $[a; b]$  kesmada chekli sondagi birinchi tur uzilish nuqtalariga ega bo'lgan funksiya bu kesmada integrallanuvchi bo'ladi.

### 7.5.3. Aniq integralning geometrik va mexanik ma'no'lari

Egri chiziqli trapetsiyaning yuzasi masalasiga qaytamiz. (5.2) tenglikning o'ng tomoni integral yig'indidan iborat. U holda

(5.5) formuladan *aniq integralning geometrik ma'nosi* kelib chiqadi: agar  $f(x)$  funksiya  $[a;b]$  kesmada integrallanuvchi va manfiy bo'lmasa, u holda  $[a;b]$  kesmada  $f(x)$  funksiyadan olingan aniq integral  $y=f(x)$ ,  $y=0$ ,  $x=a$  va  $x=b$  chiziqlar bilan chegaralangan egri chizikli trapetsiyaning yuzasiga teng.

2- misol.  $\int_{-3}^3 \sqrt{9-x^2} dx$  integralni uning geometrik ma'nosiga tayanib hisoblang.

*Yechish.* Bunda  $x$  ning  $-3$  dan  $3$  gacha o'zgarishida tenglamasi  $y=\sqrt{9-x^2}$  bo'lgan chiziq  $x^2+y^2=9$  aylananing  $Ox$  o'qidan yuqorida joylashgan bo'lagidan iborat bo'ladi. Shu sababli  $x=-3$ ,  $x=3$ ,  $y=0$  va  $y=\sqrt{9-x^2}$  chiziqlar bilan chegaralangan egri chizikli trapetsiya  $x^2+y^2=9$  doiraning yarmidan tashkil topadi.

Uning yuzi  $S=\frac{9\pi}{2}$  ga teng.

Demak,

$$\int_{-3}^3 \sqrt{9-x^2} dx = \frac{9\pi}{2}.$$

Endi bosib o'tilgan yo'l masalasiga o'tamiz. (5.3) tenglikning o'ng tomoni integral yig'indidan iborat bo'lgani uchun (5.5) formuladan ushbu xulosaga kelamiz: agar  $v(t)$  funksiya  $[a;b]$  kesmada integrallanuvchi va manfiy bo'lmasa, u holda  $v(t)$  tezlikdan  $[a;b]$  vaqt oralig'ida olingan aniq integral moddiy nuqtaning  $t=a$  dan  $t=b$  gacha vaqt oralig'ida bosib o'tgan yo'lga teng. Bu jumla *aniq integralning mexanik ma'nosini* anglatadi.

#### 7.5.4. Aniq integralning xossalari

1°. Agar integral ostidagi funksiya birga teng bo'lsa, u holda

$$\int_a^b dx = b - a$$

bo'ladi.

*Isboti.* Aniq integralning ta'rifiga ko'ra,

$$\int_a^b dx = \lim_{j \rightarrow 0} \sum_{i=1}^n 1 \cdot \Delta x_i = \lim_{j \rightarrow 0} \sum_{i=1}^n (x_i - x_{i-1}) = \lim_{j \rightarrow 0} (b - a) = b - a.$$

2°. O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, ya'ni

$$\int_a^b kf(x)dx = k \int_a^b f(x)dx, \quad k = const.$$

*Isboti.*  $\int_a^b kf(x)dx = \lim_{d \rightarrow 0} \sum_{i=1}^n kf(\xi_i)\Delta x_i = k \lim_{d \rightarrow 0} \sum_{i=1}^n f(\xi_i)\Delta x_i = k \int_a^b f(x)dx.$

3°. Chekli sondagi funksiyalar algebraik yig'indisining aniq integrali qo'shiluvchilar aniq integrallarining algebraik yig'indisiga teng, ya'ni

$$\int_a^b (f(x) \pm \varphi(x))dx = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx.$$

*Isboti.*  $\int_a^b (f(x) \pm \varphi(x))dx = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n (f(\xi_i) \pm \varphi(\xi_i))\Delta x_i =$   
 $= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i)\Delta x_i \pm \lim_{\lambda \rightarrow 0} \sum_{i=1}^n \varphi(\xi_i)\Delta x_i = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx.$

4°. Agar  $[a; b]$  kesma bir necha qismga bo'lingan bo'lsa, u holda  $[a; b]$  kesma bo'yicha olingan aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng bo'ladi.

Masalan,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad c \in [a; b]. \quad \bullet$$

*Isboti.*  $a < c < b$  bo'lsin deylik. Integral yig'indi  $[a; b]$  kesmani bo'lish usuliga bog'liq emas. Shu sababli  $c$  ni  $[a; b]$  kesmani bo'lish nuqtasi qilib olamiz. Masalan, agar  $c = x_m$  deb olsak, u holda  $w_n$  ni ikki yig'indiga ajratish mumkin:

$$w_n = \sum_{i=1}^m f(\xi_i)\Delta x_i = \sum_{i=1}^m f(\xi_i)\Delta x_i + \sum_{i=m+1}^n f(\xi_i)\Delta x_i.$$

Bunda  $d \rightarrow 0$  da limitga o'tamiz:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$a, b, c$  nuqtalarning boshqacha joylashishida ham xossa shu kabi isbotlanadi.

Masalan,  $a < b < c$  bo'lsa,  $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$  bo'ladi.

Bundan

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$$

yoki integrallash chegaralarining almashtirilishi xossaga ko'ra, ((5.6) ga qarang)

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

5°. Agar  $[a; b]$  kesmada funksiya o'z ishorasini o'zgartirmasa, u holda funksiya aniq integralning ishorasi funksiya ishorasi bilan bir xil bo'ladi, ya'ni:

$[a; b]$  da  $f(x) \geq 0$  bo'lganda,  $\int_a^b f(x)dx \geq 0$  bo'ladi;

$[a; b]$  da  $f(x) \leq 0$  bo'lganda,  $\int_a^b f(x)dx \leq 0$  bo'ladi.

*Isboti.*  $f(x) \geq 0$  funksiya uchun integral yig'indi  $w_n \geq 0$  bo'ladi, chunki  $f(\xi_n) \geq 0$  va  $\Delta x_i > 0$ . Bundan  $\int_a^b f(x)dx \geq 0$ . Shu kabi  $\Delta x_i > 0$ ,  $f(x) \leq 0$  ekanidan  $w_n \leq 0$  va  $\int_a^b f(x)dx \leq 0$  kelib chiqadi.

6°. Agar  $[a; b]$  kesmada  $f(x) \geq \varphi(x)$  bo'lsa, u holda

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$$

bo'ladi.

*Isboti.*  $f(x) \geq \varphi(x)$  dan  $f(x) - \varphi(x) \geq 0$  bo'ladi. U holda 5-xossaga ko'ra  $\int_a^b (f(x) - \varphi(x))dx \geq 0$  yoki 3-xossaga ko'ra,  $\int_a^b f(x)dx - \int_a^b \varphi(x)dx \geq 0$ .

Bundan

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx.$$

7°. Agar  $m$  va  $M$  sonlar  $f(x)$  funksiyaning  $[a; b]$  kesmadagi eng kichik va eng katta qiymatlari bo'lsa, u holda

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

bo'ladi.

*Isboti.* Shartga ko'ra,  $m \leq f(x) \leq M$ . U holda 6-xossaga ko'ra,

$$\int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx.$$

Bunda

$$\int_a^b m dx = m \int_a^b dx = m(b-a), \quad \int_a^b M dx = M \int_a^b dx = M(b-a).$$

U holda

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Bu xossa aniq integralni *baholash haqidagi teorema* deb yuritiladi.

8°. Agar  $f(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lsa, u holda shunday  $c \in [a; b]$  nuqta topiladiki,

$$\int_a^b f(x) dx = f(c)(b-a) \quad (5.8)$$

bo'ladi.

*Isboti.* 7-xossaga ko'ra,

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Bundan

$$m \leq \frac{\int_a^b f(x) dx}{b-a} \leq M.$$

$\frac{\int_a^b f(x) dx}{b-a} = \mu$  deymiz. U holda  $m \leq \mu \leq M$  bo'lgani uchun Bolsano-Koshining ikkinchi teoremasiga ko'ra, biror  $c \in [a; b]$  nuqta uchun  $f(c) = \mu$  bo'ladi.

Shu sababli

$$\frac{\int_a^b f(x) dx}{b-a} = f(c)$$

yoki

$$\int_a^b f(x) dx = f(c)(b-a).$$



8-xossa o'rtacha qiymat haqidagi teorema deb ataladi.

(5.8) formulaga o'rtacha qiymat formulasi,  $f(c)$  ga  $f(x)$  funksiyaning  $[a; b]$  kesmadagi o'rtacha qiymati deyiladi.

O'rtacha qiymat haqidagi teorema quyidagi geometrik talqinga ega: agar  $f(x) > 0$  bo'lsa, u holda  $\int_a^b f(x) dx$  integralning qiymati balandligi  $f(c)$  ga va asosi  $(b - a)$  ga teng bo'lgan to'g'ri to'rtburchakning yuzasiga teng bo'ladi.

Aniq integralning xossaligidan quyidagi natijalar kelib chiqadi.

**1-natija.**  $[a; b]$  kesmada aniqlangan  $f(x)$  funksiya uchun

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

bo'ladi.

**2-natija.** Agar  $[a; b]$  kesmada  $|f(x)| \leq k$  bo'lsa, u holda

$$\left| \int_a^b f(x) dx \right| \leq k(b - a), \quad (k = \text{const.})$$

bo'ladi.

**3-misol.**  $y = 2x + 2$  funksiyaning  $[-1; 2]$  kesmadagi o'rtacha qiymatini toping.

*Yechish.* O'rtacha qiymat haqidagi teoremdan topamiz:

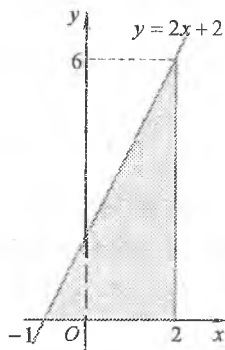
$$f_{o'n} = \frac{1}{b - a} \int_a^b f(x) dx.$$

Aniq integralning geometrik ma'nosiga ko'ra,  $\int_{-1}^2 (2x + 2) dx$  integralning qiymati 3-shaklda keltirilgan uchburchakning yuzasiga teng, ya'ni

$$S = \frac{1}{2} \cdot (2 + 1) \cdot 6 = 9.$$

Bundan

$$f_{o'n} = \frac{1}{2 - (-1)} \cdot 9 = 3.$$



3-shakl.

## 7.5.4. Mashqlar

1. Integrellarni aniq integralning geometrik ma'nosiga tayanib hisoblang:

$$1) \int_0^{\pi} \cos x dx;$$

$$2) \int_0^2 (3+x) dx;$$

$$3) \int_0^4 \sqrt{16-x^2} dx;$$

$$4) \int_{-2}^2 f(x) dx, \quad f(x) = \begin{cases} -x, & \text{agar } -2 \leq x \leq 0, \\ x, & \text{agar } 0 \leq x \leq 2. \end{cases}$$

2. Integrellarni taqqoslang:

$$1) I_1 = \int_0^{\frac{\pi}{4}} \cos x dx, \quad I_2 = \int_0^{\frac{\pi}{4}} \sin x dx;$$

$$2) I_1 = \int_{-1}^1 \sqrt{2-x^2} dx, \quad I_2 = \int_{-1}^1 x^2 dx.$$

$$3) I_1 = \int_{-2}^0 \sqrt{1-x^2} dx, \quad I_2 = \int_{-2}^0 (1-x) dx;$$

$$4) I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx, \quad I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x dx.$$

3. Integrellarni baholang:

$$1) I_1 = \int_0^{\frac{\pi}{3}} \frac{dx}{3-2\cos x};$$

$$2) I_2 = \int_1^3 \sqrt{1+3x^2} dx;$$

$$3) I_3 = \int_0^2 \sqrt{1+x^3} dx;$$

$$4) I_4 = \int_0^2 \frac{dx}{4-2x-x^2}.$$

4. Funktsiyalarning berilgan kesmalardagi o'rta qiymatini toping:

$$1) y = \sqrt{4-x^2}, \quad [-2,2];$$

$$2) y = |x|, \quad [-1,1];$$

$$3) y = 3x+2, \quad [1,3];$$

$$4) y = x^2 e^x, \quad [0,1].$$

## 7.6. ANIQ INTEGRALNI HISOBLASH

### 7.6.1. Yuqori chegarasi o'zgaruvchi aniq integral

$y = f(x)$  funksiya  $[a;b]$  kesmada aniqlangan va uzluksiz bo'lsin. U holda u ixtiyoriy  $[a;x]$  ( $a \leq x \leq b$ ) kesmada integrallanuvchi bo'ladi, ya'ni istalgan  $x \in [a;b]$  uchun

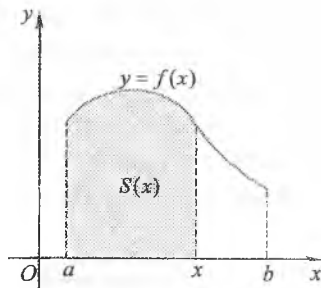
$$F(x) = \int_a^x f(t) dt \quad (6.1)$$

integral mavjud bo'ladi.

Agar ixtiyoriy  $t \in [a; b]$  da  $f(t) > 0$  bo'lsa, u holda  $F(x) = \int_a^x f(t) dt$  integral asosi  $[a; x]$  kesmadan iborat bo'lgan egri chiziqli trapetsiyaning o'zgaruvchi yuzasi  $S(x)$  ni ifodalaydi (4-shakl).

$[a; b]$  kesmada (6.1) tenglik bilan aniqlangan  $F(x)$  funksiyaga *yuqori chegarasi o'zgaruvchi aniq integral* deyiladi.

$F(x)$  funksiya  $[a; b]$  kesmada uzluksiz va differensiallanuvchi bo'ladi. Shuningdek, bunda  $F(x)$  funksiya uchun quyidagi fundamental teorema o'rinli bo'ladi.



4-shakl.

**1-teorema (Nyuton-Leybnis teoremasi).**  $[a; b]$  kesmada uzluksiz  $f(x)$  funksiyaning yuqori chegarasi o'zgaruvchi integrali  $F(x)$  dan yuqori chegara bo'yicha olingan hosila mavjud va u integral ostidagi funksiyaning yuqori chegaradagi qiymatiga teng bo'ladi, ya'ni

$$F'(x) = \left( \int_a^x f(t) dt \right)' = f(x), \quad x \in [a; b]. \quad (6.2)$$

*Isboti.*  $x \in [a; b]$  va  $x + \Delta x \in [a; b]$  bo'lsin.

U holda aniq integralning 4-xossasini qo'llab, topamiz:

$$F(x + \Delta x) = \int_a^{x + \Delta x} f(t) dt = \int_a^x f(t) dt + \int_x^{x + \Delta x} f(t) dt.$$

Bundan (6.1) tenglik va o'rta qiymat haqidagi teoreмага ko'ra,

$$\Delta F = F(x + \Delta x) - F(x) = \int_x^{x + \Delta x} f(t) dt = f(c) \Delta x, \quad \text{bu yerda } c \in [x, x + \Delta x].$$

$F(x)$  funksiyaning hosilasini aniqlaymiz:

$$F'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c) \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} f(c).$$

$\Delta x \rightarrow 0$  da  $x + \Delta x \rightarrow x$  va  $c \rightarrow x$ , chunki  $c \in [x, x + \Delta x]$ .

U holda  $f(x)$  funksiyaning uzluksizligidan

$$F'(x) = \lim_{\Delta x \rightarrow 0} f(c) = f(x)$$

bo'ladi.

Nyuton-Leybnis teoremasi matematik analizning asosiy teoremlaridan biri hisoblanadi. Bu teorema differensial bilan aniq integral tushunchalari orasidagi munosabatni ochib beradi.

Bu teoremadan  $[a; b]$  kesmada uzluksiz har qanday  $f(x)$  funksiya shu kesmada boshlang'ich funksiyaga ega bo'ladi va uning boshlang'ich funksiyalaridan biri yuqori chegarasi o'zgaruvchi  $F(x)$  integral bo'ladi degan xulosa kelib chiqadi.

$f(x)$  funksiyaning boshqa bir boshlang'ich funksiyasi  $F(x)$  funksiya o'zgarmas  $C$  songa farq qilgani uchun aniqlas va aniq integrallar orasidagi ushbu bog'lanish kelib chiqadi:

$$\int f(x)dx = \int_a^x f(t)dt + C, x \in [a; b]. \quad (6.3)$$

### 7.6.2. Nyuton-Leybnis formulasi

Aniq integralni integral yig'indining limiti sifatida hisoblash, hatto, oddiy funksiyalar uchun ham ancha qiyinchiliklar tug'diradi. Shu sababli aniq integralni hisoblashning (6.3) formulaga asoslangan, amaliy jihatdan qulay bo'lgan hamda keng qo'llaniladigan usuli bilan tanishamiz.

**2-teorema** (*integral hisobning asosiy teoremasi*). Agar  $F(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lgan  $f(x)$  funksiyaning boshlang'ich funksiyasi bo'lsa, u holda

$$\int_a^b f(x)dx = F(b) - F(a). \quad (6.4)$$

*Isboti.*  $F(x)$  funksiya  $f(x)$  funksiyaning boshlang'ich funksiyalaridan biri bo'lsin. U holda 1-teoremaga asosan,  $\int_a^x f(t)dt$  funksiya ham  $f(x)$  funksiyaning  $[a; b]$  kesmadagi boshlang'ich funksiyasi bo'ladi.

Boshlang'ich funksiyalar o'zgarmas  $C$  songa farq qilganidan

$$\int_a^x f(t)dt = F(x) + C.$$

Bu tenglikka  $x = a$  ni qo'yamiz va chegaralari teng bo'lgan aniq integralning xossasini qo'llaymiz:

$$\int_a^a f(t)dt = F(a) + C = 0.$$

Bundan  $C = -F(a)$ .

U holda istalgan  $x \in [a; b]$  uchun

$$\int_a^x f(t) dt = F(x) - F(a)$$

bo'ladi.

Oxirgi tenglikda  $x = b$  deymiz va  $t$  o'zgaruvchini  $x$  bilan almashtiramiz. Natijada (6.4) formula kelib chiqadi.

(6.4) formulaga *Nyuton-Leybnis formulasi* deyiladi.

$F(b) - F(a)$  ayirmani shartli ravishda  $F(x)|_a^b$  deb yozish kelishilgan.

Bu kelishuv natijasida Nyuton-Leybnis formulasi

$$\int_a^b f(x) dx = F(x)|_a^b \quad (6.5)$$

ko'inishda ifodalanadi.

*1-misol.*  $\int_0^3 \frac{dx}{\sqrt{1+x^2}}$  integralni hisoblang.

*Yechish.*  $\int_0^3 \frac{dx}{\sqrt{1+x^2}} = \ln|x + \sqrt{1+x^2}| \Big|_0^3 = \ln|3 + \sqrt{10}| - \ln 1 = \ln|3 + \sqrt{10}|$ .

*2-misol.*  $\int_1^4 \frac{dx}{x^2 - 2x + 10}$  integralni hisoblang.

*Yechish.*

$$\int_1^4 \frac{dx}{x^2 - 2x + 10} = \int_1^4 \frac{dx}{(x-1)^2 + 3^2} = \frac{1}{3} \arctg \frac{x-1}{3} \Big|_1^4 = \frac{1}{3} (\arctg 1 - \arctg 0) = \frac{\pi}{12}.$$

### 7.6.3. Aniq integralda o'zgaruvchini almashtirish

**3-teorema.**  $y = f(x)$  funksiya  $[a; b]$  kesmada uzluksiz bo'lsin.

Agar: 1)  $x = \varphi(t)$  funksiya  $[\alpha; \beta]$  kesmada differensiallanuvchi va  $\varphi'(t)$  funksiya  $[\alpha; \beta]$  kesmada uzluksiz; 2)  $x = \varphi(t)$  funksiyaning qiymatlar sohasi  $[a; b]$  kesmadan iborat; 3)  $\varphi(\alpha) = a$  va  $\varphi(\beta) = b$  bo'lsa, u holda

$$\int_a^b f(x) dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) dt \quad (6.6)$$

bo'ladi.

*Isboti.* Nyuton-Leybnis formulasiga ko'ra,

$$\int_a^b f(x)dx = F(b) - F(a),$$

bu yerda  $F(x)$  funksiya  $f(x)$  funksiyaning  $[a;b]$  kesmadagi boshlang'ich funksiyalaridan biri.  $\Phi(t) = F(\varphi(t))$  murakkab funksiyani qaraymiz.

Murakkab funksiyani differensiallash qoidasiga asosan,

$$\Phi'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t).$$

Demak,  $\Phi(t)$  funksiya  $[\alpha;\beta]$  kesmada  $f(\varphi(t))\varphi'(t)$  uzluksiz funksiya uchun boshlang'ich funksiya bo'ladi. Nyuton-Leybnis formulasi bilan topamiz:

$$\begin{aligned} \int_a^b f(\varphi(t))\varphi'(t)dt &= \Phi(\beta) - \Phi(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \\ &= F(b) - F(a) = \int_a^b f(x)dx. \end{aligned}$$

(6.6) formula *aniq integralda o'zgaruvchini almashtirish* formulasi deb yuritiladi. Aniq integralni hisoblashning bu usulida *aniq integralda o'rniga qo'yish* usuli deyiladi.

**Izoh.** Aniq integralni (6.6) formula bilan hisoblashda dastlabki eski o'zgaruvchiga qaytish shart emas, chunki integrallash chegarasi o'rniga qo'yishga mos tarzda o'zgaradi.

3-misol.  $\int_0^{\sqrt{3}} \sqrt{4-x^2} dx$  integralni hisoblang.

*Yechish.*  $x = 2\sin t$ ,  $0 \leq t \leq \frac{\pi}{3}$  belgilash kiritamiz. Bu o'zgaruvchini almashtirish 3-teoremaning barcha shartlarini qanoatlantiradi: birinchidan,  $f(x) = \sqrt{4-x^2}$  funksiya  $[0;\sqrt{3}]$  kesmada uzluksiz,

ikkinchidan,  $x = 2\sin t$  funksiya  $\left[0; \frac{\pi}{3}\right]$  kesmada differensiallanuvchi va  $x' = 2\cos t$  bu kesmada uzluksiz, uchinchidan  $t$  o'zgaruvchi 0 dan  $\frac{\pi}{3}$  gacha o'zgarganda  $x = 2\sin t$  funksiya 0 dan  $\sqrt{3}$  gacha o'sadi va bunda

$\varphi(0) = 0$  va  $\varphi\left(\frac{\pi}{3}\right) = \sqrt{3}$ . Bunda  $dx = 2\cos t dt$ .

(6.6) formuladan topamiz:

$$\int_0^{\sqrt{3}} \sqrt{4-x^2} dx = 4 \int_0^{\frac{\pi}{3}} \cos^2 t dt = 2 \int_0^{\frac{\pi}{3}} (1 + \cos 2t) dt = 2 \left( t + \frac{1}{2} \sin 2t \right) \Big|_0^{\frac{\pi}{3}} = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}.$$

4-misol.  $\int_0^1 x\sqrt{1+x^2} dx$  integralni hisoblang.

*Yechish.*  $t = \sqrt{1+x^2}$  o'rniga qo'yish bajaramiz.

U holda

$$x = \sqrt{t^2-1}, \quad dx = \frac{tdt}{\sqrt{t^2-1}}, \quad x=0 \text{ da } t=1, \quad x=1 \text{ da } t=\sqrt{2}.$$

[1;  $\sqrt{2}$ ] kesmada  $\sqrt{t^2-1}$  funksiya monoton o'sadi, demak o'rniga qo'yish to'g'ri bajarilgan.

Bundan

$$\int_0^1 x\sqrt{1+x^2} dx = \int_1^{\sqrt{2}} \sqrt{t^2-1} \cdot t \cdot \frac{tdt}{\sqrt{t^2-1}} = \int_1^{\sqrt{2}} t^2 dt = \frac{t^3}{3} \Big|_1^{\sqrt{2}} = \frac{2\sqrt{2}-1}{3}.$$

*Izoh.* (6.6) formulani qo'llashda teoremda sanab o'tilgan shartlarning bajarilishini tekshirish lozim. Agar bu shartlar buzilsa, keltirilgan formula bo'yicha o'zgaruvchini almashtirish xato natijaga olib kelishi mumkin.

### 7.6.4. Aniq integralni bo'laklab integrallash

**4-teorema.** Agar  $u(x)$  va  $v(x)$  funksiyalar  $u'(x)$  va  $v'(x)$  hosilalari bilan  $[a, b]$  kesmada uzluksiz bo'lsa, u holda

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du \quad (6.7)$$

bo'ladi.

*Isboti.* Teoremaning shartiga ko'ra,  $u(x)$  va  $v(x)$  funksiyalar hosilaga ega.

U holda ko'paytmanni differensiallash qoidasiga binoan,

$$(u(x)v(x))' = u(x)v'(x) + v(x)u'(x).$$

Bundan  $u(x)v(x)$  funksiya  $u(x)v'(x) + v(x)u'(x)$  funksiya uchun boshlang'ich funksiya bo'lishi kelib chiqadi.  $u(x)v'(x) + v(x)u'(x)$

funksiya  $[a; b]$  kesmada uzluksiz bo'lgani uchun u bu kesmada integrallanuvchi bo'ladi.

U holda aniq integralning 3-xossasiga va Nyuton-Leybnis formulasiga ko'ra

$$\int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx = u(x)v(x)\Big|_a^b.$$

Bundan,  $u'(x)dx = du(x)$  va  $v'(x)dx = dv(x)$  bo'lgani uchun

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du.$$

(6.7) formula *aniq integralni bo'laklab integrallash formulasi* deb ataladi.

5- misol.  $\int_0^1 xe^{-x} dx$  integralni hisoblang.

*Yechish.*

$$\begin{aligned} \int_0^1 xe^{-x} dx &= \left| \begin{array}{l} u = x, \quad dv = e^{-x} dx \\ du = dx, \quad v = -e^{-x} \end{array} \right| = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx = \\ &= -\frac{1}{e} - e^{-x}\Big|_0^1 = -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e}. \end{aligned}$$

### 7.6.5. Mashqlar

Berilgan integrallarni hisoblang:

1)  $\int_{-1}^2 (x^2 + 2x + 1) dx;$

2)  $\int_0^{\frac{\pi}{4}} \sin 4x dx;$

3)  $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x dx;$

4)  $\int_1^e \frac{dx}{x};$

5)  $\int_0^{\frac{\pi}{2}} \cos^2 x dx;$

6)  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\sin^2 x};$

7)  $\int_1^2 \frac{dx}{x+x^2};$

8)  $\int_0^1 (2x^3 + 1)x^2 dx;$

9)  $\int_0^1 x\sqrt{1+x^2} dx;$

10)  $\int_0^{\frac{\pi}{2}} \cos x \sin^3 x dx;$



$$11) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x dx}{1 + \cos x};$$

$$13) \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{3 + 4x^2};$$

$$15) \int_0^{\frac{\pi}{2}} \frac{\cos x dx}{6 - 5 \sin x + \sin^2 x};$$

$$17) \int_0^1 \arcsin x dx;$$

$$19) \int_0^{\pi} x \sin \frac{x}{2} dx;$$

$$21) \int_0^1 x^2 e^{3x} dx;$$

$$23) \int_0^{\frac{\pi}{4}} \sin \sqrt{x} dx;$$

$$12) \int_{\frac{\sqrt{3}}{3}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{4 - 9x^2}};$$

$$14) \int_0^{\frac{\pi}{2}} \sin^3 x dx;$$

$$16) \int_{\frac{\sqrt{2}}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} dx;$$

$$18) \int_1^e \ln^2 x dx;$$

$$20) \int_0^{\frac{\pi}{4}} e^x \sin 2x dx;$$

$$22) \int_1^{\sqrt{e}} x \ln x dx;$$

$$24) \int_0^{\frac{\pi}{2}} \cos(\ln x) dx.$$

## 7.7. XOSMAS INTEGRALLAR

Aniq integral qaralganida  $\int_a^b f(x) dx$  integral mavjud bo'lishi uchun ikkita shartning bajarilishi talab qilingan edi: 1) integrallash chegarasi chekli  $[a; b]$  kesmadan iborat bo'lishi; 2) integral ostidagi funksiya  $[a; b]$  kesmada aniqlangan va chegaralangan bo'lishi.

Aniq integral uchun keltirilgan shartlardan biri bajarilmaganda u *xosmas integral* deb ataladi: 1) faqat birinchi shart bajarilmasa, cheksiz chegarali xosmas integral (yoki I tur xosmas integral) deyiladi; 2) faqat ikkinchi shart bajarilmasa, uzilishga ega bo'lgan funksiyaning xosmas integrali (yoki II tur xosmas integral) deyiladi.

### 7.7.1. Cheksiz chegarali xosmas integrallar (I tur xosmas integrallar)

**1-ta'rif.**  $f(x)$  funksiya  $[a; +\infty)$  oraliqda uzluksiz bo'lsin. Agar  $\lim_{b \rightarrow +\infty} \int_a^b f(x) dx$  limit mavjud va chekli bo'lsa, bu limitga *yuqori chegarasi*

cheksiz xosmas integral deyiladi va  $\int_a^{+\infty} f(x)dx$  kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx = \lim_{b \rightarrow +\infty} \int_a^b f(x)dx. \quad (7.1)$$

Bu holda  $\int_a^{+\infty} f(x)dx$  integralga yaqinlashuvchi integral deyiladi.

Agar  $\lim_{b \rightarrow +\infty} \int_a^b f(x)dx$  limit mavjud bo'lmasa yoki cheksiz bo'lsa,

$\int_a^{+\infty} f(x)dx$  integralga uzoqlashuvchi integral deyiladi.

Quyida chegarasi cheksiz va har ikkala chegarasi cheksiz xosmas integrallar shu kabi ta'riflanadi:

$$\int_{-\infty}^b f(x)dx = \lim_{a \rightarrow -\infty} \int_a^b f(x)dx, \quad (7.2)$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a \rightarrow -\infty} \int_a^c f(x)dx + \lim_{b \rightarrow +\infty} \int_c^b f(x)dx, \quad (7.3)$$

bu yerda  $c$  — sonlar o'qining biror fiksirlangan nuqtasi. Bunda (7.3) tenglikning chap tomonidagi xosmas integral, o'ng tomondagi har ikkala xosmas integral yaqinlashgandagina yaqinlashadi.

1- misol.  $\int_1^{+\infty} \frac{dx}{x^\alpha}$  ( $\alpha > 0$ ) integralni yaqinlashishga tekshiring.

*Yechish.*  $\alpha \neq 1$  bo'lsin.

U holda

$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{b \rightarrow +\infty} \int_1^b \frac{dx}{x^\alpha} = \lim_{b \rightarrow +\infty} \left. \frac{x^{1-\alpha}}{1-\alpha} \right|_1^b = \frac{1}{1-\alpha} (\lim_{b \rightarrow +\infty} b^{1-\alpha} - 1).$$

Bunda: 1)  $\alpha < 1$  bo'lganda,  $\int_1^{+\infty} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha} (\lim_{b \rightarrow +\infty} b^{1-\alpha} - 1) = +\infty$ ,

2)  $\alpha > 1$  bo'lganda,  $\int_1^{+\infty} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha} (\lim_{b \rightarrow +\infty} b^{1-\alpha} - 1) = \frac{1}{1-\alpha}$ ,

3)  $\alpha = 1$  bo'lganda,  $\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{b \rightarrow +\infty} \int_1^b \frac{dx}{x} = \lim_{b \rightarrow +\infty} \ln x \Big|_1^b = \lim_{b \rightarrow +\infty} \ln b = +\infty$ .

Demak,  $\int_1^{+\infty} \frac{dx}{x^\alpha}$  xosmas integral  $\alpha > 1$  da yaqinlashadi va  $0 < \alpha \leq 1$  da

uzoqlashadi.

2- misol.  $\int_{-\infty}^0 x \cos x dx$  integralni yaqinlashishga tekshiring.

$$\begin{aligned} \text{Yechish. } \int_{-\infty}^0 x \cos x dx &= \lim_{a \rightarrow -\infty} \int_a^0 x \cos x dx = \lim_{a \rightarrow -\infty} \left( x \sin x \Big|_a^0 - \int_a^0 \sin x dx \right) = \\ &= \lim_{a \rightarrow -\infty} (-a \sin a + \cos a) = \lim_{a \rightarrow -\infty} (-a \sin a - \cos a + 1). \end{aligned}$$

Bu limit mavjud emas. Shu sababli  $\int_{-\infty}^0 x \cos x dx$  integral uzoqlashadi.

3- misol.  $\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 6x + 10}$  integralni yaqinlashishga tekshiring.

Yechish. Oraliq nuqtani  $c = 0$  deymiz.

U holda (7.3) tenglikga ko'ra,

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 6x + 10} = \int_{-\infty}^0 \frac{dx}{x^2 + 6x + 10} + \int_0^{+\infty} \frac{dx}{x^2 + 6x + 10}.$$

Bundan

$$\begin{aligned} \int_{-\infty}^0 \frac{dx}{x^2 + 6x + 10} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{(x+3)^2 + 1} = \lim_{a \rightarrow -\infty} \arctg(x+3) \Big|_a^0 = \\ &= \arctg 3 - \lim_{a \rightarrow -\infty} \arctg(a+3) = \arctg 3 + \frac{\pi}{2}, \end{aligned}$$

$$\int_0^{+\infty} \frac{dx}{x^2 + 6x + 10} = \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{(x+3)^2 + 1} = \lim_{b \rightarrow +\infty} \arctg(x+3) \Big|_0^b =$$

$$= \lim_{b \rightarrow +\infty} \arctg(b+3) - \arctg 3 = \frac{\pi}{2} - \arctg 3.$$

U holda

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{dx}{x^2 + 6x + 10} &= \int_{-\infty}^0 \frac{dx}{x^2 + 6x + 10} + \int_0^{+\infty} \frac{dx}{x^2 + 6x + 10} = \\ &= \arctg 3 + \frac{\pi}{2} + \frac{\pi}{2} - \arctg 3 = \pi. \end{aligned}$$

Demak, xosmas integral yaqinlashadi.

## 7.7.2. Uzulishga ega bo'lgan funksiyalarning xosmas integrallari ( II tur xosmas integrallar)

2-ta'rif.  $f(x)$  funksiya  $[a; b)$  oraliqda aniqlangan va uzluksiz bo'lib,  $x = b$  da cheksiz uzulishga ega bo'lsin. Agar  $\lim_{\varepsilon \rightarrow 0} \int_a^{b-\varepsilon} f(x) dx$  limit mavjud va chekli bo'lsa, bu limitga *uzulishga ega bo'lgan funksiyaning xosmas integrali* deyiladi va  $\int_a^b f(x) dx$  kabi belgilanadi:

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{b-\varepsilon} f(x) dx. \quad (7.4)$$

Shu kabi: 1)  $f(x)$  funksiya  $x$  ning  $a$  ga o'ngdan yaqinlashishida uzulishga ega bo'lsa,

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^b f(x) dx; \quad (7.5)$$

bo'ladi;

2) agar  $f(x)$  funksiya  $c \in [a; b]$  da uzulishga ega bo'lsa,

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{c-\varepsilon} f(x) dx + \lim_{\varepsilon \rightarrow 0} \int_{c+\varepsilon}^b f(x) dx \quad (7.6)$$

bo'ladi.

II tur xosmas integrallar uchun yaqinlashish (uzoqlashish) tushunchalari I tur integrallardagi kabi kiritiladi.

4- misol.  $\int_{-1}^1 \frac{dx}{x^3 \sqrt{x}}$  integralni yaqinlashishga tekshiring.

*Yechish.*  $x = 0$  da integral ostidagi funksiya uzulishga ega. U holda (7.6) tenglikka ko'ra,

$$\begin{aligned} \int_{-1}^1 \frac{dx}{x^3 \sqrt{x}} &= \lim_{\varepsilon \rightarrow 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^3 \sqrt{x}} + \lim_{\varepsilon \rightarrow 0} \int_{+\varepsilon}^1 \frac{dx}{x^3 \sqrt{x}} = -3 \lim_{\varepsilon \rightarrow 0} x^{-\frac{3}{2}} \Big|_{-1}^{-\varepsilon} - 3 \lim_{\varepsilon \rightarrow 0} x^{-\frac{3}{2}} \Big|_{+\varepsilon}^1 = \\ &= 3 \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\frac{3}{2}} - 1 - 1 + 3 \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\frac{3}{2}} = 6 \left( \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\frac{3}{2}} - 1 \right) = +\infty. \end{aligned}$$

Demak, xosmas integral uzoqlashadi. Berilgan integralga Nyuton-Leybnis formulasi formal qo'llanilishi xato natijaga olib keladi:

$$\int_{-1}^1 \frac{dx}{x^3 \sqrt{x}} = -\frac{3}{\sqrt{x}} \Big|_{-1}^1 = -6.$$

### 7.7.3. Xosmas integrallarning yaqinlashish alomatlari

Ko'pincha, xosmas integralni (7.1) - (7.6) formulalar orqali hisoblash shart bo'lmaganidan, faqat uning yaqinlashuvchi yoyi uzoqlashuvchi bo'lishini bilish yetarli bo'ladi. Bunday hollarda berilgan integral yaqinlashishga yaqinlashuvchi yoki uzoqlashuvchiligi oldindan ma'lum bo'lgan boshqa xosmas integral bilan taqqoslash orqali tekshiriladi. Xosmas integrallarning taqqoslash alomatlarini ifodalovchi teoremlarni isbotsiz keltiramiz.

**1-teorema** (*I tur xosmas integralning yaqinlashish alomati*).  $[a; +\infty)$  oraliqda  $f(x)$  va  $\varphi(x)$  funksiyalar uzluksiz va  $0 \leq f(x) \leq \varphi(x)$  bo'lsin. U holda:

1) agar  $\int_a^{+\infty} \varphi(x) dx$  integral yaqinlashsa,  $\int_a^{+\infty} f(x) dx$  integral ham yaqinlashadi;

2) agar  $\int_a^{+\infty} f(x) dx$  integral uzoqlashsa,  $\int_a^{+\infty} \varphi(x) dx$  integral ham uzoqlashadi.

**5-misol.**  $\int_0^{+\infty} e^{-x^2} dx$  integralni yaqinlashishga tekshiring.

*Yechish.* Puasson integrali deb ataluvchi bu integral boshlang'ich funksiyaga ega emas. Uni ikkita integralning yig'indisi ko'rinishida ifodalaymiz:

$$\int_0^{+\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{+\infty} e^{-x^2} dx.$$

Tenglikning chap qismidagi integrallardan birinchisi chekli qiymatga ega bo'lgan aniq integral. Ikkinchi xosmas  $\int_1^{+\infty} e^{-x^2} dx$  integralni yaqinlashishga tekshiramiz.

$[1; +\infty)$  oraliqda  $0 < e^{-x^2} \leq e^{-x}$  bo'ladi,  $e^{-x^2}$  va  $e^{-x}$  funksiyalar uzluksiz.

Bunda

$$\int_1^{+\infty} e^{-x^2} dx = \lim_{b \rightarrow +\infty} \int_1^b e^{-x^2} dx = \lim_{b \rightarrow +\infty} (-e^{-x}) \Big|_1^b = \frac{1}{e} - \lim_{b \rightarrow +\infty} \frac{1}{e^b} = \frac{1}{e}.$$

Demak, bu integral yaqinlashadi va 1-teoremaning 1) bandiga asosan, Puasson integrali ham yaqinlashadi.

6-misol.  $\int_0^1 \frac{dx}{e^x - \cos x}$  integralni yaqinlashishga tekshiring.

Yechish. Integral ostidagi funksiya  $x=0$  da uzulishga ega.

$$x \in (0;1] \text{ da } \frac{1}{e^x - \cos x} \geq \frac{1}{xe}.$$

Bundan

$$\int_0^1 \frac{dx}{xe} = \frac{1}{e} \lim_{\varepsilon \rightarrow 0} \int_{+\varepsilon}^1 \frac{dx}{x} = \frac{1}{e} \lim_{\varepsilon \rightarrow 0} \ln x \Big|_{+\varepsilon}^1 = \frac{1}{e} (0 - \lim_{\varepsilon \rightarrow 0} \ln |\varepsilon|) = +\infty.$$

Demak,  $\int_0^1 \frac{dx}{xe}$  integral uzoqlashadi va 1-teoremaning 2) bandiga

asosan berilgan integral ham uzoqlashadi.

**2-teorema (II tur xosmas integralning yaqinlashish alomati).**  $[a;b]$  oraliqda  $f(x)$  va  $\varphi(x)$  funksiyalar uzluksiz bo'lsin va  $0 \leq f(x) \leq \varphi(x)$  tengsizlikni qanoatlantirsin,  $x=b$  da  $f(x)$  va  $\varphi(x)$  funksiyalar cheksiz uzilishga ega bo'lsin. U holda:

1) agar  $\int_a^b \varphi(x) dx$  integral yaqinlashsa,  $\int_a^b f(x) dx$  integral ham yaqinlashadi;

2) agar  $\int_a^b f(x) dx$  integral uzoqlashsa,  $\int_a^b \varphi(x) dx$  integral ham uzoqlashadi.

Taqqoslash teoremlari integral ostidagi funksiya bir xil ishorali bo'lganida o'rinli bo'ladi. Ishorasi almashinadigan funksiyalarning xosmas integrallari uchun quyidagi alomat o'rinli bo'ladi.

**3-teorema.** Agar  $\int_a^{+\infty} |f(x)| dx$   $\left( \int_a^b |f(x)| dx \right)$  integral yaqinlashsa,

$\int_a^{+\infty} f(x) dx$   $\left( \int_a^b f(x) dx \right)$  integral ham yaqinlashadi.

**3-ta'rif.** Agar  $\int_a^{+\infty} |f(x)| dx$   $\left( \int_a^b |f(x)| dx \right)$  integral yaqinlashsa,

$\int_a^{+\infty} f(x) dx$   $\left( \int_a^b f(x) dx \right)$  integralga *absolut yaqinlashuvchi xosmas integral* deyiladi.

**4-ta'rif.** Agar  $\int_a^{+\infty} f(x) dx \left( \int_a^h f(x) dx \right)$  integral yaqinlashsa va

$\int_a^{+\infty} |f(x)| dx \left( \int_a^h |f(x)| dx \right)$  integral uzoqlashsa,  $\int_a^{+\infty} f(x) dx \left( \int_a^h f(x) dx \right)$

integralga *shartli yaqinlashuvchi xosmas integral* deyiladi.

7 misol.  $\int_1^{+\infty} \frac{\cos x}{x^2} dx$  integralni yaqinlashishga tekshiring.

*Yechish.* Integral ostidagi funksiya  $[1; +\infty)$  oraliqda ishorasini almashtiradi.

Ma'lumki,  $\left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2}$ .

1-misolga ko'ra,  $\int_1^{+\infty} \frac{dx}{x^2}$  integral yaqinlashuvchi.

U holda 1-teoreмага asosan,  $\int_1^{+\infty} \left| \frac{\cos x}{x^2} \right| dx$  integral yaqinlashuvchi va

3-teorema va 3-ta'rifga ko'ra,  $\int_1^{+\infty} \frac{\cos x}{x^2} dx$  integral absolut yaqinlashuvchi

bo'ladi.

(7.2), (7.3) ko'rinishdagi ((7.5),(7.6) ko'rinishdagi) xosmas integrallar uchun taqqoslash alomatleri hamda absolut va shartli yaqinlashish tushunchalari yuqorida (7.1) ko'rinishdagi ((7.4) ko'rinishdagi) integrallar uchun keltirilgandagi kabi kiritiladi.

### 7.7.4. Mashqlar

1. Berilgan integrallarni hisoblang yoki uzoqlashuvchi ekanini aniqlang:

1)  $\int_1^{+\infty} \frac{dx}{1+x^2}$ ;

2)  $\int_0^{+\infty} x e^{-x^2} dx$ ;

3)  $\int_{-\infty}^0 x \cos x dx$ ;

4)  $\int_2^{+\infty} \frac{\ln x dx}{x}$ ;

5)  $\int_2^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$ ;

6)  $\int_1^{+\infty} \frac{\arctg x dx}{x^2}$ ;

7)  $\int_0^{+\infty} e^{-x} \sin x dx;$

8)  $\int_1^e \frac{dx}{x\sqrt{\ln x}};$

9)  $\int_0^1 \frac{dx}{\sqrt{1-x^2}};$

10)  $\int_1^3 \frac{x dx}{\sqrt{(x-1)}};$

11)  $\int_{-1}^1 \frac{3x^2+2}{\sqrt{x^2}} dx;$

12)  $\int_0^2 \frac{dx}{x^2-4x+3};$

13)  $\int_{-1}^1 \frac{dx}{x^3\sqrt{x}};$

14)  $\int_{-\infty}^{+\infty} \frac{dx}{x^2+6x+10}.$

2. Integrellarni yaqinlashishga tekshiring:

1)  $\int_1^{+\infty} \frac{dx}{x^\alpha};$

2)  $\int_0^{+\infty} \frac{dx}{\sqrt{1+x^3}};$

3)  $\int_0^{+\infty} \sqrt{x} e^{-x} dx;$

4)  $\int_1^{+\infty} \frac{\sin x dx}{x^2};$

5)  $\int_1^{+\infty} \frac{x^3+1}{x^4} dx;$

6)  $\int_0^1 \frac{dx}{e^{\sqrt{x}}-1};$

7)  $\int_{\frac{\pi}{2}}^1 \frac{e^x dx}{\sqrt{1-\cos x}};$

8)  $\int_0^1 \frac{dx}{e^x - \cos x};$

9)  $\int_0^{\frac{\pi}{2}} \frac{3+\sin x}{(x-1)^3} dx;$

10)  $\int_0^1 \frac{\sqrt{x} dx}{\sqrt{1-x^4}};$

11)  $\int_1^{+\infty} \frac{\cos x}{x^2} dx;$

12)  $\int_0^{+\infty} e^{-x} \sin x dx.$

## 7.8. ANIQ INTEGRALNING TATBIQLARI

### 7.8.1. Aniq integralning qo'llanilish sxemalari

$x$  o'zgaruvchi aniqlangan  $[a;b]$  kesmaga bog'liq biror geometrik yoki fizik  $A$  kattalikning qiymatini (tekis shakl yuzasini, jism hajmini, kuchning bajargan ishini va hokazo) hisoblash talab qilingan bo'lsin. Bunda  $A$  kattalik additiv deb faraz qilinadi, ya'ni  $[a;b]$  kesmaning  $c \in [a;b]$  nuqta bilan  $[a;c]$  va  $[c;b]$  qismlarga bo'linishida  $A$  kattalikning  $[a;b]$  kesmaga mos qiymati uning  $[a;c]$  va  $[c;b]$  kesmalarga mos qiymatlarining yig'indisiga teng deb qaraladi.



$A$  kattalikning qiymatini hisoblash ma'lum tartibda (sxema asosida) bajariladi. Bunda ikki sxemadan biriga amal qilish mumkin: I sxema (integral yig'indilar usuli) va II sxema (differensial usuli).

I sxema aniq integralning ta'rifi asoslanadi. Bunda:

1°.  $[a; b]$  kesma  $a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = b$  nuqtalar bilan  $n$  ta kichik kesmalarga bo'linadi. Bunda  $A$  kattalik mos  $n$  ta  $\Delta A_i$ , ( $i = \overline{1, n}$ ) «elementar qo'shiluvchilar» ga bo'linadi:

$$A = \Delta A_1 + \Delta A_2 + \dots + \Delta A_n;$$

2°. Har bir «elementar qo'shiluvchi» masalaning shartidan aniqlanuvchi funksiyaning mos kesma istalgan nuqtasida hisoblangan qiymati bilan kesmaning uzunligi ko'paytmasi ko'rinishi keltiriladi:

$$\Delta A_i \approx f(\xi_i) \Delta x_i;$$

$\Delta A_i$  ning taqribiy qiymatini topishda ayrim soddalashtirishlar qilish mumkin: kichik bo'lakda yoyni uning chekkalarini tortib turuvchi vatar bilan almashtirish mumkin; kichik bo'lakda o'zgaruvchi tezlikni o'zgarmas deyish mumkin va hokazo.

Bunda  $A$  kattalikning taqribiy qiymati integral yig'indidan iborat bo'ladi:

$$A \approx \sum_{i=1}^n f(\xi_i) \Delta x_i.$$

3°.  $A$  kattalikning haqiqiy qiymati bu integral yig'indining  $n \rightarrow \infty$  dagi (bunda  $\lambda = \max_{a \leq x \leq b} \Delta x_i$ ) limitiga teng bo'ladi:

$$A = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \int_a^b f(x) dx.$$

II sxema I sxemaning o'zgaragan ko'rinishi hisoblanadi va «differensial usul» yoki «yuqori tartibli cheksiz kichiklarni tashlab yuborish usuli» deb ataladi.

Bunda:

1°.  $[a; b]$  kesmada ixtiyoriy  $x$  qiymatni tanlaymiz va o'zgaruvchi  $[a; x]$  kesmani qaraymiz. Bu kesmada  $A$  kattalik  $x$  ning funksiyasi bo'ladi:  $A = A(x)$ , ya'ni  $A$  kattalikning bo'lagi noma'lum  $A(x)$  funksiya bo'ladi;

2°.  $x$  ning kichik  $\Delta x = dx$  kattalikka o'zgarishida  $\Delta A$  orttirmaning bosh qismini topamiz:  $dA = f(x) dx$ , bu yerda  $f(x) = x$

o'zgaruvchining masala shartidan kelib chiquvchi funksiyasi (bunda mumkin bo'lgan soddalashtirishlar qilinishi mumkin);

3°.  $\Delta A \approx dA$  deb,  $dA$  ni  $a$  dan  $b$  gacha integrallash orqali  $A$  ning izlanayotgan qiymati topiladi:

$$A = \int_a^b f(x) dx.$$

### 7.8.2. Tekis shakl yuzasini hisoblash

#### *Yuzani dekart koordinatalarida hisoblash*

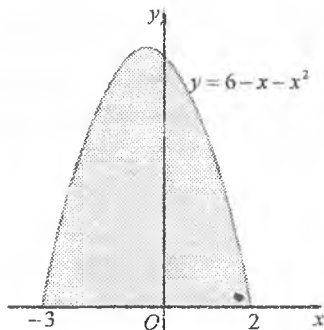
Aniq integralning geometrik ma'nosiga asosan, absissalar o'qidan yuqorida yotgan, ya'ni yuqoridan  $y = f(x)$  ( $f(x) \geq 0$ ) funksiya grafiği bilan, quyidan  $Ox$  o'q bilan, yon tomonlaridan  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan egri chizikli trapetsiyaning yuzasi

$$S = \int_a^b f(x) dx \quad (8.1)$$

integralga teng bo'ladi.

Shu kabi, absissalar o'qidan pastda yotgan, ya'ni quyidan  $y = f(x)$  ( $f(x) \leq 0$ ) funksiya grafiği bilan, yuqoridan  $Ox$  o'q bilan, yon tomonlaridan  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan egri chizikli trapetsiyaning yuzasi

$$S = -\int_a^b f(x) dx \quad (8.2)$$



5-shakl.

integregalga teng bo'ladi.

*1-misol.*  $Ox$  o'q va  $y = 6 - x - x^2$  parabola bilan chegaralangan tekis shakl yuzasini toping.

*Yechish.* Parabolaning  $Ox$  o'q bilan kesishish nuqtasini topamiz (5-shakl):

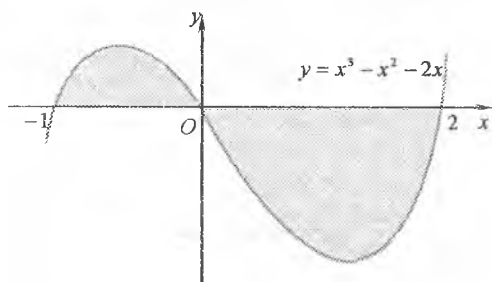
$$y = 0 = 6 - x - x^2 = (3 + x)(2 - x), \quad x_1 = -3, \quad x_2 = 2.$$

Tekis shakl yuzasini (8.1) formula bilan topamiz:

$$s = \int_{-3}^2 (6 - x - x^2) dx = \left( 6x - \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_{-3}^2 =$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 27\right) = 20\frac{5}{6}.$$

Yuzani hisoblashga oid murakkabroq masalalar yuzaning additivlik xossasiga asoslangan holda yechiladi. Bunda tekis shakl kesishmaydigan qismlarga ajratiladi va aniq integralning 4-xossasiga ko'ra tekis shaklning yuzasi qismlar yuzalarining yig'indisiga teng bo'ladi.



2-misol.  $y = x^3 - x^2 - 2x$   
va  $y = 0$  chiziqlar bilan

6-shakl.

chegaralangan tekis shakl yuzasini hisoblang (6-shakl).

*Yechish* Berilgan tekis shaklni yuzalari  $S_1$  va  $S_2$  bo'lgan kesishmaydigan qismlarga ajratamiz. U holda yuzaning additivlik xossasiga asosan, berilgan tekis shaklning yuzasi qismlar yuzalarining yig'indisiga teng bo'ladi.

Demak,

$$\begin{aligned} S &= S_1 + S_2 = \int_{-1}^0 (x^3 - x^2 - 2x) dx - \int_0^2 (x^3 - x^2 - 2x) dx = \\ &= \left( \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_{-1}^0 - \left( \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_0^2 = \\ &= -\left( \frac{1}{4} + \frac{1}{3} - 1 \right) + \left( 4 - \frac{8}{3} - 4 \right) = \frac{37}{12}. \end{aligned}$$

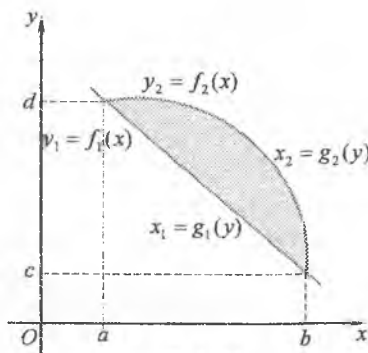
$[a; b]$  kesmada ikkita  $y_1 = f_1(x)$  va  $y_2 = f_2(x)$  uzliksiz funksiyalar berilgan va  $x \in [a; b]$  da  $f_2(x) \geq f_1(x)$  bo'lsin. Bu funksiyalarning grafiklari va  $x = a$ ,  $x = b$  to'g'ri chiziqlar bilan chegaralangan tekis shaklni qaraymiz.

Bu tekis shaklning yuzasi yuqoridan  $y_2 = f_2(x)$  va  $y_1 = f_1(x)$  funksiyalar grafiklari bilan, quyidan  $Ox$  o'q bilan, yon tomonlardan  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyalar yuzalarining ayirmasiga teng bo'ladi:

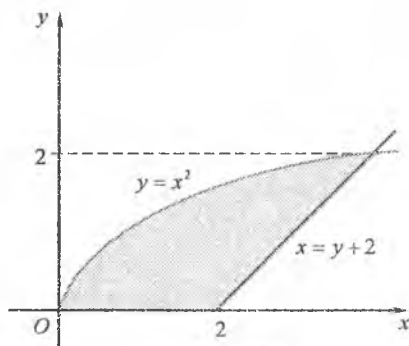
$$S = \int_a^b f_2(x) dx - \int_a^b f_1(x) dx = \int_a^b (f_2(x) - f_1(x)) dx. \quad (8.3)$$

Ayrim hollarda yuzani hisoblashga oid masalalar yuzaning ko'chishga nisbatan invariantlik xossasidan foydalangan holda soddalashtiriladi. Bunda tekis shakl yuzasi (8.3) formulada  $x$  va  $y$  o'zgaruvchilar ( $Ox$  va  $Oy$  o'qlar) ning o'rnini almashtirish yo'li bilan hisoblanadi, ya'ni (7-shakl).

$$S = \int_a^b (f_2(x) - f_1(x)) dx = \int_c^d (g_2(y) - g_1(y)) dy. \quad (8.4)$$



7-shakl.



8-shakl.

**3-misol.**  $x = y^2$  va  $x = y + 2$  chiziqlar bilan chegaralangan tekis shaklning absissalar o'qidan yuqorida yotgan qismining yuzasini hisoblang (8-shakl).

*Yechish.* Berilgan chiziqlarning kesishish nuqtalarini topamiz:

$$y^2 = y + 2, \quad y^2 - y - 2 = 0, \quad y_1 = 0, \quad y = 2.$$

Tekis shakl yuzasini (8.4) formula bilan topamiz:

$$s = \int_0^2 (y + 2 - y^2) dy = \left( \frac{y^2}{2} + 2y - \frac{y^3}{3} \right) \Big|_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}.$$

Agar egri chiziqli trapetsiya yuqoridan  $x = \varphi(t)$ ,  $y = \psi(t)$ ,  $\alpha \leq t \leq \beta$  parametrik tenglamalar bilan berilgan funksiya grafigi bilan chegaralangan va  $a = \varphi(\alpha)$ ,  $b = \varphi(\beta)$  bo'lsa, (8.1) formulada  $x = \varphi(t)$ ,  $dx = \varphi'(t)dt$  ko'rinishdagi o'rniga qo'yish orqali o'zgaruvchi almashtiriladi.

U holda

$$S = \int_a^b \psi(t) \varphi'(t) dt \quad (8.5)$$

bo'ldi, bu yerda,  $a = \varphi(\alpha)$  va  $b = \varphi(\beta)$ .

*4-misol.* Radiusi  $R$  ga teng doira yuzasini hisoblang.

*Yechish.* Koordinatalar boshini doiraning markaziga joylashtiramiz. Bu doiraning aylanasi  $x = R \cos t$ ,  $y = R \sin t$  parametrik tenglamalar bilan aniqlanadi va doira koordinata o'qlariga nisbatan simmetrik bo'ldi. Shu sababli uning birinchi chorakdagi yuzasini hisoblaymiz (bunda  $x$  o'zgaruvchi 0 dan  $R$  gacha o'zgaranda  $t$  parametr  $\frac{\pi}{2}$  dan 0 gacha o'zgaradi) va natijani to'rtga ko'paytiramiz. U holda (8.5) formulaga ko'ra:

$$\begin{aligned} S &= 4S_1 = 4 \int_0^{\frac{\pi}{2}} R \sin t (-R \sin t) dt = 4R^2 \int_0^{\frac{\pi}{2}} \sin^2 t dt = \\ &= 2R^2 \int_0^{\frac{\pi}{2}} (1 - \cos 2t) dt = 2R^2 \left( t - \frac{\sin 2t}{2} \right) \Big|_0^{\frac{\pi}{2}} = \pi R^2. \end{aligned}$$

### *Yuzani qutb koordinatalarida hisoblash*

Qutb koordinatalar ( $r$  – qutb radiusi,  $\varphi$  – qutb burchagi) sistemasida berilgan  $r = r(\varphi)$  funksiya  $\varphi \in [\alpha; \beta]$  kesmada uzluksiz bo'lsin.

$r = r(\varphi)$  funksiya grafigi hamda  $O$  qutbdan chiqqan  $\varphi = \alpha$  va  $\varphi = \beta$  nurlar bilan chegaralangan tekis shaklga *egri chiziqli sektor* deyiladi.

$AOB$  egri chiziqli sektor yuzasini hisoblaymiz. Bunda *II* sxemadan foydalanamiz.

1°. Qutbdan chiqqan  $\varphi$  va  $\alpha$  burchaklarga mos nurlar bilan chegaralangan egri chiziqli sektor yuzi  $\varphi$  burchakning  $S = S(\varphi)$  funksiyasi bo'lsin, bu yerda  $\alpha \leq \varphi \leq \beta$  ( $\varphi = \alpha$  bo'lganda  $S(\alpha) = 0$ ,  $\varphi = \beta$  bo'lganda  $S(\beta) = S$ ).

2°. Joriy  $\varphi$  qutb burchak  $\Delta\varphi = d\varphi$  orttirma olganida  $\Delta S$  yuzi  $OAB$  «elementar egri chiziqli sektor» yuziga teng orttirma oladi.

Bunda  $dS$  differensial  $\Delta S$  orttirmaning  $d\varphi \rightarrow 0$  dagi orttirmaning bosh qismini ifodalaydi va radiusi  $r$  ga, markaziy burchagi  $d\varphi$  ga teng bo'lgan  $OAC$  doiraviy sektor yuziga teng bo'ldi (9-shakl).

Shu sababli

$$dS = \frac{1}{2} r^2 d\varphi.$$

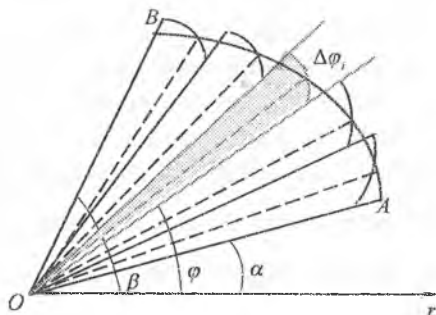
3°. Oxirgi ifodani  $\varphi = \alpha$  dan  $\varphi = \beta$  gacha integrallab izlanayotgan yuzani topamiz:

$$S = \frac{1}{2} \int_{\alpha}^{\beta} r^2(\varphi) d\varphi. \quad (8.6)$$

5-misol.  $r = 2\cos 3\varphi$  egri chiziq bilan chegaralangan figuraning yuzasini hisoblang.

*Yechish.*  $r = 2\cos 3\varphi$  egri chiziq bilan chegaralangan figuraga uch yaproqli gul deyiladi (1-ilovaga qarang).

Uning oltidan bir qismi yuzasini hisoblaymiz:



9-shakl.

$$\frac{1}{6} S = \frac{1}{2} \int_0^{\frac{\pi}{6}} 4 \cos^2 3\varphi d\varphi = \int_0^{\frac{\pi}{6}} (1 + \cos 6\varphi) d\varphi = \left( \varphi + \frac{\sin 6\varphi}{6} \right) \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{6}.$$

Bundan  $S = \pi$ .

Agar egri chiziqli sektor  $r_1 = r_1(\varphi)$  va  $r_2 = r_2(\varphi)$  ( $\varphi \in [\alpha; \beta]$  da  $r_2(\varphi) > r_1(\varphi)$ ) funksiyalar grafiklari bilan chegaralangan bo'lsa,

$$S = \frac{1}{2} \int_{\alpha}^{\beta} (r_2^2(\varphi) - r_1^2(\varphi)) d\varphi \quad (8.7)$$

bo'ladi.

### 7.8.3. Yassi egri chiziq yoyi uzunligini hisoblash

$AB$  egri chiziq  $y = f(x) \geq 0$  funksiyaning grafigi bo'lsin. Bunda  $x \in [a; b]$ ,  $y = f(x)$  funksiya va  $y' = f'(x)$  hosila bu kesmada uluksiz bo'lsin.

$AB$  egri chiziq uzunligi  $l$  ni  $\Pi$  sxemadan foydalangan holda topamiz.

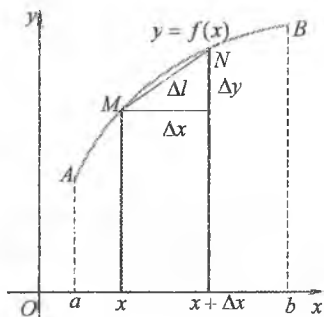
1°.  $[a; b]$  kesmada ixtiyoriy  $x$  nuqtani tanlaymiz va o'zgaruvchi  $[a; x]$  kesmani qaraymiz.  $(x, f(x))$  nuqta  $M$  bo'lsin.

$\overline{AM}$  yoy uzunligi  $l_{AM}$   $x$  ning funksiyasi bo'ladi:  $l=l(x)$  ( $l(a)=0$  va  $l(b)=l$ ).

2°.  $x$  ning kichik  $\Delta x = dx$  kattalikka o'zgarishida  $dl$  differensialni topamiz:  $dl = l'(x)dx$ .  $\overline{MN}$  yoyni uni tortib turuvchi vatar bilan almashtiramiz va  $l'(x)$  ni topamiz (10-shakl):

$$l'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta l}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} = \sqrt{1 + (y'_x)^2}.$$



10-shakl.

Demak,

$$dl = \sqrt{1 + (y'_x)^2} dx.$$

3°.  $dl$  ni  $a$  dan  $b$  gacha integrallaymiz:

$$l = \int_a^b \sqrt{1 + (y'_x)^2} dx. \quad (8.8)$$

(8.7) tenglikka yoy differensialining to'g'ri burchakli koordinatalardagi formulasi deyiladi.

Agar egri chiziq  $x = g(y)$ ,  $y \in [c; d]$ , tenglama bilan berilgan bo'lsa, yuqorida keltirilganlarni takrorlab,  $AB$  yoy uzunligini hisoblashning quyidagi formulasini hosil qilamiz:

$$l = \int_c^d \sqrt{1 + (x'_y)^2} dy. \quad (8.9)$$

6-misol.  $y = \frac{3}{8}x^{\frac{4}{3}} - \frac{3}{4}\sqrt{x^2}$  egri chiziq yoyining  $Ox$  o'q bilan

kesishish nuqtalari orasidagi uzunligini hisoblang.

*Yechish.*  $y=0$  deb egri chiziqning  $Ox$  oq bilan kesishish nuqtalarini aniqlaymiz:  $x_1 = 0$ ,  $x_2 = 2\sqrt{2}$ .

Hosilani topamiz:

$$y' = \frac{3}{8} \cdot \frac{4}{3} x^{\frac{1}{3}} - \frac{3}{4} \cdot \frac{2}{3} x^{-\frac{1}{2}} = \frac{1}{2} \left( x^{\frac{1}{3}} - x^{-\frac{1}{2}} \right).$$

Yoy uzunligini hisoblaymiz:

$$l = \int_0^{2\sqrt{2}} \sqrt{1 + \frac{1}{4} \left( x^{\frac{1}{3}} - x^{-\frac{1}{3}} \right)^2} dx = \frac{1}{2} \int_0^{2\sqrt{2}} \sqrt{\left( x^{\frac{1}{3}} + x^{-\frac{1}{3}} \right)^2} dx =$$

$$= \frac{1}{2} \int_0^{2\sqrt{2}} \left( x^{\frac{1}{3}} + x^{-\frac{1}{3}} \right) dx = \frac{1}{2} \left( \frac{3}{4} x^{\frac{4}{3}} + \frac{3}{2} x^{\frac{2}{3}} \right) \Big|_0^{2\sqrt{2}} = 3.$$

Agar egri chiziq  $x = \varphi(t)$ ,  $y = \psi(t)$ ,  $\alpha \leq t \leq \beta$ , parametrik tenglamalar bilan berilgan bo'lsa, (8.8) formulada  $x = \varphi(t)$ ,  $y = \psi(x)$ ,  $dx = \varphi'(t)dt$  o'rniga qo'yish orqali o'zgaruvchi almashtiriladi.

Bunda

$$l = \int_{\alpha}^{\beta} \sqrt{\varphi'^2(t) + \psi'^2(t)} dt \quad (8.10)$$

kelib chiqadi, bu yerda  $a = \varphi(\alpha)$  va  $b = \varphi(\beta)$ .

Egri chiziq qutb koordinatalar sistemasida  $r = r(\varphi)$ ,  $\alpha \leq \varphi \leq \beta$ , tenglama bilan berilgan bo'lsin, bunda  $r(\varphi)$ ,  $r'(\varphi)$  funksiyalar  $[\alpha; \beta]$  kesmada uzluksiz va  $A, B$  nuqtalarga qutb koordinatalarida  $\alpha, \beta$  burchaklar mos keladi.

$x = r \cos \varphi$ ,  $y = r \sin \varphi$  ekanligidan

$$x'(\varphi) = r'(\varphi) \cos \varphi - r(\varphi) \sin \varphi,$$

$$y'(\varphi) = r'(\varphi) \sin \varphi + r(\varphi) \cos \varphi.$$

(8.10) formulaga  $x'(\varphi)$  va  $y'(\varphi)$  hosilalarni qo'yamiz va almashtirishlar bajarib, topamiz:

$$l = \int_{\alpha}^{\beta} \sqrt{r^2(\varphi) + r'^2(\varphi)} d\varphi. \quad (8.11)$$

7-misol.  $r = a(1 + \cos \varphi)$ ,  $a > 0$  kardioida uzunligini toping.

*Yechish.* Egri chiziqning simmetrikligini (1-ilovaga qarang) hisobga olib, (8.11) formula bilan topamiz:

$$l = 2 \int_0^{\pi} \sqrt{a^2(1 + \cos \varphi)^2 + a^2(-\sin \varphi)^2} d\varphi = 4a \int_0^{\pi} \sqrt{\frac{1 + \cos \varphi}{2}} d\varphi =$$

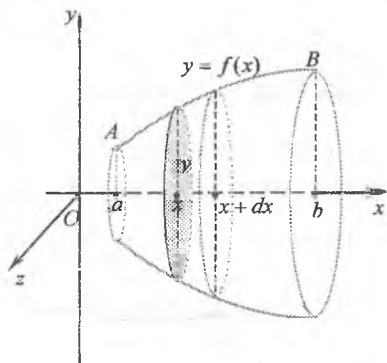
$$= 4a \int_0^{\pi} \cos \frac{\varphi}{2} d\varphi = 8a \sin \frac{\varphi}{2} \Big|_0^{\pi} = 8a.$$



### 7.8.4. Aylanish sirti yuzini hisoblash

$AB$  egri chiziq  $y = f(x) \geq 0$  funksiyaning grafigi bo'lsin. Bunda  $x \in [a; b]$ ,  $y = f(x)$  funksiya va  $y' = f'(x)$  hosila bu kesmada uzluksiz bo'lsin.

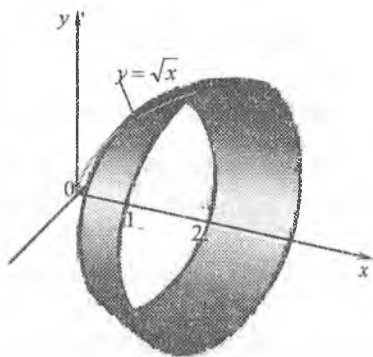
$AB$  egri chiziqning  $Ox$  o'q atrofida aylanishidan hosil bo'lgan jism sirti yuzini hisoblaymiz. Buning uchun II sxemani qo'llaymiz.



11-shakl.

1°. Istalgan  $x \in [a; b]$  nuqta orqali  $Ox$  o'qqa perpendikular tekislik o'tkazamiz. Bu tekislik aylanish sirtini radiusi  $y = f(x)$  bo'lgan aylana bo'ylab kesadi (11-shakl). Bunda aylanish sirtidan  $Ox$  o'qidagi  $a$  va  $x$  nuqtalardan  $zOy$  tekisligiga parallel ravishda o'tkazilgan tekisliklar ajratgan sirt yuzi  $x$  ning funksiyasi bo'ladi:  $S = S(x)$  ( $S(a) = 0$  va  $S(b) = S$ ).

2°.  $x$  argumentga  $\Delta x = dx$  ortirma beramiz va  $x + \Delta x \in [a; b]$  nuqta orqali  $Ox$  o'qqa perpendikular tekislik o'tkazamiz. Bunda  $S = S(x)$  funksiya «belbog'» ko'rinishida  $\Delta S$  ortirma oladi.



12-shakl.

Kesimlar orasidagi jismni yasovchisi  $dl$  bo'lgan va asoslarining radiuslari  $y$  va  $y + dy$  bo'lgan kesik konus bilan almashtiramiz. Bu kesik konusning yon sirti  $dS = \pi(y + y + dy)dl = 2\pi y dl + \pi dy dl$  ga teng.  $dy dl$  ko'paytmani  $dS$  ga nisbatan

yuqori tartibli cheksiz kichik sifatida tashlab yuboramiz:  $dS = 2\pi y dl$ .

Bunda  $dl = \sqrt{1 + (y'_x)^2} dx$  ekanini hisobga olamiz:  $dS = 2\pi y \sqrt{1 + (y'_x)^2} dx$ .

3°.  $dS$  ni  $a$  dan  $b$  gacha integrallab, topamiz:

$$S = 2\pi \int_a^b y \sqrt{1 + (y'_x)^2} dx \quad (8.12)$$

Shu kabi  $x = g(y)$ ,  $y \in [c; d]$  funksiya grafigining  $Oy$  o'q atrofida aylantirshdan hosil bo'lgan jism sirtining yuzi ushbu

$$S = 2\pi \int_c^d x \sqrt{1 + (x')^2} dy \quad (8.13)$$

formula bilan hisoblanadi.

*5-misol.*  $y = 2\sqrt{x}$ ,  $1 \leq x \leq 2$  egri chiziqning  $Ox$  o'qi atrofida aylanishidan hosil bo'lgan sirt (12-shakl) yuzini toping.

*Yechish.* Misol shartidan topamiz:  $y' = \frac{1}{\sqrt{x}}$ .

(8.12) formula bilan topamiz:

$$S = 2\pi \int_1^2 2\sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} dx = 4\pi \int_1^2 \sqrt{x+1} dx = 4\pi \cdot \frac{2}{3} (x+1)^{\frac{3}{2}} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}).$$

Agar  $AB$  egri chiziq  $x = \varphi(t)$ ,  $y = \psi(t)$ ,  $\alpha \leq t \leq \beta$ , parametrik tenglamalar bilan berilgan bo'lsa, u holda  $AB$  egri chiziqning  $Ox$  ( $Oy$ ) o'q atrofida aylanishidan hosil bo'lgan jism sirti yuzi quyidagicha hisoblanadi:

$$S = 2\pi \int_a^b \psi(t) \sqrt{\varphi'^2(t) + \psi'^2(t)} dt \quad \left( S = 2\pi \int_{\alpha_1}^{\alpha_2} \varphi(t) \sqrt{\psi'^2(t) + \varphi'^2(t)} dt \right), \quad (8.14)$$

bu yerda  $a = \varphi(\alpha)$  va  $b = \varphi(\beta)$  ( $c = \psi(\alpha_1)$  va  $d = \psi(\beta_1)$ ).

$AB$  egri chiziq qutb koordinatalar sistemasida  $r = r(\varphi)$ ,  $\alpha \leq \varphi \leq \beta$  tenglama bilan berilgan bo'lganida quyidagi formulalar o'rinli bo'ladi:

$$Ox: S = 2\pi \int_a^b r \sin \varphi \sqrt{r^2 + r'^2} d\varphi, \quad Oy: S = 2\pi \int_a^b r \cos \varphi \sqrt{r^2 + r'^2} d\varphi. \quad (8.15)$$

*6-misol.* Radiusi  $R$  ga teng bo'lgan shar sirti yuzini hisoblang.

*Yechish.* Shar parametrik tenglamasi  $x = R \cos t$ ,  $y = R \sin t$  bo'lgan yarim aylananing  $Ox$  o'q atrofida aylanishidan hosil bo'ladi. Sharning koordinata o'qlariga simmetrik bo'lishini inobatga olib, hisoblaymiz:

$$\begin{aligned} S &= 2 \cdot 2\pi \int_0^{\frac{\pi}{2}} R \sin t \sqrt{(-R \sin t)^2 + (R \cos t)^2} dt = \\ &= 4\pi R^2 \int_0^{\frac{\pi}{2}} \sin t dt = -4\pi R^2 \cos t \Big|_0^{\frac{\pi}{2}} = 4\pi R^2. \end{aligned}$$

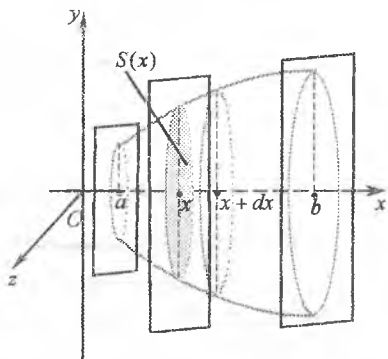
## 7.8.5. Hajmlarni hisoblash

### Hajmni ko'ndalang kesim yuzasi bo'yicha hisoblash

Hajmi hisoblanishi lozim bo'lgan qandaydir jism (13-shakl) uchun uning istalgan ko'ndalang kesim yuzasi  $S$  ma'lum bo'lsin. Bu yuza ko'ndalang kesim joylashishiga bog'liq bo'ladi:  $S = S(x)$ ,  $x \in [a; b]$ , bu yerda  $S(x) \in [a; b]$  kesmada uzluksiz funksiya.

Izlanayotgan hajmni II sxema asosida topamiz.

1°. Istalgan  $x \in [a; b]$  nuqta orqali  $Ox$  o'qqa perpendikular tekislik o'tkazamiz. Jismning bu tekislik bilan kesimi yuzasini  $S(x)$  bilan va



13-shakl.

jismning bu tekislikdan chapda yotgan bo'lagining hajmini  $V(x)$  bilan belgilaymiz (13-shakl). Bunda  $V$  kattalik  $x$  ning funksiyasi bo'ladi:  $V = V(x)$  ( $V(a) = 0$  va  $V(b) = V$ ).

2°.  $V(x)$  funksiyaning  $dV$  differensialini topamiz. Bu differensial  $Ox$  o'q bilan  $x$  va  $x + \Delta x$  nuqtalarda kesishuvchi parallel tekisliklar orasidagi «elementar qatlam»dan iborat bo'ladi. Bu differensialni asosi  $S(x)$ ga va balandligi  $dx$  ga teng silindr bilan taqriban almashtirish mumkin. Demak,  $dV = S(x)dx$ .

3°.  $dV$  ni  $a$  dan  $b$  gacha integrallab, izlanayotgan hajmni topamiz:

$$V = \int_a^b S(x)dx. \quad (8.16)$$

7-misol.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  ellipsoidning hajmini hisoblang.

*Yechish.* Ellipsoidning  $Ox$  o'qqa perpendikulyar va koordinatalar boshidan  $x$  ( $-a \leq x \leq a$ ) masofada o'tuvchi tekislik bilan kesamiz.

Kesimda yarim o'qlari  $b(x) = b\sqrt{1 - \frac{x^2}{a^2}}$  va  $c(x) = c\sqrt{1 - \frac{x^2}{a^2}}$  bo'lgan ellips

hosil bo'ladi.

Uning yuzasi

$$S(x) = \pi b(x)c(x) = \pi bc \left(1 - \frac{x^2}{a^2}\right).$$

U holda

$$V = \int_{-a}^a \pi bc \left(1 - \frac{x^2}{a^2}\right) dx = \pi bc \left(x - \frac{x^3}{3a^2}\right) \Big|_{-a}^a = \frac{4}{3} \pi abc.$$

### Aylanish jismlarining hajmini hisoblash

Yuqoridan  $y = f(x)$  uzluksiz funksiya grafigi bilan, quyidan  $Ox$  o'q bilan, yon tomonlaridan  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning  $Ox$  o'q atrofida aylantirishdan hosil bo'lgan jism hajmini hisoblaymiz. Bu jismning ixtiyoriy ko'ndalang kesimi doiradan iborat. Shu sababli jismning  $X = x$  tekislik bilan kesimining yuzasi  $S(x) = \pi y^2$  bo'ladi.

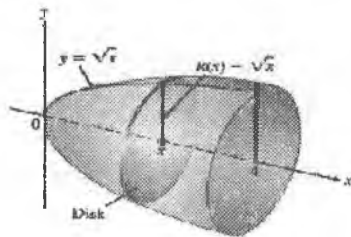
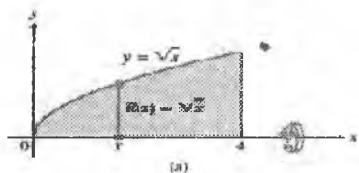
U holda (8.16) formulaga ko'ra,

$$V = \pi \int_a^b y^2 dx. \quad (8.17)$$

Shu kabi yuqoridan  $y = f(x)$  uzluksiz funksiya grafigi bilan, quyidan  $Ox$  o'q bilan, yon tomonlaridan  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning  $Oy$  o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmi quyidagi formula bilan hisoblanadi:

$$V = 2\pi \int_a^b yx dx. \quad (8.18)$$

*8-misol.*  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 0$ ,  $x = 4$  chiziqlar bilan chegaralangan yassi shaklning  $Ox$  o'qi atrofida aylanishidan hosil bo'lgan jism hajmini toping (14-shakl).



14-shakl.

*Yechish.* Hajmni (8.17) formula bilan topamiz:

$$S = \pi \int_0^4 x dx = \pi \cdot \frac{x^2}{2} \Big|_0^4 = 8\pi.$$

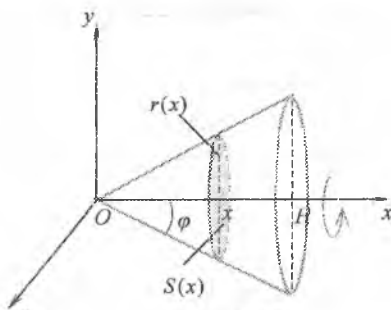
Agar egri chiziqli trapetsiya  $x = \varphi(y)$  uzluksiz funksiya grafigi,  $Oy$  o'qi,  $y = c$  va  $y = d$  to'g'ri chiziqlar bilan chegaralangan bo'lsa,

$$V = \pi \int_c^d x^2 dy (Oy) \quad \left( V = 2\pi \int_c^d xy dy (Ox) \right) \quad (8.19)$$

bo'ladi.

*9-misol.* Radiusi  $R$  ga va balandligi  $H$  ga teng bo'lgan konusning hajmini hisoblang.

*Yechish.* Konusni katetlari  $R$  va  $H$  bo'lgan to'g'ri burchakli uchburchakning balandlik bo'ylab yo'nalgan  $Ox$  o'q atrofida aylanishidan hosil bo'lgan jism deyish mumkin (15-shakl). Gipotenuza tenglamasi  $y = kx$  bo'lsin.



15-shakl.

U holda

$$y = kx, \quad k = \operatorname{tg} \varphi = \frac{R}{H}, \quad y = \frac{R}{H} x.$$

Bundan

$$V = \pi \int_0^H y^2 dx = \pi \int_0^H \frac{R^2}{H^2} x^2 dx = \frac{\pi R^2}{H^2} \cdot \frac{x^3}{3} \Big|_0^H = \frac{1}{3} \pi R^2 H.$$

### 7.8.6. Momentlar va og'irlik markazini hisoblash

Tekis shakl va aylanish sirti yuzalarini, yassi egri chiziq yoyi uzunligini, hajmlarini hisoblashga oid yuqorida keltirilgan masalalarni yechishda aynan bir xil usul qo'llanildi. Bunda izlanilayotgan  $Q$  kattalikni hisoblash integral yig'indi limitini topishga keltirildi. Barcha masalalarda  $Q$  kattalik biror  $[a; b]$  kesma va bu kesmadagi uzluksiz funksiyalarga bog'liq holda o'rganildi.

Shu kabi  $Q$  kattalik quyidagi xossalarga ega deb faraz qilamiz:

1°. *Additivlik xossasi.*  $[a; b]$  kesma istalgancha bo'laklarga

bo'linganida ham  $Q = \sum_{i=1}^n Q_i$  bo'ladi, bu yerda  $Q_i$  —  $Q$  ning  $i$ -bo'lakdagi qiymati;

2°. *Kichiklikdagi chiziqlilik xossasi.*  $[a; b]$  kesmadagi istalgan kichik  $[x_{i-1}; x_i]$  kesmada  $Q_i \approx k\Delta x_i$  bo'ladi, bu yerda  $\Delta x_i = x_i - x_{i-1}$ .

Quyida keltiriladigan momentlarni, og'irlik markazi va kuchning bajargan ishini hisoblash formulalari ham yuqoridagi singari hosil qilinadi. Shu sababli bu formulalarni keltirib chiqarmaymiz va ulardan masalalarni yechishda foydalanamiz.

$Oxy$  tekislikda massalari mos ravishda  $m_1, m_2, \dots, m_n$  bo'lgan  $A_1(x_1; y_1), A_2(x_2; y_2), \dots, A_n(x_n; y_n)$  nuqtalar sistemasi berilgan bo'lsin.

Sistemaning  $Ox$  ( $Oy$ ) o'qqa nisbatan *statik momenti*  $M_x$  ( $M_y$ ) deb nuqtalar massalarini ularning ordinalariga (absissalariga) ko'paytmalari yig'indisiga aytiladi, ya'ni

$$M_x = \sum_{i=1}^n m_i y_i \quad \left( M_y = \sum_{i=1}^n m_i x_i \right).$$

Sistemaning  $Ox$  ( $Oy$ ) o'qqa nisbatan *inersiya momenti*  $J_x$  ( $J_y$ ) deb nuqtalar massalarini ularning ordinalari (absissalari) kvadratiga ko'paytmalari yig'indisiga aytiladi, ya'ni

$$J_x = \sum_{i=1}^n m_i y_i^2 \quad \left( J_y = \sum_{i=1}^n m_i x_i^2 \right).$$

Sistemaning *og'irlik markazi* deb koordinatalari  $\left( \frac{M_y}{m}; \frac{M_x}{m} \right)$  bo'lgan nuqtalarga aytiladi, bu yerda  $m = \sum_{i=1}^n m_i$ .

### ***Yassi egri chiziqning momentlari va og'irlik markazi***

$Oxy$  tekislikda  $AB$  egri chiziq  $y = f(x)$  ( $a \leq x \leq b$ ) tenglama bilan berilgan bo'lib, egri chiziqning har bir  $(x, f(x))$  nuqtasidagi zichlik  $\gamma = \gamma(x)$  ga teng va  $f(x)$  funksiya  $f'(x)$  hosilasi bilan birga uzluksiz bo'lsin.

U holda  $AB$  egri chiziqning momentlari va og'irlik markazining

koordinatalari quyidagi formulalar bilan aniqlanadi:

$$M_x = \int_a^b \gamma y dl, \quad M_y = \int_a^b \gamma x dl; \quad (8.20)$$

$$J_x = \int_a^b \gamma y^2 dl, \quad J_y = \int_a^b \gamma x^2 dl; \quad (8.21)$$

$$x_c = \frac{\int_a^b \gamma x dl}{\int_a^b \gamma dl}, \quad y_c = \frac{\int_a^b \gamma y dl}{\int_a^b \gamma dl}, \quad (8.22)$$

bu yerda  $y = f(x)$ ,  $\gamma = \gamma(x)$ ,  $dl = \sqrt{1 + y'^2} dx$ ,  $a \leq x \leq b$ .

10-misol. Zichligi  $\gamma = 1$  ga teng bo'lgan  $y = \sqrt{R^2 - x^2}$ ,  $|x| \leq R$  yarim aylananing momentlari va og'irlik markazini toping.

*Yechish.*  $y' = -\frac{x}{\sqrt{R^2 - x^2}}$  bo'lgani uchun  $dl = \frac{R dx}{\sqrt{R^2 - x^2}}$ .

U holda (8.20) - (8.22) formulalardan topamiz:

$$M_x = \int_{-R}^R \sqrt{R^2 - x^2} \frac{R dx}{\sqrt{R^2 - x^2}} = R \int_{-R}^R dx = R x \Big|_{-R}^R = 2R^2,$$

$$M_y = \int_{-R}^R \frac{x R dx}{-\sqrt{R^2 - x^2}} = -R \sqrt{R^2 - x^2} \Big|_{-R}^R = 0.$$

$$J_x = \int_{-R}^R (R^2 - x^2) \frac{R dx}{\sqrt{R^2 - x^2}} = R \int_{-R}^R \sqrt{R^2 - x^2} dx = R \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} R^2 \cos^2 t dt =$$

$$= R^3 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1 + \cos 2t}{2} dt = \frac{R^3}{2} \left( t + \frac{\sin 2t}{2} \right) \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{\pi R^3}{2},$$

$$J_y = \int_{-R}^R \frac{x^2 R dx}{-\sqrt{R^2 - x^2}} = R \left( -x \sqrt{R^2 - x^2} \Big|_{-R}^R + \int_{-R}^R \sqrt{R^2 - x^2} dx \right) = R \left( 0 + \frac{\pi R^2}{2} \right) = \frac{\pi R^3}{2}.$$

$$\int_{-R}^R dl = \int_{-R}^R \frac{R dx}{-\sqrt{R^2 - x^2}} = R \arcsin \frac{x}{R} \Big|_{-R}^R = \pi R,$$

$$x_c = 0, \quad y_c = \frac{2R^2}{\pi R} = \frac{2R}{\pi}.$$

## Tekis shaklning momentlari va og'irlik markazi

$Oxy$  tekislikda  $[a; b]$  kesmada uzluksiz bo'lgan  $y = f(x)$  funksiya grafigi,  $Ox$  o'q,  $x = a$  va  $x = b$  to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya (tekis shakl) berilgan va tekis shaklning har bir nuqtasida  $\gamma = \gamma(x)$  zichlik uzluksiz bo'lsin. U holda tekis shaklning momentlari va og'irlik markazining koordinatalari quyidagi formulalar orqali topiladi:

$$M_x = \frac{1}{2} \int_a^b \gamma y^2 dx, \quad M_y = \int_a^b \gamma xy dx; \quad (8.23)$$

$$J_x = \frac{1}{2} \int_a^b \gamma y^3 dx, \quad J_y = \int_a^b \gamma x^2 y dx; \quad (8.24)$$

$$x_c = \frac{\int_a^b \gamma xy dx}{\int_a^b \gamma y dx}, \quad y_c = \frac{\frac{1}{2} \int_a^b \gamma y^2 dx}{\int_a^b \gamma y dx}, \quad (8.25)$$

bu yerda  $\gamma = \gamma(x)$ ,  $y = y(x)$ ,  $a \leq x \leq b$ .

*11-misol.*  $y = \sin x$  sinusoida yoyi va  $Ox$  o'qining  $0 \leq x \leq \pi$  bo'lagi bilan chegaralangan, zichligi  $\gamma = 1$  ga teng tekis shaklning og'irlik markazini toping.

*Yechish.* Sinusoidaning simmetrikligidan  $x_c = \frac{\pi}{2}$  bo'ladi. U holda

$$M_x = \frac{1}{2} \int_0^{\pi} y^2 dx = \frac{1}{2} \int_0^{\pi} \sin^2 x dx = \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2x}{2} dx = \frac{1}{4} \left( x - \frac{\sin 2x}{2} \right) \Big|_0^{\pi} = \frac{\pi}{4}.$$

$$\int_0^{\pi} y dx = \int_0^{\pi} \sin x dx = -\cos x \Big|_0^{\pi} = 2, \quad y_c = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}.$$

Demak,

$$x_c = \frac{\pi}{2}, \quad y_c = \frac{\pi}{8}.$$

### 7.8.7. Kuchning bajargan ishini hisoblash

Moddiy nuqta o'zgaruvchan  $\vec{F}$  kuch ta'sirida  $Ox$  o'qi bo'ylab harakatlanayotgan bo'lsin va bunda kuchning yo'nalishi harakat



yoʻnalishi bilan bir xil boʻlsin. U holda  $\bar{F}$  kuchning moddiy nuqtani  $dx$  oʻqi boʻylab  $x=a$  nuqtadan  $x=b$  ( $a < b$ ) nuqtaga koʻchirishda bajarilgan ishi quyidagi formula bilan hisoblanadi:

$$A = \int_a^b F(x) dx, \quad (8.26)$$

bu yerda  $F(x)$  funksiya  $[a; b]$  kesmada uzluksiz.

*12-misol.*  $m$  massali kosmik kemani yerdan  $h$  masofaga uchirish uchun qancha ish bajarish kerak?

*Yechish.* Butun olam tortishish qonuniga koʻra, yerning jismini tortish kuchi  $F = k \frac{mM}{x^2}$  ga teng boʻladi, bu yerda  $M$  – yerning massasi,  $x$  – yer markazidan kosmik kemagacha boʻlgan masofa,  $k$  – gravitatsiya doimiyligi. Yer sirtida, yaʼni  $x=R$  da  $F = mg$  ga teng, bu yerda  $g$  – erkin tushish tezlanishi.

U holda

$$mg = k \frac{mM}{R^2}.$$

Bundan

$$kM = gR^2, \quad F = mg \frac{R^2}{x^2}.$$

Izlanayotgan ishni (8.26) formula bilan topamiz:

$$A = \int_R^{R+h} mg \frac{R^2}{x^2} dx = -mgR^2 \frac{1}{x} \Big|_R^{R+h} = -mgR^2 \left( \frac{1}{R+h} - \frac{1}{R} \right) = mgR \frac{h}{R+h}.$$

### 7.8.8. Jismning bosib oʻtgan yoʻli

Moddiy nuqta (jism) toʻgʻri chiziq boʻylab oʻzgaruvchan  $v = v(t)$  tezlik bilan harakatlanayotgan boʻlsin. Bu nuqtaning  $t_1$  dan  $t_2$  gacha vaqt oraligʻida bosib oʻtgan yoʻlini topamiz.

Hosilaning fizik maʼnosiga koʻra, nuqtaning bir tomonga harakatida «toʻgʻri chizikli harakat tezligi yoʻldan vaqt boʻyicha olingan hosilaga teng», yaʼni  $v(t) = \frac{dS}{dt}$ . Bundan  $dS = v(t)dt$ . Bu tenglikni  $t_1$  dan  $t_2$  gacha integrallaymiz:

$$S = \int_{t_1}^{t_2} v(t) dt. \quad (8.27)$$

**Izoh.** Bu formulani aniq integralni qo'llash sxemalari bilan ham topish mumkin.

### 7.8.9. Suyuqlikning vertikal plastinkaga bosimi

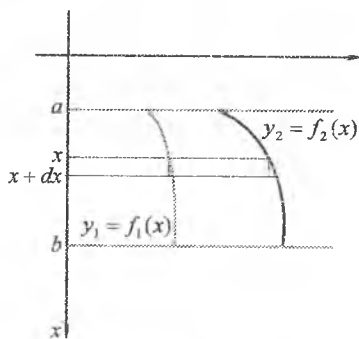
Paskal qonuniga ko'ra, suyuqlikning gorizontaal plastinkaga bosimi

$$P = g \cdot \gamma \cdot S \cdot h$$

formula bilan topiladi, bu yerda  $g$ — erkin tushish tezlanishi,  $\gamma$ — suyuqlik zichligi,  $S$ — plastinkaning yuzasi,  $h$ — plastinkaning botish chuqurligi.

Plastinkaning vertikal botishida suyuqlikning plastinkaga bosimini bu formula bilan topib bo'lmaydi, chunki plastinkaning har xil nuqtalari turli chuqurlikda yotadi.

Suyuqlikka  $x=a$ ,  $x=b$ ,  $y_1 = f_1(x)$ ,  $y_2 = f_2(x)$  chiziqlar bilan chegaralangan plastinka vertikal botirilayotgan bo'lsin. Bunda koordinatalar sistemasi 16-shakida ko'rsatilganidek tanlangan bo'lsin. Suyuqlikning plastinkaga  $P$  bosimini topish uchun II sxemadan foydalanamiz.

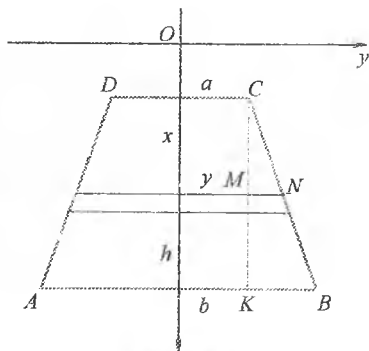


16-shakl.

Bunda:

1°. Izlanayotgan  $P$  kattalikning bir qismi  $x$  ning funksiyasi bo'lsin:  $p = p(x)$ , ya'ni  $p = p(x)$  — plastinkaning  $x$  o'zgaruvchi  $[a; x]$  kesmasiga mos qismiga suyuqlik bosimi, bu yerda  $x \in [a; b]$  ( $p(a) = 0$  va  $p(b) = P$ ).

2°.  $x$  argumentga  $\Delta x = dx$  orttirma beramiz. Bunda  $p(x)$  funksiya  $\Delta p$  orttirma oladi (16-shaklda  $dx$  qalinlikdagi tasma). Funksiyaning  $dp$  differensialni topamiz.  $dx$  kichik ekanidan tasmani barcha nuqtalari bitta chuqurlikda yotuvchi to'g'ri to'rtburchak deb hisoblaymiz, ya'ni plastiuka gorizontaal bo'lsin deymiz.



17-shakl.

U holda Paskal qonuniga ko'ra,

$$dp = g \cdot \gamma \cdot \frac{(y_2 - y_1) \cdot dx \cdot x}{h}$$

3°.  $dp$  ni  $x=a$  dan  $x=b$  gacha integrallaymiz:

$$P = g \cdot \gamma \cdot \int_a^b (y_2 - y_1) x dx$$

yoki

$$P = g \cdot \gamma \cdot \int_a^b (f_2(x) - f_1(x)) x dx. \quad (8.28)$$

*13-misol.* Asoslari  $a$  va  $b$  ga, balandligi  $h$  ga teng bo'lgan teng yonli trapetsiya shaklidagi plastinka suyuqlikka  $c$  chuqurlikda botirilgan (17-shakl). Suyuqlikning plastinkaga bosimini toping.

*Yechish.* Izlanayotgan bosim (8.28) formulaga ko'ra,

$$P = g\gamma \int_c^{c+h} y x dx$$

bo'ladi.

$y$  o'zgaruvchini  $x$  o'zgaruvchi orqali ifodalash uchun  $CMN$  va  $CKB$  uchburchaklarning o'xshashligidan foydalanamiz:

$$\frac{y-a}{b-a} = \frac{x-c}{h}$$

Bundan  $y = a + \frac{b-a}{h}(x-c)$ .

U holda

$$\begin{aligned} P &= g\gamma \int_c^{c+h} \left( a + \frac{b-a}{h}(x-c) \right) x dx = g\gamma \left( \frac{ax^2}{2} + \frac{b-a}{h} \left( \frac{x^3}{3} - \frac{cx^2}{2} \right) \right) \Big|_c^{c+h} = \\ &= g\gamma \left( \frac{a+b}{2} ch + \frac{h^2}{6} (a+2b) \right) \end{aligned}$$

### 7.8.10. Mashqlar

1 Berilgan chiziqlar bilan chegaralangan tekis shakl yuzalarini hisoblang:

1)  $y=9-x^2$ ,  $y=0$ ;

2)  $y=-x$ ,  $y=2x-x^2$ ;

3)  $y=\ln(x+6)$ ,  $y=3\ln x$ ,  $y=0$ ,  $x=0$ ;

4)  $y=\ln x$ ,  $y=0$ ,  $x=e^2$ ;

5)  $x = y^2, x = |y + 2|;$

6)  $xy = 4, x = 5 - y;$

7)  $y = x^2, y^2 = -x;$

8)  $y = x^2, y = x^3, x = -1, x = 1;$

9)  $x = 4 \cos t, y = 3 \sin t, 0 \leq t \leq 2\pi;$

10)  $x = 3(t - \sin t), y = 3(1 - \cos t)$  (sikloida bitta arkasi)

11)  $r = 3\sqrt{\cos 2\varphi};$

12)  $r = 3 \sin 2\varphi.$

13)  $r = 2 + 3 \cos \varphi;$

14)  $r = 2\varphi,$  bir o'rami.

2. Berilgan egri chiziqning argumentning ko'rsatilgan oralig'iga mos yoylari uzunliklarini toping:

1)  $y = \frac{x^2}{2}, x = 0$  dan  $x = \sqrt{3}$  gacha;

2)  $y = chx, x = 0$  dan  $x = 1$  gacha;

3)  $y^2 = x^3, x = 0$  dan  $x = 5$  gacha;

4)  $y = \arccos \sqrt{x} - \sqrt{x - x^2}, x = 0$  dan  $x = 1$  gacha;

5)  $x = \frac{1}{4}y^2 - \frac{1}{2} \ln y, y = 1$  dan  $y = 2$  gacha;

6)  $x = 1 - \ln(y^2 - 1), y = 3$  dan  $y = 4$  gacha;

7)  $x = t^2, y = \frac{t^3}{3} - t,$  koordinata o'qlari bilan kesishish nuqtalari orasidagi;

8)  $x = t^2, y = t^3, t = 0$  dan  $t = 1$  gacha;

9)  $x = 2(t - \sin t), y = 2(1 - \cos t)$  (sikloida bitta arkasi);

10)  $x = 3(2 \cos t - \cos 2t), y = 3(2 \sin t - \sin 2t);$

11)  $r = a(1 - \cos \varphi), r \leq \frac{a}{2}$  kardioida bo'lagining;

12)  $r = 8 \cos^3 \frac{\varphi}{3}, \varphi = 0$  dan  $\varphi = \frac{\pi}{2}$  gacha.

3. Chiziqning berilgan o'q atrofida aylanishidan hosil bo'lgan sirt yuzini hisoblang:

1)  $y^2 = 4x, x = 0$  dan  $x = 3$  gacha,  $Ox$  o'q;

2)  $x^2 + y^2 = 9, Oy$  o'q;

3)  $x = 2(t - \sin t), y = 2(1 - \cos t),$  bitta arkasi,  $Ox$  o'q;

4)  $x = \sqrt{2} \cos t, y = \sin t, Ox$  o'qi;

4. R radiusli shar hajmini hisoblang.

5. Asesi  $\frac{x^2}{16} + \frac{y^2}{9} = 1$  ellipsdan iborat bo'lgan va balandligi  $h=3$  ga teng elliptik konusning hajmini hisoblang.

6.  $x^2 + y^2 + z^2 = 16$  shar hamda  $x=2$  va  $x=3$  tekisliklar bilan chegaralangan jism hajmini hisoblang.

7.  $\frac{y^2}{4} + \frac{z^2}{9} - x^2 = 1$  bir pallali giperboloid hamda  $x=-1$  va  $x=2$  tekisliklar bilan chegaralangan jism hajmini hisoblang.

8. Berilgan chiziqlar bilan chegaralangan tekis shaklning berilgan o'q atrofida aylanishidan hosil bo'lgan jism hajmini hisoblang:

1)  $x^2 = 4 - y, y = 0, Ox$  o'qi;

2)  $x^2 + y^2 = 4$  yarim aylana ( $x \geq 0$ ) va  $y^2 = 3x$  parabola,  $Ox$  o'qi;

3)  $y = \arcsin x, y = 0, x = 1, Oy$  o'qi;

4)  $y^2 = x^3, x = 1, y = 0, Oy$  o'qi;

5)  $x^2 = 4y, x = 0, y = 1, Oy$  o'qi;

6)  $\frac{x^2}{25} + \frac{y^2}{9} = 1, Oy$  o'qi;

7)  $x = 2(t - \sin t), y = 2(1 - \cos t)$ , bitta arkasi,  $Ox$  o'qi;

8)  $x = t^2, y = t^3, x = 0, y = 1, Oy$  o'qi;

9)  $r = 3(1 + \cos \varphi)$ , qutb o'qi;

10)  $r = 2R \cos \varphi$ , yarim aylana, qutb o'qi;

9.  $r = 2R \sin \varphi$  bir jinsli aylananing og'irlik markazini toping.

10.  $x = a \cos^3 t, y = a \sin^3 t$  bir jinsli astroidaning  $Ox$  o'qdan yuqorida yotgan yoyining og'irlik markazini toping.

11.  $4x + 3y - 12 = 0$  bir jinsli to'g'ri chiziqning koordinata o'qlari orasida joylashgan kesmasining koordinata o'qlariga nisbatan statik momentlarini toping.

12.  $x = 0, y = 0, x + y = 2$  chiziqlar bilan chegaralangan bir jinsli tekis shaklning koordinata o'qlariga nisbatan statik va inersiya momentlarini, og'irlik markazini toping.

13.  $y = 4 - x^2$  va  $y = 0$  bir jinsli chiziqlar bilan chegaralangan tekis shaklning og'irlik markazini toping.

14. Yarim o'qlari  $a = 5$  va  $b = 4$  bo'lgan bir jinsli ellipsning koordinata o'qlariga nisbatan inersiya momentini toping.

15.  $x^2 + y^2 = R^2$  aylananing birinchi chorakda joylashgan bo'lagining o'girlik markazini toping. Bunda aylananing har bir nuqtasidagi chiziqli zichligi shu nuqta koordinatalarining ko'paytmasiga proporsional.

16.  $x = 8 \cos^3 t$ ,  $y = 4 \sin^3 t$  astroida birinchi chorakda yotgan yoyining koordinata o'qlariga nisbatan statik momentlarini va massasini toping. Bunda astroidaning har bir nuqtasidagi chiziqli zichligi  $x$  ga teng.

17. Prujinani  $4 \text{ sm}$  ga cho'zish uchun  $24 \text{ J}$  ish bajariladi.  $150 \text{ J}$  ish bajarilsa, prujina qanday uzunlikka cho'ziladi?

18. Agar prujinani  $1 \text{ sm}$  ga siqish uchun  $1 \text{ kG}$  kuch sarf qilinsa, prujinaning  $8 \text{ sm}$  ga siqishda sarf bo'ladigan  $F$  kuch bajargan ishni toping.

19. Uzunligi  $0,5 \text{ m}$  va radiusi  $4 \text{ mm}$  bo'lgan mis simni  $2 \text{ mm}$  cho'zish uchun qancha ish bajarish kerak? Bunda  $F = E \frac{\Delta x}{l}$ ,  $E = 12 \cdot 10^4 \text{ N/mm}^2$ .

20. Og'irligi  $P = 1,5 \text{ T}$  bo'lgan kosmik kemani yer sirtidan  $h = 2000 \text{ km}$  masofaga uchirish uchun bajarilishi kerak bo'ladigan ishni toping.

21. Jismning to'g'ri chiziqli harakat tezligi  $v = 2t + 3t^2 \text{ (m/s)}$  formula bilan ifodalanadi. Jismning harakat boshlanishidan  $5 \text{ s}$  davomida bosib o'tgan yo'lini toping.

22. Nuqtaning harakat tezligi  $v = 0,1t^3 \text{ (m/s)}$  ga teng. Nuqtaning  $10 \text{ s}$  davomidagi o'rtacha tezligini toping.

23. Sportchining parashutdan tushish tezligi  $v = \frac{mg}{k} \left( 1 - e^{-\frac{kt}{m}} \right)$  formula bilan ifodalanadi, bu yerda  $g$  – erkin tushish tezlanishi,  $m$  – sportchining massasi,  $k$  – proporsionallik koeffitsiyenti. Agar parashutdan tushish  $3 \text{ min}$  davom etgan bo'lsa, sportchi qanday balandlikdan sakragan?

24. Nuqtaning harakat tezligi  $v = 0,1e^{-0,01t} \text{ (m/s)}$  ga teng. Nuqtaning harakat boshlanishidan harakat to'xtaguncha bosib o'tgan yo'lini toping.

25. Suyuqlikka vertikal botirilgan asoslari  $a$  va  $b$  ( $b > a$ ) ga, balandligi  $h$  ga teng bo'lgan teng yonli trapetsiya shaklidagi plastinkaga suyuqlikning bosimini toping.

26. Asosi  $18 \text{ m}$  va balandligi  $6 \text{ m}$  bo'lgan to'rtburchakli shlyuzga suv bosimini toping.

27. Diametri 6 m bo'lgan va suv sathida joylashgan yarim doira shaklidagi vertikal devorga suv bosimini toping. Suv zichligi  $\gamma = 1000 \text{ kg/m}^3$ .

## 7.9. ANIQ INTEGRALNI TAQRIBIY HISOBLASH

Ma'lumki, agar  $f(x)$  funksiyaning  $[a; b]$  kesmada boshlang'ich funksiyasi  $F(x)$  mavjud bo'lsa,  $f(x)$  funksiyaning aniq integrali Nyuton-Leybnis formulasi bilan topiladi. Elementar funksiyalarda olinmaydigan aniq integrallar amalda taqribiy hisoblash usullari bilan topiladi. Bunday usullardan aniq integralning integral yig'indining limiti haqidagi ta'rifiga va geometrik ma'nosiga asoslangan usullarni ko'rib chiqamiz.

### 7.9.1. To'g'ri to'rtburchaklar formulasi

$[a; b]$  kesmada uzluksiz  $y = f(x)$  funksiya uchun  $\int_a^b f(x) dx$  integralni hisoblash talab qilingan bo'lsin. Aniqlik uchun barcha  $x \in [a; b]$  da  $f(x)$  funksiya musbat va monoton o'suvchi deb faraz qilamiz.

$[a; b]$  kesmani  $a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = b$  nuqtalar bilan uzunliklari  $\Delta x = \frac{b-a}{n}$  bo'lgan  $n$  ta teng kesmalarga ajratamiz.

Funksiyaning  $x_0, x_1, \dots, x_i, \dots, x_n$  nuqtalardagi qiymatlarini  $y_0, y_1, \dots, y_i, \dots, y_n$  bilan belgilab,  $y_0 = f(x_0), y_1 = f(x_1), \dots, y_i = f(x_i), \dots, y_n = f(x_n)$  larni hisoblaymiz va quyidagi integral yig'indilarni tuzamiz:

$$y_0 \Delta x + y_1 \Delta x + \dots + y_i \Delta x + \dots + y_{n-1} \Delta x = \sum_{i=0}^{n-1} y_i \Delta x,$$

$$y_1 \Delta x + y_2 \Delta x + \dots + y_i \Delta x + \dots + y_n \Delta x = \sum_{i=1}^n y_i \Delta x.$$

U holda 18-shaklga ko'ra,

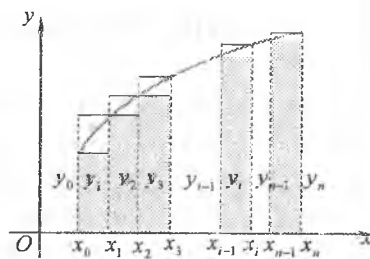
$$\int_a^b f(x) dx \approx \frac{b-a}{n} (y_0 + y_1 + \dots + y_i + \dots + y_{n-1}) = \frac{b-a}{n} \sum_{i=0}^{n-1} y_i, \quad (9.1)$$

$$\int_a^b f(x) dx \approx \frac{b-a}{n} (y_1 + y_2 + \dots + y_i + \dots + y_n) = \frac{b-a}{n} \sum_{i=1}^n y_i. \quad (9.2)$$

(9.1) va (9.2) formulalar aniq integralni taqribiy hisoblashning to'g'ri to'rtburchaklar formulalari deyiladi.

18-shakldan ko'rinadiki, agar  $f(x)$  funksiya  $[a;b]$  kesmada musbat va o'suvchi bo'lsa, (9.1) formula berilgan integralning qiymatini kami bilan, (9.2) formula esa ortig'i bilan beradi.

Agar  $[a;b]$  kesmada  $f'(x)$  chekli hosila mavjud bo'lsa, to'g'ri to'rtburchaklar formulalarining absolut xatoliklari  $|\delta_n| \leq \frac{M_1(b-a)^2}{2n}$  tengsizlik



18-shakl.

bilan baholanadi, bu yerda  $M_1 = \max_{a \leq x \leq b} |f'(x)|$ .

### 7.9.2. Trapetsiyalar formulasi

$[a;b]$  kesmani bo'lishlar sonini avvalgidek qoldiramiz va  $\Delta x$  intervalga mos keladigan  $y = f(x)$  funksiyaning har bir yoyini bu yoyning chetki nuqtalarini tutashtiruvchi  $N_0N_1, N_1N_2, \dots, N_{i-1}N_i, \dots, N_{n-1}N_n$  vatarlar bilan almashtiramiz. Bunda berilgan egri chiziqli trapetsiya  $n$  ta to'g'ri chiziqli trapetsiya bilan almashtiriladi (19-shakl).

Bu to'g'ri chiziqli trapetsiyalar har birining yuzasi  $\frac{y_{i-1} + y_i}{2} \Delta x$  ( $i = \overline{1, n}$ ) ga teng. Bu yuzalarning barchasini qo'shib, *trapetsiyalar formulasini* hosil qilamiz:

$$\int_a^b f(x) dx \approx \sum_{i=1}^n \frac{y_{i-1} + y_i}{2} \Delta x = \frac{b-a}{n} \left( \frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right) \quad (9.3)$$

Bu formulaning xatoligi  $|\delta_n| \leq \frac{M_2(b-a)^3}{12n^2}$  tengsizlik bilan baholanadi, bu yerda  $M_2 = \max_{a \leq x \leq b} |f''(x)|$ .

### 7.9.3. Simpson formulasi (parabolalar usuli)

$[a;b]$  kesmani

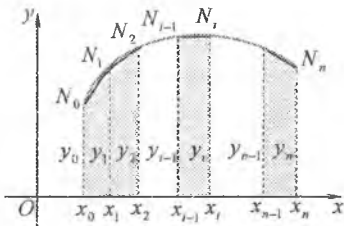
$$a = x_0 < x_1 < x_2 < \dots < x_{2i-2} < x_{2i-1} < x_{2i} < \dots < x_{2m-2} < x_{2m-1} < x_{2m} = b$$

nuqtalar bilan uzunliklari  $h = \frac{b-a}{2m}$  bo'lgan  $n = 2m$  ta teng kesmalarga

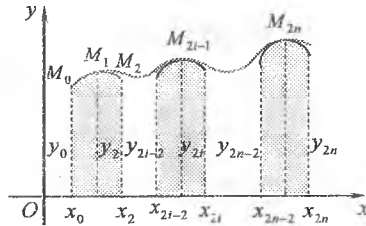


ayratamiz va har bir ketma-ket kelgan  $M_{2m-2}, M_{2m-1}, M_{2m}$  nuqtalar orqali parabolalar o'tkazamiz. Egri chiziqli  $aM_0M_{2m}b$  trapetsiyaning yuzasini parabolalarning  $M_{2i-2}, M_{2i-1}$  va  $M_{2i}$  ( $i = \overline{1, m}$ ) nuqtalarini tutashtiruvchi yoylari bilan chegaralangan  $2m$  ta parabolik trapetsiyalar yuzalarining yig'indisi bilan almashtiramiz.

$M_{2m-2}, M_{2m-1}, M_{2m}$  nuqtalar orqali o'tkazilgan parabola tenglamasini  $y = Ax^2 + Bx + C$  ko'rinishda izlaymiz.  $A, B, C$  koeffitsiyentlarni parabolaning berilgan uchta nuqtadan o'tishi shartidan topamiz. Hisoblashlar qulay bo'lishi uchun koordinatalar boshini o'qlarning yo'nalishini o'zgartirmasdan  $[x_{2i-2}, x_{2i}]$  kesmaning o'rtasiga joylashtiramiz (20-shakl).



19-shakl.



20-shakl.

Parabolaning  $(-h; y_{2i-2}), (0; y_{2i-1}), (h; y_{2i})$  nuqtalardan o'tishi shartidan  $y_{2i-2} = Ah^2 - Bh + C$ ,  $y_{2i-1} = C$ ,  $y_{2i} = Ah^2 + Bh + C$  kelib chiqadi.

Bundan

$$A = \frac{1}{2h^2}(y_{2i-2} - 2y_{2i-1} + y_{2i}), \quad B = \frac{1}{2h}(y_{2i} - y_{2i-2}), \quad C = y_{2i-1}.$$

Endi  $2i$ -parabolik trapetsiya yuzini aniq integral orqali hisoblaymiz:

$$S_{2i} = \int_{-h}^h (Ax^2 + Bx + C) dx = \left( A \frac{x^3}{3} + B \frac{x^2}{2} + Cx \right) \Big|_{-h}^h = \frac{h}{3}(2Ah^2 + 6C).$$

Formulaga  $A, B, C$  koeffitsiyentlarning qiymatlarini qo'yib, topamiz:

$$S_{2i} = \frac{b-a}{6m}(y_{2i-2} + 4y_{2i-1} + y_{2i}), \quad i = \overline{1, n}.$$

Parabolik trapetsiyalarning yuzlarini qo‘shib, izlanayotgan integralning taqribiy qiymatini beruvchi formulani hosil qilamiz:

$$\int_a^b f(x) dx \approx \frac{b-a}{6m} (y_0 + y_{2m} + 4(y_1 + y_3 + \dots + y_{2m-1}) + 2(y_2 + y_4 + \dots + y_{2m-2})). \quad (9.4)$$

(9.4) formulaga *Simpson formulasi* (yoki *parabolalar formulasi*) deyiladi.

Bu formulaning xatoligi  $|\delta_n| \leq \frac{M_4(b-a)^5}{2880n^4}$  tengsizlik bilan baholanadi,

bu yerda  $M_4 = \max_{a \leq x \leq b} |f^{(4)}(x)|$ .

*1-misol.*  $\int_0^1 \frac{dx}{1+x^2}$  integralni taqribiy hisoblash usullari bilan

( $n=10$  da) aniqlang.

*Yechish.* Quyidagi jadvalni tuzamiz:

$x$	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$y$	1	0,9901	0,9615	0,9174	0,8621	0,8	0,7353	0,6711	0,6098	0,5526	0,5

1) (9.1) va (9.2) formulalar yordamida to‘g‘ri to‘rtburchaklar usuli bilan topamiz:

$$\int_0^1 \frac{dx}{1+x^2} \approx 0,1(1 + 0,9901 + 0,9615 + 0,9174 + 0,8621 + 0,8 + 0,7353 + 0,6711 + 0,6098 + 0,5525) = 0,1 \cdot 8,0998 = 0,80998 \text{ (ortig'i bilan),}$$

$$\int_0^1 \frac{dx}{1+x^2} \approx 0,1(0,9901 + 0,9615 + 0,9174 + 0,8621 + 0,8 + 0,7353 + 0,6711 + 0,6098 + 0,5525 + 0,5) = 0,1 \cdot 7,5998 = 0,75998 \text{ (kami bilan).}$$

2) Trapetsiyalar formulasi bilan hisoblaymiz:

$$\int_0^1 \frac{dx}{1+x^2} \approx 0,1 \left[ \frac{1+0,5}{2} + 0,9901 + 0,9615 + 0,9174 + 0,8621 + 0,8 + 0,7353 + 0,6711 + 0,6098 + 0,5525 \right] = 0,1 \cdot 7,8498 = 0,78498.$$

3) Simpson formulasi bilan hisoblaymiz:

$$\int_0^1 \frac{dx}{1+x^2} = (1 + 0,5 + 4(0,9901 + 0,9174 + 0,8 + 0,6711 + 0,5525) + 290,9615 + 0,8621 + 0,7353 + 0,6098) = 0,78539$$

Berilgan integralni aniq qiymatini topamiz:

$$\int_0^1 \frac{dx}{1+x^2} = \arctg x \Big|_0^1 = \frac{\pi}{4} = 0,787571.$$

Taqribiy hisoblashlarning nisbiy va absolt xatoliklarini aniqlaymiz:

To'g'ri to'rtburchaklar (kami bilan)		To'g'ri to'rtburchaklar (ortig'i bilan)		Trapetsiyalar		Parabolalar	
3,3%	0.026	3,1%	0.024	0,09%	0.0003	0,04%	0,0003

Berilgan integralning *Maple* paketida yechimini beramiz.

1) To'g'ri to'rtburchaklar usuli bo'yicha:

```
> restart;
> with(Student[Calculus1]);
> RiemannSum(1/(1+x^2), x=0..1, method=left);
1579799420518583
1950414208136225
> evalf(%);
0.8099814972
> RiemannSum(1/(1+x^2), x=0..1, method=right);
5929114840447087
7801656832544900
> evalf(%);
0.7599814972
```

2) Trapetsiyalar usuli bo'yicha:

```
> with(Student[Calculus1]);
> ApproximateInt(1/(1+x^2), x=0..1, method=trapezoid);
12248312522521419
15603313665089800
> evalf(%);
0.7849814972
```

3) Simpson formulasi bo'yicha:

```
> with(Student[Calculus1]);
```

```
> ApproximateInt( (1/(1+x^2)) - x = 0..1, method = simpson );  
5988585315838311774901484536676836463  
7624903642650463520301694141655283000
```

```
> evalf(%);
```

0.7853981632

#### 7.9.4. Mashqlar

1.  $\int_0^1 e^{-x^2} dx$  aniq integralni 0,001 aniqlikda taqribiy hisoblang: 1) to'g'ri to'rtburchaklar usullari bilan; 2) trapetsiyalar usuli bilan; 3) Simpson formulasi bilan.

# FOYDALANILGAN ADABIYOTLAR

1. David C. Lay. Linear algebra and its applications. Copyright, 2012.
2. W.Keith Nicholson. Linear Algebra. Third Edition. Copyright, 1995.
3. T.S.Blyth, E.F.Robertson. Basic Linear Algebra. Springer-Verlag London Limited, 2007.
4. Erving Kreyszig, Herbert Kreyszig, Edward Normuton. Advanced engineering Mathematics. New York, Copyright, 2011.
5. W.Keith Nicholson. Linear algebra with Applications. December.2011. [www.mccc.edu/course/203/documents/student Soliton Monual](http://www.mccc.edu/course/203/documents/student%20Soliton%20Manual).
- 6.A.K.Lal, S. Pati. Lektüre Notes on Linear Algebra. Februaire 10, 2015. [https://www.Conrsehero.com>...>MATH 211](https://www.Conrsehero.com>...>MATH%20211).
7. David J.Jeffey, Robert M.Corbess. Linear Algebra in Maple. [www.apmaths.uwo.ca/~ deffrey/Offprint/C5106\\_C072](http://www.apmaths.uwo.ca/~deffrey/Offprint/C5106_C072).
8. Jr. Thomas. Calculus. Copyright, 2005
9. Izu Vaisman. Analytical Geometry. Coryright, 1997.
10. L.P.Siceloff, G.Wentworth, D.E.Smith. Analytic Geometry. Copyrigt.)
11. Additional Topics in Analytic Geometry. <http://salkhateeb.kau.edu.sa/.../20section-chapter5-6>
12. Analytic geometry in calculus. [http://higherdbcs.wiley.com/.../analytic\\_geometry\\_in\\_calculus](http://higherdbcs.wiley.com/.../analytic_geometry_in_calculus)
13. Plane and sold Analytic Geometry. Copyrigt, 2016, [www.forgottenbooks.com](http://www.forgottenbooks.com)
14. Claudio Canuto, Anita Tabacco. Mathematical Analysis I. Sprinder-Verlag Italia, Milan 2008.
15. Claudio Canuto, Anita Tabacco. Mathematical Analysis II. Sprinder-Verlag Italia, Milan 2010.
16. Chapter 11: Sequences and Series [https://www.whitman.edu/.../calculus/calculus\\_11\\_Sequences...](https://www.whitman.edu/.../calculus/calculus_11_Sequences...)
17. Hyperbolic functions - Mathcentre [www.mathcentre.ac.uk/resources/.../hyperbolicfunctions](http://www.mathcentre.ac.uk/resources/.../hyperbolicfunctions).
18. Understanding and computing limits by means of infinitesimal quantities [kifri.fri.uniza.sk/ojs/index.php/JICMS/article/viewFile/.../598](http://kifri.fri.uniza.sk/ojs/index.php/JICMS/article/viewFile/.../598)

19. J.Stewart. Calculus, Broks/Cole, Cengage Learning, 2012.
20. Complex Numbers - Stewart Calculus  
[www.stewartcalculus.com/data/.../ess\\_at\\_12\\_cn\\_stu.pdf](http://www.stewartcalculus.com/data/.../ess_at_12_cn_stu.pdf)
21. Section 4-5. Partial Fractions  
[www.mhhe.com/.../barnettcat7/.../barnett07cat/ch04/.../bar68...](http://www.mhhe.com/.../barnettcat7/.../barnett07cat/ch04/.../bar68...)
22. Advanced Integration Techniques  
[faculty.swosu.edu/michael.dougherty/book/chapter07](http://faculty.swosu.edu/michael.dougherty/book/chapter07)
23. Lecture 5: Numerical integration [https://web.stanford.edu/.../numerical\\_methods.../lecture5](https://web.stanford.edu/.../numerical_methods.../lecture5)
24. The Newton-Raphson Method - UBC Mathematics  
<https://www.math.ubc.ca/~anste/.../104newtonmethod.pdf>
25. Gerd Baumann. Mathematics for Engineers I. Munchen, 2010.
26. Wolfgang Ertel. Advanced Mathematics for ingeneers. 2012
27. V.V.Konev. Linear algebra, Vektor algebra, Analitical geometry. TextBook. Tomsk, TPU Press, 2009.
28. А.Б.Соболев, А.Ф.Рыбалко. Математика. Екатеринбург, Часть 1, 2004.
29. Д.Т.Писменный. Конспект лекций по высшей математике: польный курс. – М: Айрис-пресс, 2009.
30. Т. Jo‘rayev, A.Sa‘dullayev, G.Xudoyberganov, X. Mansurov, A.Vorisov. Oliy matematika asoslari. 1-qism. – T., «O‘zbekiston», NMIU, 1998.
31. Soatov Yo.U. Oliy matematika. I-tom, – T.: «O‘qituvchi», 1992.
32. Xurramov Sh.R. Oliy matematika. Misollar. Nazorat topshiriqlari. 1-qism. – T.: «Fan va texnologiyalar», 2015.
33. В.С.Шипачев. Высшая математика. Базовый курс. – М.: Юрист. 2002. – 447 с.

# JAVOBLAR

## Matritsalar

2.  $A=3X+4Y+5Z$ . 3.  $a=2, b=-4$ . 4.  $1 \times 30, 30 \times 1, 2 \times 15, 15 \times 2, 3 \times 10, 10 \times 3, 5 \times 6, 6 \times 5$ .

$$5. \begin{pmatrix} 3 & 7 & -1 \\ -4 & 3 & 4 \end{pmatrix} \quad 6. \begin{pmatrix} 3 & -12 \\ -13 & 5 \\ -4 & 23 \end{pmatrix} \quad 7. \begin{pmatrix} 0 & 1 & -2 \\ 3 & -7 & 6 \\ 2 & -3 & -7 \end{pmatrix} \quad 8. \begin{pmatrix} 2-v & -1 & 2 \\ 5 & -3-v & 3 \\ -1 & 0 & -2-v \end{pmatrix}$$

$$10. x=3, y=0. \quad 11. 3 \times 5. \quad 12. \begin{pmatrix} 0 & 0 \\ b_{21} & b_{22} \end{pmatrix} \quad 13. \begin{pmatrix} 2 & -2 & -4 \\ 8 & 7 & 2 \end{pmatrix} \quad 14. \begin{pmatrix} 10 & -1 \\ -2 & -3 \\ 16 & 0 \end{pmatrix}$$

$$15. \begin{pmatrix} 7 & 6 \\ -1 & 10 \\ -2 & 5 \end{pmatrix} \quad 16. \begin{pmatrix} 2 & -1 & 4 \\ -8 & -3 & 13 \\ 2 & 1 & -2 \end{pmatrix} \quad 17. \begin{pmatrix} -8 & 20 \\ -38 & 30 \end{pmatrix} \quad 18. \begin{pmatrix} 35 & 67 \\ 154 & 166 \end{pmatrix}$$

$$19. \begin{pmatrix} -12 & 0 \\ -9 & 0 \\ 2 & 5 \end{pmatrix}; \begin{pmatrix} 13 & 6 \\ 6 & 5 \end{pmatrix}; \begin{pmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{pmatrix} \quad 20. \begin{pmatrix} 4 & -34 \\ 0 & 38 \end{pmatrix} \quad 21. \begin{pmatrix} 1 & 0 \\ 20 & 1 \end{pmatrix}$$

$$22. I, -I, \begin{pmatrix} 1 & a_{12} \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & a_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_{21} & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ a_{21} & 1 \end{pmatrix}$$

## Determinantlar

1.  $\lambda^n \det A$ . 5. 1. 6. 1)  $-2$ ; 2)  $-125$ ; 3)  $1$ ; 4)  $8$ . 7. 1)  $-12$ ; 2)  $-243$ ; 3)  $64$ ; 4)  $4$ . 8.  $-x^2$ . 9.  $-b(a+b)$ . 10.  $\sin(\alpha-\beta)\sin(\alpha+\beta)$ . 11.  $2\sin\alpha$ . 12.  $-47$ . 13.  $-18$ . 14.  $b^2(b-2)$ . 15.  $4x$ . 16.  $-2\sin\alpha\sin\beta\sin\gamma$ . 17.  $-tg\alpha - tg\beta$ . 18.  $(a-b)(a-c)(b-c)$ . 19.  $a(x-y)(x-z)(z-y)$ . 20.  $a^2(a+3b)$ . 21.  $-xyz$ . 22.  $0$ . 23.  $\cos 2\alpha$ . 4.  $63$ . 25.  $100$ . 26.  $2a-8b+c+5d$ . 27.  $-6$ .

## Matritsalar ustida almashtirishlar

$$3. ad-bc \neq 0. \quad 7. B, D. \quad 8. A^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \quad 9. 1); 2) 3). \quad 10. \frac{1}{3} \begin{pmatrix} -5 & -6 \\ -2 & -3 \end{pmatrix}$$

$$11. \frac{1}{22} \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix} \quad 12. 1) A = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad 2) A = \frac{1}{10} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}; \quad 3) A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

$$4) A = \frac{1}{2} \begin{pmatrix} -9 & 14 & -3 \\ -4 & 8 & -2 \\ 3 & -4 & 1 \end{pmatrix} \quad 13. \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 14. 0 \quad 15. -10$$

$$16. \frac{1}{2} \begin{pmatrix} 10 & -2 & -3 \\ -6 & 2 & 2 \\ -2 & 0 & 1 \end{pmatrix} \quad 17. \frac{1}{6} \begin{pmatrix} 8 & 14 & -10 \\ -4 & -1 & 2 \\ 2 & -4 & 2 \end{pmatrix} \quad 18. \frac{1}{8} \begin{pmatrix} 0 & 0 & 4 & -4 \\ -4 & 6 & -3 & 5 \\ 0 & -4 & 6 & -2 \\ 4 & 2 & -5 & 3 \end{pmatrix}$$

$$19. \begin{pmatrix} 2 & -1 & -1 & 1 \\ -4 & 0 & 2 & -1 \\ 2 & 0 & -1 & 1 \\ -5 & 1 & 3 & -2 \end{pmatrix} \quad 20. \quad 1) \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}; \quad 2) \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix}$$

$$3) \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}; \quad 4) \text{ mavjud emas}; \quad 5) \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 5 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

$$6) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{pmatrix} \quad 21. \quad 3. \quad 22. \quad 2. \quad 23. \quad 2. \quad 24. \quad 3.$$

### Chiziqli tenglamalar sistemasi

1. Birgalikda emas. 2. Birgalikda, aniqmas. 3. Birgalikda, aniq. 4. Birgalikda emas.

5.  $x_1 = -1, x_2 = -2, x_3 = -3.$  6.  $x_1 = 2, x_2 = -2, x_3 = 1.$

7.  $x_1 = 1, x_2 = -i, x_3 = -1, x_4 = 1.$  8.  $x_1 = 2, x_2 = -1, x_3 = -2, x_4 = 1.$

9.  $x_1 = 2, x_2 = c+1, x_3 = 2c-1, x_4 = c.$  10.  $x_1 = 5c_2 - 13c_1 - 3, x_2 = 5c_2 - 8c_1 - 1,$

$x_3 = c_1, x_4 = c_2.$  11.  $x_1 = 3, x_2 = 0, x_3 = 6.$  12.  $x_1 = 1, x_2 = -1, x_3 = 0.$

13.  $x_1 = 3, x_2 = 4, x_3 = -6.$  14.  $x_1 = -5, x_2 = -4, x_3 = 0.$  15.  $x_1 = -5, x_2 = 1, x_3 = 3.$

16.  $x_1 = -11, x_2 = -6, x_3 = -3.$  17.  $x_1 = -1, x_2 = 3, x_3 = 2.$  18.  $x_1 = 3, x_2 = -3, x_3 = 1.$

19.  $x_1 = 3, x_2 = 2, x_3 = 1.$  20.  $x_1 = 1, x_2 = 1, x_3 = -1.$  21.  $x_1 = 3, x_2 = -2.$

22.  $x_1 = 3, x_2 = -2.$  23.  $x_1 = 1, x_2 = 2, x_3 = 0.$  24.  $x_1 = 1, x_2 = -2, x_3 = 2.$

25.  $x_1 = -2, x_2 = 1, x_3 = 2.$  26.  $x_1 = 0, x_2 = \frac{1}{a}, x_3 = 0, a(a-1)(a+2) \neq 0.$

27.  $x_1 = -c, x_2 = 0, x_3 = c.$  28.  $x_1 = -15c, x_2 = 11c, x_3 = 14c.$

29.  $x_1 = 7c, x_2 = -11c, x_3 = -5c.$  30.  $x_1 = x_2 = x_3 = 0.$  31.  $x_1 = x_2 = x_3 = x_4 = 0.$

32.  $x_1 = -2c, x_2 = 7c, x_3 = 0, x_4 = 3c.$  33.  $\left(1, 0 - \frac{3}{2}, -\frac{1}{2}\right); (0, 1, 0, -1).$  34.  $(1, 0 - 2, 3), (0, 1, 1, 2).$



## Vektorlar

1.  $\vec{a} \perp \vec{b}$ . 2. 22. 5.  $\overline{AN} = \frac{\vec{a} + 2\vec{b}}{3}$ . 6.  $\overline{NM} = \vec{n} - 2\vec{m}$ . 7.  $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ .
8.  $-\frac{1}{2}(\vec{a} + \vec{b}), \frac{1}{2}(\vec{a} - \vec{b})$ . 9.  $m = 2\sqrt{3}$ . 10.  $m = 1, n = -3$ . 11.  $\Pi p_1 \overline{AB} = 2\sqrt{2}$ ,  
 $\Pi p_1 \overline{AD} = -\sqrt{2}$ ,  $\Pi p_1 \overline{DC} = \sqrt{2}$ ,  $\Pi p_1 \overline{AC} = 0$ . 12.  $\Pi p_1 \overline{AB} = 3$ ,  $\Pi p_1 \overline{BC} = 0$ ,  $\Pi p_1 \overline{CA} = -3$ ,  
 $\Pi p_1 \overline{AD} = 3$ ,  $\Pi p_1 \overline{BF} = -\frac{3}{2}$ ,  $\Pi p_1 \overline{CE} = -\frac{3}{2}$ . 1.13.  $\left\{ \frac{1}{2}; -\frac{1}{2} \right\}$ . 14.  $\left\{ \frac{2}{3}; \frac{2}{3} \right\}$ . 15.  $\left\{ \frac{1}{2}; \frac{1}{2}; -\frac{1}{2} \right\}$ .
16.  $\left\{ \frac{1}{3}; \frac{1}{3}; \frac{1}{3} \right\}$ . 17.  $\vec{a} = 2\vec{b} + \vec{c}$ ,  $\vec{b} = \frac{\vec{a} - \vec{c}}{2}$ ,  $\vec{c} = \vec{a} - 2\vec{b}$ . 1.18.  $\vec{d} = 2\vec{a} - 3\vec{b} + \vec{c}$ .
- 19.)  $\{-7; 17; -12\}$ ; 2)  $\left\{ \frac{5}{3}; -\frac{7}{3}; \frac{8}{3} \right\}$ . 20. 1)  $\{-8; -4; 0\}$ ; 2)  $\{7; -3; 2\}$ . 21.  $|\vec{a} + \vec{b}| = 6$ ,  $|\vec{a} - \vec{b}| = 14$ .
22.  $\sqrt{10}$ . 23.  $M(\pm\sqrt{3}; \pm\sqrt{3}; \pm\sqrt{3})$ . 24.  $\vec{a} = \{2; \pm 2\sqrt{2}; -2\}$ . 25.  $B(5; -3; -3)$ . 26.  $A(-3; -1; -3)$ .
27. 1)  $|\overline{AB}|^{\circ} = \left\{ \frac{12}{25}; \frac{3}{5}; -\frac{16}{25} \right\}$ ; 2)  $|\overline{AB}|^{\circ} = \left\{ -\frac{4}{13}; -\frac{3}{13}; -\frac{12}{13} \right\}$ . 28.  $\vec{a}^{\circ} = \left\{ \frac{2}{7}; \frac{6}{7}; \frac{3}{7} \right\}$ . 29.  $\alpha = -3$ .
30.  $\vec{b} = \left\{ \frac{48}{5}; -\frac{36}{5}; 9 \right\}$ . 31.  $p = 12$ . 33.  $O\left(\frac{1}{2}; \frac{13}{2}\right)$ ,  $R = \frac{\sqrt{130}}{2}$ . 34.  $M(-3; -5)$ .
35.  $(-2; 1)$ . 36.  $D(-3; -7)$ . 37.  $AD = 7b$ .

## Vektorlarni ko'paytirish

1.  $-\frac{3}{2}$ . 2.  $-19$ . 3. 1) 112; 2) 252. 4. 1)  $-89$ ; 2) 86. 5. 1)  $m = 1$ ; 2)  $m = 2$ ,  $m = 3$ .
6.  $-\frac{3}{2}$ . 7.  $\frac{\pi}{2}$ . 8.  $\frac{\pi}{4}$ . 9.  $\frac{\pi}{3}$ . 10. 1)  $\frac{\pi}{3}$ ; 2)  $\pi$ . 11. 1)  $\frac{21}{13}$ ; 2)  $\frac{261}{13}$ . 12. 10 (ish.b.).
13.  $\vec{x} = 2\vec{i} - 3\vec{j}$ . 14.  $\vec{x} = 7\vec{i} + 5\vec{j} + \vec{k}$ . 15. 1)  $12\vec{e}^0$ ; 2) 132. 16. 1)  $\frac{3}{2}(y.b.)$ ; 2)  $66\sqrt{3}(y.b.)$ .
17.  $25\sqrt{3}$ . 18.  $\pm 15$ . 19. 1)  $\{9; 9; -3\}$ ; 2)  $\{63; 63; -21\}$ . 20. 1)  $9\sqrt{2}$ ; 2)  $\frac{49}{2}$ .
21.  $h = \frac{14}{\sqrt{13}}(u.b.)$ . 22.  $\vec{M} = \{-8; -9; -4\}$ ;  $\vec{M} = \{1; -4; -7\}$ . 23. 28.  $\cos \alpha = -\frac{3}{7}$ ,  $\cos \beta = \frac{6}{7}$ ,  
 $\cos \gamma = \frac{2}{7}$ . 24.  $\alpha = -9$ . 25.  $\alpha = \frac{3}{2}$ ,  $\beta = 2$ . 26. 1)  $\{3; 2\}$ ; 2)  $\{-2; 3\}$ . 27.  $\{-6; -24; 8\}$ .
28. 1) yo'q; 2) ha. 29. 1)  $\alpha = \frac{1}{3}$ ; 2)  $\alpha = -3$ . 30. 1)  $V = 14(h.b.)$ ,  $h = \sqrt{14}(u.b.)$ ; 2)  
 $V = 2(h.b.)$ ,  $h = 3\sqrt{2}(u.b.)$ . 31. 1) o'ng uchlik,  $V = 12(h.b.)$ ; 2) chap uchlik,  $V = 27(h.b.)$ .

## Tekislikdagi to'g'ri chiziq

1. 1)  $k = -\frac{3}{4}$ ,  $a = 4$ ,  $b = 3$ ; 2)  $k = \frac{1}{3}$ ,  $a = -2$ ,  $b = \frac{2}{3}$ ; 3)  $k = \frac{1}{2}$ ,  $a = 5$ ,  $b = -\frac{5}{2}$ .

- 4)  $k = -\frac{3}{5}, a = \frac{5}{2}, b = \frac{3}{2}$ . 2. 1)  $3x + 4y + 6 = 0$ ; 2)  $3x + y + 9 = 0$ ; 3)  $x + 2 = 0$ ;
- 4)  $x + y - 5 = 0$ . 3.  $2\sqrt{3}$ . 4. 1)  $M_0(1; 2), \varphi = 45^\circ$ ; 2)  $M_0(2; -1), \varphi = 90^\circ$ ;
- 3)  $M_0 \in \emptyset, \varphi = 0$ ; 4)  $M_0(2; 2), \varphi = 45^\circ$ . 5. 1)  $m = -6, n \neq 3$  va  $m = 6, n \neq -3$ ;
- 2)  $m = -6, n = 3$  va  $m = 6, n = -3$ ; 3)  $m = 0, n$  - chekli son. 6. 1)  $m = -\frac{3}{2} da \parallel$ ,  
 $m = \frac{2}{3} da \perp$ ; 2)  $m = 4 da \parallel$ ,  $m = -9 da \perp$ . 7. (1; 6). 8.  $x - y - 2 = 0$  va  $x - 4y + 4 = 0$ .
9.  $3x + 2y - 11 = 0$ . 10.  $x - 5y + 2 = 0$ . 11.  $12x + 9y - 17 = 0$ . 12.  $5x - y + 3 = 0$ ,  
 $x + 5y + 11 = 0$ . 13.  $3x + y - 4 = 0$ ,  $x + 5y + 8 = 0$ ,  $3x + y + 10 = 0$ ,  $x + 5y - 6 = 0$ .
14.  $M(4; 4), \varphi = \frac{\pi}{2}$ . 15.  $3x - 3y - 8 = 0$ . 16.  $3x + 4y - 12 = 0$ .
17.  $x + 2y - 7 = 0$ ,  $7x + 2y - 37 = 0$ ,  $5x - 2y + 1 = 0$ . 18.  $y = 2x$ .
19.  $x - y + 7 = 0$ ,  $7x + 4y - 6 = 0$ ,  $6x + 5y + 9 = 0$ . 20.  $2x + y + 9 = 0$ ,  $x - y - 3 = 0$ .
21.  $29x - 2y + 33 = 0$ . 22.  $2x - 3y + 8 = 0$ . 23.  $29(y; b)$ . 24.  $\frac{23}{10}(u; b)$ . 25.  $6\sqrt{2}(u; b)$ .
26.  $x - 2 = 0$ ,  $3x - 4y + 10 = 0$ . 27.  $y - 3 = 0$ ,  $4x + y + 5 = 0$ . 8.  $(-12; 5)$ .

### *Tekislikdagi ikkinchi tartibli chiziqlar*

1. 1)  $(x+1)^2 + (y-3)^2 = 36$ ; 2)  $(x+3)^2 + (y-5)^2 = 50$ ; 3)  $(x+2)^2 + (y-4)^2 = 2$ ;
- 4)  $(x-4)^2 + (y+4)^2 = 16$ ,  $(x-20)^2 + (y+20)^2 = 400$ ; 5)  $(x-2)^2 + (y+1)^2 = 1$ .
2. 1)  $M_0(-4; 7), R = 7$ ; 2)  $M_0(-2; 3), R = 4$ ; 3)  $M_0\left(\frac{1}{2}; -1\right), R = \frac{3}{2}$ ; 4)  $M_0\left(-\frac{3}{2}; \frac{7}{2}\right), R = 4$ . 3. 1) aylanada; 2) aylana ichkarisida; 3) aylanada; 4) aylana ichkarisida.
4.  $5\sqrt{2}(u; b)$ . 5.  $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{25}{4}$ . 6.  $(x-5)^2 + (y-1)^2 = 13$ . 7.  $M_0(3; 2), R = 5$ . 8.  $0 < k < \frac{8}{15}, k_1 = 0$  va  $k_2 = \frac{8}{15}$ . 9.  $y = 0$  va  $4x - 3y = 0$ .
10. 1)  $\begin{cases} x = 8(1 + \cos t), \\ y = 8 \sin t, t \in [0; 2\pi]; \end{cases}$  2)  $\begin{cases} x = 2 \cos t, \\ y = 2(1 + \sin t), t \in [0; 2\pi]; \end{cases}$  3)  $\begin{cases} x = 1 + \sqrt{2} \cos t, \\ y = 1 + \sqrt{2} \sin t, t \in \{0; 2\pi\}. \end{cases}$
11. 1)  $\frac{x^2}{36} + \frac{y^2}{100} = 1$ ; 2)  $\frac{x^2}{24} + \frac{y^2}{49} = 1$ ; 3)  $\frac{x^2}{36} + \frac{y^2}{81} = 1$ ; 4)  $\frac{x^2}{16} + \frac{y^2}{25} = 1$ .

12. 12(*kyb*). 13.  $x+y+5=0$  va  $x+y-5=0$ . 14.  $\frac{32}{5}(u, b)$ .

15.  $M_1\left(-\frac{15\sqrt{2}}{4}; \frac{\sqrt{126}}{4}\right)$ ,  $M_2\left(-\frac{15\sqrt{2}}{4}; -\frac{\sqrt{126}}{4}\right)$ . 16.  $M(3;0)$ . 17.  $16x^2+25y^2=400$ .

18. 1)  $\begin{cases} x=5\cos t, \\ y=4\sin t, t \in [0; 2\pi]; \end{cases}$  2)  $\begin{cases} x=5\cos t, \\ y=12\sin t, t \in [0; 2\pi]. \end{cases}$  19.  $\left(4; \frac{3}{2}\right)$ ,  $(3; 2)$ .

20. 1)  $\frac{y^2}{9} - \frac{x^2}{16} = 1$ ; 2)  $\frac{y^2}{144} - \frac{x^2}{25} = 1$ ; 3)  $\frac{y^2}{16} - \frac{x^2}{9} = 1$ ; 4)  $\frac{y^2}{25} - \frac{x^2}{24} = 1$ .

21. 1)  $\frac{x^2}{24} - \frac{y^2}{8} = 1$ ; 2)  $\frac{x^2}{8} - \frac{y^2}{4} = 1$ ; 3)  $\frac{x^2}{12} - \frac{y^2}{27} = 1$ ; 4)  $\frac{x^2}{24} - \frac{y^2}{18} = 1$ . 22.  $\frac{2\pi}{3}$ .

23.  $\sqrt{2}$ . 24.  $x^2 - y^2 = 6$ . 25.  $\frac{x^2}{4} - \frac{y^2}{12} = 1$ . 26.  $y = \frac{-3x+10}{x-2}$ . 27.  $y' = -\frac{29}{9x'}$ .

28. )  $x = -\frac{1}{16}y^2 + \frac{1}{2}y$ ; 2)  $y = \frac{1}{10}x^2 - x + 3$ . 29. 1)  $A(-4; 1)$ ,  $y = 1$ ; 2)  $A(2; 3)$ ,  $x = 2$ .

30. 1)  $x^2 - y^2 = 1$  - giperbola; 2)  $y^2 = \frac{9}{2}x$  - parabola; 3) giperbolaning pastgi yarim tekislikdagi tarmog'i; 4) giperbolaning chap yarim tekislikdagi tarmog'i.

33.  $\frac{x'^2}{4} + \frac{y'^2}{9} = 1$ . 34.  $\frac{x'^2}{2} - y'^2 = 1$ . 36.  $x - 3y + 12 = 0$ . (Ko'rsatma. urinma

lenglamalari:  $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$  - ellips,  $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$  - giperbola,

$yy_0 = p(x+x_0)$  - parabola). 37.  $4x + y - 4 = 0$ . 37.  $2\sqrt{5}x - 3y + 4 = 0$ . 38.  $2x - y - 16 = 0$ .

### Qutb koordinatalarda chiziqlar

1.  $A\left(2; \frac{\pi}{6}\right)$ ,  $B\left(2; \frac{5\pi}{6}\right)$  2.  $A(1; -\sqrt{3})$ ,  $B\left(-\frac{1}{2}; \frac{\sqrt{3}}{2}\right)$ . 3. 7 b. 4.  $S = \frac{1}{2}r_1r_2 \sin(\varphi_2 - \varphi_1)$ . 5. 4 y.b.

6. 64 y.b. 7. 1)  $x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$ ; 2)  $\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$ ; 3)  $(x^2 + y^2)^2 = 2xy$ ;

4)  $(x^2 + y^2)^3 = 4(x^2 - y^2)$ ; 5)  $x^2 = 4(1 - y)$ ; 6)  $y^2 = 3(3 + 2x)$ ; 7)  $\frac{\left(x + \frac{3}{8}\right)^2}{25} + \frac{y^2}{16} = \frac{1}{64}$ ;

8)  $\frac{x^2}{16} - \frac{(y-5)^2}{9} = \frac{13}{3}$ . 8. 1)  $r = \frac{5}{\sin \varphi}$ ; 2)  $r = \frac{1}{2\cos \varphi - \sin \varphi}$ ; 3)  $r = \frac{4\sin \varphi}{\cos^2 \varphi}$ ; 4)

$r^2 = \frac{a^2}{\cos 2\varphi}$ ; 5)  $r = 2a\cos \varphi$ ; 6)  $r^2 = \frac{2}{\sin 2\varphi}$ ; 7)  $r = \frac{16}{1 - \frac{9}{25}\cos^2 \varphi}$ ; 9)  $r = \frac{-4}{1 - \frac{10}{9}\cos^2 \varphi}$ .

9. 1) ellips; 2) giperbola; 3) parabola; 4) ellips.

### Tekislik

1.  $2x - y + 3z - 14 = 0$ . 2.  $2x - 3y + 4z + 20 = 0$ . 3.  $2x - 3y + 5z + 10 = 0$ .

4.  $x + 3y - 4z - 21 = 0$ . 5. 1)  $2y + 3z = 0$ ; 2)  $y + 1 = 0$ ; 3)  $z - 4 = 0$ ;

4)  $x + z - 3 = 0$ ; 5)  $22x + 14y - 5z = 0$ . 6. 1)  $2x - z = 0$ ; 2)  $x - 3 = 0$ ; 3)  $y + 3 = 0$ ;

4)  $x + 3z + 4 = 0$ ; 5)  $8x - y + 6z = 0$ . 7.  $A(-3; 0; 0), B(0; -6; 0), C(0; 0; 2)$ . 1).

8.  $a = -15, b = -10, c = 6$ . 9.  $x + y + z - 4 = 0$ . 10.  $x + 3y + z - 15 = 0$ .

11. 1)  $x + 3y - z - 6 = 0$ ; 2)  $x - y - z = 0$ . 12.  $x + y + z - 6 = 0$ .

13.  $x + y - 2z = 0$ . 14.  $x - 2y - 3z + 3 = 0$ . 15. 1)  $2x - 5y + z - 15 = 0$ ;

2)  $2x + 4y + 9z - 21 = 0$ . 16.  $x - y - 2z + 5 = 0$ . 17.  $\frac{x}{9} + \frac{y}{2} + \frac{z}{6} = 1$ ;

$\frac{9}{11}x - \frac{2}{11}y + \frac{6}{11}z - 1 = 0$ . 18.  $\frac{x}{-6} + \frac{y}{7} + \frac{z}{7} = 1$ ;  $-\frac{7}{11}x + \frac{6}{11}y + \frac{6}{11}z - \frac{42}{11} = 0$ .

19. 1)  $45^\circ$ ; 2)  $90^\circ$ ; 3)  $90^\circ$ ; 4)  $\arccos(0,4)$ . 20. 1)  $m = -\frac{6}{5}, n = -\frac{15}{2}$ ; 2)  $m = 3, n = -4$ .

21. 1)  $m = 13$ ; 2)  $m = 1$ . 22. 1) a)  $x - 2y - 3z - 4 = 0$ ; b)  $2x + 3y + z - 8 = 0$ ;

2) a)  $2x + 3y + 4z - 3 = 0$ ; b)  $4x + y - 7z + 19 = 0$ ; 3) a)  $5x + 7y + 3 = 0$ ; b)  $y - z + 7 = 0$ ;

c)  $5x + 7z - 46 = 0$ . 23.  $7x + 14y - 2z + 6 = 0$ . 24.  $x - y + z + 1 = 0$ .

25.  $x + 2y + \sqrt{5}z - 2 = 0$  va  $x + 2y - \sqrt{5}z - 2 = 0$ . 26. 1)  $M(-2; 1; 2)$ ; 2)  $M(2; -1; 1)$ . 27. 4(b).

28.  $M(-15; 0; 0)$  va  $M(1; 0; 0)$ . 29.  $2x - y - 2z = 0$  va  $2x - y - 2z - 18 = 0$ . 30. 8(h.b.).

31. -2. 32. -3. 33.  $\frac{\sqrt{3}}{6}$ .

### Fazodagi to'g'ri chiziq

1. 1)  $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+2}{-1}$ ; 2)  $\frac{x-2}{0} = \frac{y+3}{1} = \frac{z+1}{0}$ ; 3)  $\frac{x-2}{3} = \frac{y+1}{1} = \frac{z+2}{-2}$ .

2. 1)  $\frac{x-2}{4} = \frac{y+3}{1} = \frac{z-2}{3}$ ; 2)  $\frac{x-2}{2} = \frac{y+3}{3} = \frac{z-2}{-1}$ ; 3)  $\frac{x-2}{-11} = \frac{y+3}{6} = \frac{z-2}{-7}$ .

3.  $\frac{x}{-1} = \frac{y}{2} = \frac{z-2}{0}$ . 4.  $\frac{x+1}{1} = \frac{y-2}{\sqrt{2}} = \frac{z+3}{-1}$ . 5. 1)  $\begin{cases} x+4y-7=0, \\ x+z-1=0; \end{cases}$  2)  $\begin{cases} 3x-2y-7=0, \\ 2y+3z+1=0. \end{cases}$

6.  $\frac{x-2}{-5} = \frac{y-2}{-3} = \frac{z+1}{4}$ . 7. 1)  $x = 13t, y = 1 + 19t, z = 2 + 28t$ ; 2)  $x = t, y = 1 - 3t, z = -2t$ .

$$8. 1) \frac{x}{1} = \frac{y-7}{2} = \frac{z-5}{2}; 2) \frac{x}{1} = \frac{y+1}{2} = \frac{z-2}{3}. 9. \frac{x+1}{2} = \frac{y-2}{-1} = \frac{z-3}{0}. 10. \frac{x+3}{-5} = \frac{y}{1} = \frac{z-4}{-3}.$$

$$11. 1) \varphi = \frac{\pi}{4}; 2) \varphi = \frac{\pi}{2}. 12. 1) \frac{x+2}{-5} = \frac{y-3}{1} = \frac{z+1}{3}; 2) \frac{x+2}{6} = \frac{y-3}{16} = \frac{z+1}{17}.$$

$$13. \frac{x-1}{1} = \frac{y+1}{3} = \frac{z-2}{2}. 14. 1) \text{ parallel}; 2) \text{ ayqash}. 15. 1) \varphi = \frac{\pi}{4}; 2) \varphi = \frac{\pi}{6}.$$

$$16. 1) \text{ parallel}; 2) \text{ to'g'ri chiziq tekisligida yotadi}. 17. 1) M(3;2;1); 2) M(2;4;6).$$

$$18. 1) \frac{x-4}{1} = \frac{y-5}{-2} = \frac{z+6}{0}; 2) \frac{x-4}{1} = \frac{y-5}{-1} = \frac{z+6}{1}. 19. 1) m=3, n=-23; 2) m=12,$$

$$n=-i2. 20. 1) 2x-3y+4z-1=0; 2) 4x-y-2z-7=0; 3) z+1=0.$$

$$21. 3x+5y+2z-9=0. 22. (-2;0;-3). 23. M(2;3;4). 24. 5x-2y+2z+5=0.$$

$$25. 8x+2y-9z+12=0. 26. 1) \frac{\sqrt{102}}{10}(ub.); 2) \frac{\sqrt{41}}{3}(ub.).$$

### Ikkinchi tartibli sirtlar

$$1. 1) (x-3)^2 + (y+1)^2 + (z-1)^2 = 21; 2) (x-3)^2 + (y+5)^2 + (z+2)^2 = 56.$$

$$2. 1) M\left(-\frac{1}{2}; \frac{1}{2}; -\frac{1}{2}\right), R = \frac{\sqrt{3}}{2}; 2) M(3, -4, -5), R = 5. 3. 1) x^2 + y^2 + z^2 = 9.$$

$$4. x^2 + z^2 = 10y. 5. \frac{x^2 + y^2 + z^2}{9} = 1. 6. \frac{y^2 + z^2}{16} - \frac{x^2}{9} = -1. 7. 1) 4y^2 - x^4 + 4z^2 = 0,$$

$$z = -\frac{x^2 + y^2}{2}, 2) \frac{x^2}{16} - \frac{y^2 + z^2}{25} = 1, \frac{y^2}{25} - \frac{x^2 + z^2}{16} = -1; 3) \frac{y^2}{64} + \frac{x^2 + z^2}{16} = 1,$$

$$\frac{x^2 + y^2}{64} + \frac{z^2}{16} = 1. 8. x^2 + y^2 - z^2 = 0. 9. 1) m \neq 0 \text{ va } m \geq -\frac{1}{4}; 2) m = 0. 10. 1) \text{ ellips};$$

$$2) \text{ giperbola}; 3) \text{ parabola}; 4) \text{ nuqta}. 11. y^2 + z^2 - 2x^2 = -6 (\text{ikki pallali giperboloid}).$$

$$12. 1) M_1(3;4;-2), M_2(6;-2;2); 2) M(4;-3;2). 13. 1) \text{ ikki pallali giperboloid}; 2) \text{ sfera}; 3) \text{ elliptik paraboloid}; 4) \text{ aylanish ellipsoidi}; 5) \text{ giperbolik silindr}; 6) \text{ giperbolik paraboloid}; 7) \text{ ikki pallali giperboloid}; 8) \text{ doiraviy silindr}.$$

### Haqiqiy sonlar

$$1. 1) A \cap B = \{4\}, A \cup B = \{1,2,3,4,5,6\}, A/B = \{1,2,3\}, B/A = \{5,6\};$$

$$2) A \cap B = \{1,4,8\}, A \cup B = \{1,2,3,4,5,6,7,8,9\}, A/B = \{3\}, B/A = \{2,5,7,9\};$$

$$3) A \cap B = \{4\}, A \cup B = \{-5,3,4\}, A/B = \{-5\}, B/A = \{3\}.$$

$$2. 1) A \cap B = \emptyset, A \cup B = N. 3. A \cap B - \text{barcha } 10 \text{ ga bo'linadigan sonlar}.$$

$$6. 1) A = \{1,2,3,4\}; 2) A = (1,2,3). 7. 1) \left\{ \frac{7}{6}, \frac{1}{3} \right\}; 2) \emptyset; 3) \{-1,0\}; 4) \{-\infty, 2\}.$$

8. 1)  $(-\infty; 1] \cup [3; \infty)$ ; 2) (3,4); 3)  $[-\infty; -1]$ ; 4)  $\{-1, 0\}$ ; 4)  $\{-\infty, 2\}$ . 9. 1) 0, -5;  
2) 0, mavjud emas.

### Sonli ketma-ketliklar

1. 1)  $x_n = \frac{1}{3n-1}$ ; 2)  $x_n = \frac{5^n}{n!}$ ; 3)  $x_n = \cos n\pi$ ; 4)  $x_n = 3 + 2(-1)^n$ . 2. 1); 2); 4); 6).

3. 2), 5)-monoton, 1), 3), 4), 6)-qat'iy monoton. 6. 1)  $-\frac{1}{2}$ ; 2) 0; 3)  $\infty$ ; 4) 8; 5) 4;

6) 2; 7)  $\frac{1}{5}$ ; 8)  $-\frac{5}{2}$ ; 9) 0; 10) 1; 11)  $-\frac{5}{2}$ ; 12)  $-\frac{4}{3}$ ; 13)  $\infty$ ; 14)  $\infty$ ; 15) 1; 16) 0;

17) -3; 18)  $\frac{1}{2}$ ; 19)  $\frac{1}{6}$ ; 20)  $\frac{1}{4}$ ; 21) 0; 22)  $-\frac{3}{2}$ ; 23)  $\frac{4}{3}$ ; 24)  $\frac{1}{36}$ ; 25)  $-\frac{1}{2}$ ; 26) 2;

27)  $\frac{1}{2}$ ; 28)  $\frac{1}{e^4}$ ; 29)  $e^3$ ; 30)  $e^2$ .

### Bir o'zgaruvchining funksiyasi

1. 1)  $(-\infty; -2) \cup (-2; +\infty)$ ; 2)  $(-\infty; -3) \cup (-3; -2) \cup (-2; +\infty)$ ; 3)  $[-2; 2]$ ; 4)  $(-2; 1) \cup (1; +\infty)$ ;

5) 4)  $(-\infty; 2) \cup (9; 10]$ ; 6)  $\left[-1; -\frac{1}{2}\right] \cup \left(-\frac{1}{2}; \frac{1}{2}\right) \cup \left(\frac{1}{2}; 1\right]$ ; 7)  $[7; 10]$ ; 8)  $\left[-\frac{1}{2}; +\infty\right)$ ; 9)  $\{2\}$ ;

10)  $(2; +\infty)$ ; 11)  $\emptyset$ ; 12)  $(2; 3]$ ; 13)  $(10; +\infty)$ ; 14)  $(2n\pi; (2n+1)\pi), n \in \mathbb{Z}$ ; 15)  $\left[0; \frac{2}{3}\right)$ ;

16)  $[3; 6) \cup (6; 7]$ ; 17)  $\left[-\frac{3}{4}; \frac{3}{4}\right]$ ; 18)  $[-5; 0) \cup (0; 1]$ ; 19)  $(-\infty; 1) \cup (1; 2) \cup (2; +\infty)$ ;

20)  $(-3; 2)$ . 2. 1)  $[-2; +\infty)$ ; 2)  $[2; +\infty)$ ; 3)  $[-7; -3]$ ; 4)  $[-\sqrt{2}; \sqrt{2}]$ ; 5)  $[0; +\infty)$ ; 6)  $(1; 3]$ ;

7)  $[0; 3]$ ; 8)  $\left(-\frac{1}{2}; \frac{1}{2}\right)$ ; 9)  $\left[-\frac{1}{5}; +\infty\right)$ ; 10)  $\{-1\} \cup \{1\}$ ; 11)  $(0; 3]$ ; 12)  $(0; 2]$ . 3. 1) 3;

1)  $-\frac{4}{3\sqrt{4}}$ ; 3)  $-\frac{x^3}{3^x}$ ; 4)  $\frac{3^x}{x^3}$ . 4. 1)  $\left(-\infty; \frac{5}{2}\right)$  da kamayadi,  $\left(\frac{5}{2}; +\infty\right)$  da o'sadi;

2)  $(-\infty; +\infty)$  da o'sadi; 3)  $(-\infty; 0) \cup (0; +\infty)$  da kamayadi; 4)  $(-\infty; +\infty)$  da

kamayadi. 5. 1) toq; 2) juft; 3) juft; 4) umumiy ko'rinishda; 5) toq; 6) toq;

7) juft; 8) toq; 9) toq; 10) juft. 6. 1)  $M = n, m = k$ ; 2)  $M = 4, m = -4$ ;

3)  $M = \sqrt{2}, m = -\sqrt{2}$ ; 4)  $M = \sqrt{5}, m = -\sqrt{5}$ ; 5)  $M = 1, m = \frac{1}{2}$ ; 6)  $M = 1, m = 0$ .

7. 1) chegaralangan; 2) qat'iy monoton; 3) qat'iy monoton; 4) monoton.

8. 1)  $6\pi$ ; 2)  $\frac{\pi}{2}$ ; 3)  $4\pi$ ; 4)  $\frac{\pi}{2}$ ; 5)  $\pi$ ; 6)  $2\pi$ ; 7)  $\frac{\pi}{2}$ ; 8)  $\frac{\pi}{3}$ ; 9)  $12\pi$ ; 10)  $6\pi$ .

9. 1)  $y = \frac{x-5}{3}$ ; 2)  $y = \frac{x}{1-x}$ ; 3)  $y = 3^{x-4}$ ; 4)  $y = \frac{1}{3} \arcsin \frac{x}{2}$ .

10. 1)  $f(g(x)) = 3x^3 + 1$ ,  $g(f(x)) = (3x+1)^3$ ; 2)  $f(g(x)) = \sin |x|$ ,  $g(f(x)) = |\sin x|$ .

$f(g(x)) = 5-x$ ,  $g(f(x)) = \frac{x}{3x-1}$ ; 4)  $f(g(x)) = x^3$ ,  $g(f(x)) = 3x$ .

13. A; C; D. 14. A; B. 15. 1)  $y = x^2 + 1$ ; 2)  $\frac{x^2}{9} + \frac{y^2}{4} = 1$ .

### Funksiyaning limiti

2.  $f(x_0 - 0) = 2$ ,  $f(x_0 + 0) = 3$ ; 2)  $f(x_0 - 0) = 0$ ,  $f(x_0 + 0) = +\infty$ ; 3)  $f(x_0 - 0) = 2$ ,

$f(x_0 + 0) = 0$ ; 4)  $f(x_0 - 0) = \frac{1}{5}$ ,  $f(x_0 + 0) = 1$ . 5. 1) 8; 2) 0; 3)  $\frac{3}{2}$ ; 4)  $\frac{1}{3}$ ; 5)  $\frac{4}{3}$ ; 6) 2;

7)  $-\frac{1}{12}$ ; 8)  $\frac{1}{3}$ ; 9) -1; 10)  $+\infty$ ; 11) -2; 12) -1; 13)  $-\frac{4}{3}$ ; 14) -3; 15) 0; 16)  $+\infty$ ;

17)  $-\frac{1}{4}$ ; 18) 2; 19) 0; 20)  $\frac{2}{25}$ ; 21) 2; 22) 0; 23) 1; 24)  $-\frac{3}{2}$ ; 25)  $\frac{3}{4}$ ; 26)  $\frac{1}{2}$ ; 27)  $6\sqrt{2}$ ;

28)  $\frac{\sqrt{2}}{8}$ ; 29) 0; 30) 0; 31)  $\frac{1}{\pi}$ ; 32)  $\frac{1}{\pi}$ ; 33) -1; 34)  $\frac{1}{2}$ ; 35)  $e^{-3}$ ; 36)  $e$ ; 37)  $+\infty$ ; 38) 0;

39)  $e^2$ ; 40)  $e^{-1}$ ; 41)  $e$ ; 42)  $e^{-2}$ ; 43)  $e$ ; 44)  $e$ ; 45) 1; 37) 3; 46)  $\frac{1}{2}$ ; 47) 4; 48) 1.

6. 1)  $\frac{2}{3}$ ; 2)  $\frac{1}{2}$ ; 3) -1; 4)  $\ln 3$ ; 5) 1; 6) 5; 7)  $\frac{\ln 3}{2}$ ; 8) 2; 9)  $\frac{2}{3}$ ; 10)  $\frac{1}{6}$ ; 11) 1; 12) 2;

13)  $\frac{1}{2}$ ; 14)  $\frac{1}{2} \ln \frac{9}{5}$ ; 15)  $-\frac{1}{4}$ ; 16)  $-\frac{1}{2}$ ; 17)  $\frac{1}{2}$ ; 18) -9; 19) 3; 20)  $\frac{2}{\pi}$ ; 21) 0;

22)  $\ln 2$ ; 23) -1; 24)  $\frac{3}{2}$ .

### Funksiyaning uzluksizligi

4. 1) -3,3; 2) -1. 5. 1) ikkinchi tur uzilish nuqtasi; 2) birinchi tur (bartaraf qilinadigan) uzilish nuqtasi; 3) birinchi tur uzilish (sakrash) nuqtasi; 4) ikkinchi tur uzilish nuqtasi; 6. 1)  $x=0$  birinchi tur (bartaraf qilinadigan) uzilish nuqtasi; 2)  $x = \frac{\pi}{2} + n\pi (n \in \mathbb{Z})$  birinchi tur (bartaraf qilinadigan) uzilish nuqtasi. 7. 1)  $x = -3$  da ikkinchi tur uzilishga ega; 2) uzluksiz. 8. 1)  $[4,5]$  da uzluksiz,  $[0,2]$  da

$x = 1$  – ikkinchi tur uzilishga ega,  $[-3; 1]$  da  $x = -3, x = 1$  – ikkinchi tur uzilishga ega; 2) hech bir kesmada aniqlanmagan. 10.1)  $\ln a; 2) m$ .

### Funksiyaning hosilasi va differensiallari

1. 1)  $f'(x) = \frac{3}{2\sqrt{3x-1}}$ ; 2)  $f'(x) = \frac{5}{(1-5x)^2}$ ; 3)  $f'(x) = -\frac{2}{\sin^2 2x}$ ; 4)  $f'(x) = 2\operatorname{sh} 2x$ .

2. 1)  $-3$ ; 2)  $-4$ ; 3)  $4$ ; 4)  $-\frac{1}{2}$ . 3. 1)  $-3, 3$ ; 2)  $0, 2$ ; 3)  $1, -2x+3$ ; 4)  $-1, 1$ .

4.  $t_1 = 1, t_2 = 3$ . 5. 1)  $t = 2c$ ; 2)  $t = 1c$ . 6.  $l = 12a$ .

### Differensiallash qoidalari va formulalari

1. 1)  $y' = 12x^3 - x^2$ ; 2)  $y' = x^5 + 12x^3 - 2$ ; 3)  $y' = -\frac{1}{x\sqrt{x}} + 7x\sqrt{x} + \frac{4}{x^3\sqrt{x^2}}$ ;

4)  $y' = \frac{1}{2\sqrt{x}} + \frac{3}{x^2} - \frac{1}{x^4}$ ; 5)  $y' = \frac{xe^x(x-1) + e^{-x}(x+2)}{x^3}$ ; 6)  $y' = \frac{2 \cdot 6^x \ln 3}{(2^x - 3^x)^2}$ ;

7)  $y' = \frac{\ln^2 x - \ln x - 1}{(\ln x - 1)^2}$ ; 8)  $y' = \frac{2e^x(x \ln x - 1)}{x(\ln x - e^x)^2}$ ; 9)  $y' = -\frac{2 \sin x}{(1 - \cos x)^2}$ ; 10)  $y' = \frac{2}{1 - \sin 2x}$ ;

11)  $y' = \frac{4}{\sin^2 2x}$ ; 12)  $y' = \frac{x^2 + 2}{(x \cos x + \sin x)^2}$ ; 13)  $y' = -\left(\frac{x}{x \operatorname{sh} x - \operatorname{ch} x}\right)^2$ ; 14)  $y' = -\frac{4}{\operatorname{sh}^2 2x}$ ;

15)  $y' = -\frac{1}{x \ln^2 x}$ ; 16)  $y' = -\frac{3}{x \ln 10}$ ; 17)  $y' = -\frac{3x}{\sqrt{4-3x^2}}$ ; 18)  $y' = \frac{1}{2\sqrt{x-x^2}}$ ;

19)  $y' = -2 \sin 2x$ ; 20)  $y' = \frac{1}{x^2 - 9}$ ; 21)  $y' = \arcsin x$ ; 22)  $y' = \frac{2e^x(e^x - 1)}{e^{2x} + 1}$ ;

23)  $y' = \frac{3^x \ln 3}{1 - 9^x}$ ; 24)  $y' = \frac{1}{3}$ ; 25)  $y' = (1 - \operatorname{tg} 3x)^2$ ; 26)  $y' = -6e^{-3x} \sin 3x$ ; 27)  $y' = \frac{\sqrt{e^x - 1}}{2}$ ;

28)  $y' = -\frac{1}{\cos x}$ ; 29)  $y' = -\frac{x}{\sqrt{6x-4-x^2}}$ ; 30)  $y' = \frac{x^3 + x - 1}{(x^2 + 2)^2}$ . 2. 1)  $y' = -\frac{2}{(1+x)^2}$ ; 2)

$y' = -\frac{1}{x}$ ; 3)  $y' = \frac{1}{\sqrt{4-x^2}}$ ; 4)  $y' = -\frac{3}{x^2+9}$ . 3. 1) 2,0125; 2) 1,009; 3) 0,9942;

4) 27,351. 4. 1) 2,03; 2) 0,97; 3) 0,31; 4) 1,01. 5. 1)  $dy = \ln x dx$ ; 2)  $dy = \frac{1 - \ln x}{x^2} dx$ ;

3)  $dy = -2 \sin 4x dx$ ; 4)  $dy = 3a \sin^2 x \cos x dx$ ; 5)  $dy = -\sin x 3^{\cos x} \ln 3 dx$ ;

6)  $dy = -3 \operatorname{tg} x \ln^2 \cos x dx$ . 6. 1)  $y'' = 24x(5x^2 - 3)$ ; 2)  $y'' = e^{2x}(2 \cos x - 11 \sin x)$ ;



$$1) y'' = \frac{4}{(1+x^2)^2}; \quad 4) y''' = \frac{2}{x}. \quad 7. 1) \sin \frac{n\pi}{2}; \quad 2) n \sin \frac{n\pi}{2}; \quad 3) -n(n-1) \sin \frac{n\pi}{2}; \quad 4) n(n-1).$$

$$8. 1) y' = -\frac{b^2 x}{a^2 y}; \quad 2) y' = \frac{x^2 + y}{y^2 - x}; \quad 3) y' = \frac{y(1-x)}{x(y-1)}; \quad 4) y' = -\frac{2x + y \sin(xy)}{x \sin(xy)};$$

$$5) y' = -\frac{y}{e^y + x}; \quad 6) y' = -\frac{y \cos x + \sin y}{x \cos y + \sin x}. \quad 9. 1) \frac{3}{4t}; \quad 2) -\frac{1}{a \sin^3 t}; \quad 3) \frac{1+t^2}{4t}; \quad 4) -\sqrt{1-t^2}.$$

$$10. 1) 3x - 3y + 2 = 0, 3x + 3y + 4 = 0; \quad 2) x + y - \pi = 0,$$

$$x - y - \pi = 0; \quad 3) 5x - y - 4 = 0, x + 5y - 6 = 0; \quad 4) 5x + 4y - 25 = 0, 20x - 25y + 64 = 0;$$

$$5) x - y = 0, \quad x + y - 4 = 0; \quad 6) 4x + 2y - 3 = 0, 2x - 4y + 1 = 0.$$

### Differensial hisobining asosiy teoremlari

$$1. 1) c = \frac{2\sqrt{3}}{3}; \quad 2) c = \frac{3\pi}{4}; \quad 3) yo'q; \quad 4) yo'q. \quad 2. 1) c = \frac{\sqrt{3}}{3}; \quad 2) c = \ln(e-1); \quad 3) c = e-1;$$

$$4) c = \frac{1}{2}. \quad 3. 1) \left(-\frac{1}{2}; -\frac{5}{4}\right); \quad 2) \left(\frac{5}{4}; \frac{3}{2}\right). \quad 4. 1) c = \frac{\pi}{8}; \quad 2) c = \frac{3}{2}. \quad 6. 1) -\pi; \quad 2) \frac{1}{3}; \quad 3) 1;$$

$$4) 0; \quad 5) 0; \quad 6) 2; \quad 7) \frac{1}{2}; \quad 8) -\frac{1}{4}; \quad 9) 3; \quad 10) -3; \quad 11) 0; \quad 12) 0; \quad 13) 1; \quad 14) e; \quad 15) e^{\frac{2}{3}};$$

$$16) e^{-2}; \quad 17) 1; \quad 18) 3e. \quad 7. 1) P(x) = 19 - 11(x+2) - (x+2)^2 + (x+2)^3;$$

$$2) P(x) = 4 + 13(x-2) + 12(x-2)^2 + 6(x-2)^3 + (x-2)^4;$$

$$8. 1) 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 - \frac{5(x-3)^4}{128\sqrt{(1+c)^7}}, \quad c = x_0 + \theta(x-x_0), \quad 0 < \theta < 1;$$

$$2) \frac{1}{2} - \frac{(x+2)}{4} - \frac{(x+2)^2}{8} - \frac{(x+2)^3}{16} + \frac{(x+2)^4}{c^5}, \quad c = x_0 + \theta(x-x_0), \quad 0 < \theta < 1.$$

$$9. 1) f(x) = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \dots + \frac{x^n}{(n-1)!} + \frac{x^{n+1}}{n!} (\theta x + n + 1) e^{\theta x}, \quad 0 < \theta < 1;$$

$$2) f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{e^{\theta x} - e^{-\theta x}}{2}, \quad 0 < \theta < 1. \quad 10. 1) 0,587; \quad 2) 0,868;$$

$$3) 1,395; \quad 4) 1,004. \quad 11. 1) 1; \quad 2) \frac{1}{24}; \quad 3) 1; \quad 4) -1.$$

### Funksiyalarni tekshirish va grafiklarini chizish

$$1. 1) (-\infty; 1) \cup (5; +\infty) \text{ intervalda } o'sadi, (1; 5) \text{ intervalda kamayadi, } f_{\max} = f(1) = 7,$$

$$f_{\min} = f(5) = -25; \quad 2) (-\infty; -1) \cup (2; +\infty) \text{ intervalda } o'sadi, (-1; 2) \text{ intervalda kamayadi,}$$

$$f_{\max} = f(-1) = \frac{7}{6}, f_{\min} = f(2) = -\frac{10}{3}; \quad 3) (0; 2) \cup (2; +\infty) \text{ intervalda } o'sadi, (-\infty; -2) \cup (2; 0)$$

- intervalda kamayadi,  $f_{\min} = f(0) = 0$ ; 4)  $(-2; 2)$  intervalda o'sadi,  $(-\infty; -2) \cup (2; +\infty)$  intervalda kamayadi,  $f_{\max} = f(2) = 1$ ,  $f_{\min} = f(-2) = -1$ ; 5)  $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$  intervalda o'sadi,  $\left(-1; -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}; 1\right)$  intervalda kamayadi,  $f_{\max} = f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ ,  $f_{\min} = f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$ ; 6)  $(-\infty; -1) \cup (0; 1)$  intervalda o'sadi,  $(-1; 0) \cup (1; +\infty)$  intervalda kamayadi,  $f_{\max 1} = f(-1) = 2$ ,  $f_{\max 2} = f(1) = 2$ ,  $f_{\min} = f(0) = 0$ ; 7)  $(-\infty; 1)$  intervalda o'sadi,  $(1; +\infty)$  intervalda kamayadi,  $f_{\max} = f(1) = \frac{1}{e}$ ; 8)  $(0; +\infty)$  intervalda o'sadi,  $(-\infty; 0)$  intervalda kamayadi,  $f_{\min} = f(0) = 1$ ; 9)  $(0; +\infty)$  intervalda o'sadi,  $(-\infty; 0)$  intervalda kamayadi,  $f_{\min} = f(0) = 0$ ; 10)  $(e; +\infty)$  intervalda o'sadi,  $(0; 1) \cup (1; e)$  intervalda kamayadi,  $f_{\min} = f(e) = e$ ; 11)  $\left(\frac{\pi}{3}; \frac{5\pi}{3}\right)$  intervalda o'sadi,  $\left(0; \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}; 2\pi\right)$  intervalda kamayadi,  $f_{\max} = f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$ ,  $f_{\min} = f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$ ; 12)  $\left(0; \frac{\pi}{12}\right) \cup \left(\frac{5\pi}{12}; \pi\right)$  intervalda o'sadi,  $\left(\frac{\pi}{12}; \frac{5\pi}{12}\right)$  intervalda kamayadi,  $f_{\max} = f\left(\frac{\pi}{12}\right) = \frac{\pi + 6\sqrt{3} + 12}{12}$ ,  $f_{\min} = f\left(\frac{5\pi}{12}\right) = \frac{5\pi - 6\sqrt{3} + 12}{12}$ .
2. 1)  $M = 2$ ,  $m = -2$ ; 2)  $M = 17$ ,  $m = -10$ ; 3)  $M = \frac{\pi + 6\sqrt{3}}{12}$ ,  $m = \frac{2\pi - 3}{6}$ ;
- 4)  $M = e^3$ ,  $m = 0$ . 3.  $v = 24$  (tez.birl.). 4.  $\frac{\sqrt{3}}{3} D$  (eni),  $\sqrt{\frac{2}{3}} D$  (bo'yi). 5.  $\frac{1}{4}, \frac{1}{4}$ .
6.  $S = 24$  (yuz birl.). 7.  $H = R\sqrt{2}$ . 8.  $H = \sqrt{\frac{4V_0}{\pi}}$ . 9.  $\alpha = \frac{\pi}{3}$ . 10. 13,5 so'm.
11. 1)  $(-\infty; 0) \cup (2; +\infty)$  intervalda botiq,  $(0; 2)$  intervalda qavariq,  $M_1(0; 0)$ ,  $M_2(2; -4)$  egilish nuqtalari; 2)  $(5; +\infty)$  intervalda botiq,  $(-\infty; 5)$  intervalda qavariq,  $M(5; 7)$  egilish nuqtasi; 3)  $(-\infty; 0) \cup (0; +\infty)$  intervalda botiq, egilish nuqtasi yo'q; 4)  $(3; +\infty)$  intervalda botiq,  $(-\infty; 3)$  intervalda qavariq,  $M(3; 1)$  egilish nuqtasi; 5)  $(-1; +\infty)$  intervalda botiq, egilish nuqtasi yo'q; 6)  $(-1; 1)$  intervalda botiq,  $(-\infty; -1) \cup (1; +\infty)$  intervalda qavariq,  $M_1(-1; \ln 2)$ ,  $M_2(1; \ln 2)$  egilish nuqtalari; 7)  $\left(-\infty; -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}; +\infty\right)$  intervalda botiq,  $\left(-\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}\right)$  intervalda qavariq,  $M_1\left(-\frac{1}{\sqrt{3}}; \frac{3}{4}\right)$ ,  $M_2\left(\frac{1}{\sqrt{3}}; \frac{3}{4}\right)$  egilish nuqtalari; 8)  $(-1; 0) \cup (1; +\infty)$  intervalda botiq,  $(-\infty; -1) \cup (0; 1)$  intervalda qavariq,  $M_1(-1; 2)$ ,  $M_2(1; -2)$  egilish nuqtalari.
12. 1)  $x = \pm 1$ ,  $y = 0$ ; 2)  $x = 0$ ,  $y = 1$  ( $x \rightarrow +\infty$  da),  $y = -1$  ( $x \rightarrow -\infty$  da), 3)  $y = x$ ;

$$1) y = -\frac{\pi}{2}x + 1 \quad (x \rightarrow +\infty \text{ da}), \quad y = \frac{\pi}{2}x + 1 \quad (x \rightarrow -\infty \text{ da}). \quad 5)$$

$$2) y = 0 \quad (x \rightarrow -\infty \text{ da}); \quad 6) x = 0, \quad y = 0;$$

$$7) x = 0, \quad y = 3x. \quad 8) x = 1, \quad y = x + \frac{1}{2} \quad (x \rightarrow +\infty \text{ da}), \quad y = -x - \frac{1}{2} \quad (x \rightarrow -\infty \text{ da}).$$

### Hosilarning geometrik tatbiqlari

$$1. \quad 1) 4; \quad 2) 2; \quad 3) (1 + 8\sin^2 t)^{\frac{3}{2}}; \quad 4) \frac{1}{16|\sin \frac{t}{2}|}; \quad 5) \frac{3}{8}; \quad 6) \frac{3}{a}. \quad 2. \quad 1) \frac{25}{3}; \quad 2) \frac{\sqrt{(145)^3}}{12};$$

$$3) 2; \quad 4) \frac{\alpha\sqrt{(1+\theta^2)^3}}{2+\theta^2}. \quad 3. \quad 1) (2;2); \quad 2) \left(\frac{\pi}{2}+1; -1\right). \quad 4. \quad 1) \eta^2 = \frac{8}{27}\xi^3; \quad 2) \eta^2 = \frac{8}{27}(\xi-1)^3;$$

$$3) \xi^2 = \frac{4}{27}\eta^3; \quad 4) \xi^2 + \eta^2 = a^2. \quad 5. \quad 1) \frac{x+1}{2} = \frac{y-2}{-3} = \frac{z}{4}; \quad 2) \frac{x^2}{16} - \frac{z^2}{9} = 1, y = -1;$$

$$3) x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{2}{3}}, z = 1; \quad 4) \frac{x^2}{36} + \frac{y^2}{25} = 1, z = 3. \quad 6. \quad 1) \frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{6},$$

$$2x + 3y + 6z - 11 = 0; \quad 2) \frac{x-2}{1} = \frac{y-8}{2} = \frac{z-4}{4}, 3x + 6y + 12z - 70 = 0;$$

$$3) \frac{x-1}{1} = \frac{y-1}{0} = \frac{z-3}{-3}, x - 3z + 4 = 0; \quad 4) \frac{x-1}{2} = \frac{y+1}{0} = \frac{z-1}{1}, 2x + z - 3 = 0.$$

$$7. \quad 1) 5dt; \quad 2) 13dt.$$

### Tenglamalarni taqribiy yechish

$$1. \quad 1) -2,2583; \quad 2) 0,6823; \quad 3) 2,5062; \quad 4) -0,8879.$$

### Kompleks sonlar

$$1. \quad x = 2, \quad y = -1. \quad 2. \quad x = 5, \quad y = 10. \quad 3. \quad x = -1, \quad y = 21. \quad 4. \quad x = \frac{1}{5}, \quad y = \frac{1}{3}.$$

$$5. \quad 1) z_1 = -3 + 4i, \quad z_2 = -3 - 4i; \quad 2) z_1 = \frac{1}{2}i, \quad z_2 = -i; \quad 3) z_1 = i, \quad z_2 = -3i; \quad 4) z_1 = 5i, \quad z_2 = i.$$

$$6. \quad 1) x = a \text{ to'g'ri chiziq}; \quad 2) y = b \text{ to'g'ri chiziq}; \quad 3) \text{markazi } O(0;0) \text{ nuqtada bo'lgan va } r, R \text{ radiusli aylanalar orasidagi halqa}; \quad 4) \varphi \text{ va } \psi \text{ nurlar orasidagi sektor};$$

$$5) 3) \text{markazi } O(0;0) \text{ nuqtada bo'lgan va } r, R \text{ radiusli aylanalar orasidagi halqaning } \varphi$$

$$\text{va } \psi \text{ nurlar orasidagi bo'lagi.} \quad 7.1) -2 + 2\sqrt{3}i = 4 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = 4e^{i\frac{2\pi}{3}};$$

$$2) \sqrt{3} - i = 2 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right) = 2e^{-\frac{\pi i}{6}}; 3) \frac{-\sqrt{6}}{2} + \frac{\sqrt{6}}{2}i = \sqrt{3} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = \sqrt{3}e^{\frac{3\pi i}{4}};$$

$$4) 2 + 2i = 2\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 2\sqrt{2}e^{\frac{\pi i}{4}}; 5) 1 - i = \sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right) = \sqrt{2}e^{-\frac{\pi i}{4}};$$

$$6) -3 - 2i = \sqrt{13} \left( \cos \left( \operatorname{arctg} \frac{2}{3} - \pi \right) + i \sin \left( \operatorname{arctg} \frac{2}{3} - \pi \right) \right) = \sqrt{13}e^{i \left( \operatorname{arctg} \frac{2}{3} - \pi \right)};$$

$$7) 1 - \sqrt{3}i = 2 \left( \cos \left( -\frac{\pi}{3} \right) + i \sin \left( -\frac{\pi}{3} \right) \right) = 2e^{-\frac{\pi i}{3}};$$

$$8) -\sqrt{2} - \sqrt{2}i = 2 \left( \cos \left( -\frac{3\pi}{4} \right) + i \sin \left( -\frac{3\pi}{4} \right) \right) = 2e^{-\frac{3\pi i}{4}}.$$

$$8. 1) z_1 + z_2 = -3 - i, z_1 - z_2 = -7 + 7i, z_1 \cdot z_2 = 2 + 26i, \frac{z_1}{z_2} = -\frac{11}{10} - \frac{7}{10}i;$$

$$2) z_1 + z_2 = -1 - i, z_1 - z_2 = -5 - 7i, z_1 \cdot z_2 = 6 - 17i, \frac{z_1}{z_2} = -\frac{18}{13} + \frac{1}{13}i. \quad 9. 1) 5\sqrt{2}; 2) 5\sqrt{2};$$

$$3) \sqrt{5}; 4) 10. \quad 10. 1) \frac{6}{5} - \frac{17}{5}i; 2) -\frac{9}{5} - \frac{2}{5}i; 3) 24i; 4) 48i. \quad 11. 1) \operatorname{Re} z = \frac{1}{2}, \operatorname{Im} z = \frac{\sqrt{3}}{2};$$

$$2) \operatorname{Re} z = 0, \operatorname{Im} z = \frac{1}{8}; 3) \operatorname{Re} z = \frac{4}{5}, \operatorname{Im} z = \frac{3}{5}; 4) \operatorname{Re} z = -\frac{37}{5}, \operatorname{Im} z = -\frac{29}{5}.$$

$$12. 1) z_1 \cdot z_2 = -4 + 4\sqrt{3}i, \frac{z_1}{z_2} = \sqrt{3} - i; 2) z_1 \cdot z_2 = 3\sqrt{3} + 3i, \frac{z_1}{z_2} = -6i; 3) z_1 \cdot z_2 = -16,$$

$$\frac{z_1}{z_2} = 4i; 4) z_1 \cdot z_2 = -4 + 4\sqrt{3}i, \frac{z_1}{z_2} = 4 + 4\sqrt{3}i. \quad 13. 1) 16(1 + i); 2) -1; 3) 2^{13}(1 - i); 4)$$

$$-32(1 + \sqrt{3}i). \quad 14. 1) \pm(\sqrt{3} - i); 2) i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i, \frac{\sqrt{3}}{2} - \frac{1}{2}i; 3) \pm(\sqrt{3} + i), \pm(1 - \sqrt{3}i);$$

$$4) \sqrt[4]{2} \left( \cos \frac{\pi + 8k\pi}{20} + i \sin \frac{\pi + 8k\pi}{20} \right), k = 0, 1, 2, 3, 4.$$

### Ко'phadlar

$$1. 1) P + Q = x^5 + 5x^3 - x^2 - x + 4, P - Q = -x^5 - x^3 - x^2 + x + 4,$$

$$P \cdot Q = 2x^8 - x^7 + 6x^6 + x^5 - 2x^4 + 13x^3 - 4x; 2) P + Q = 4x^4 + x^3 - 5,$$

$$P - Q = -2x^4 - x^3 + 2x^2 - 5, P \cdot Q = 3x^8 + x^7 + 2x^6 + x^5 - 16x^4 - 5x^3 + 5x^2.$$

$$2. 1) R = x + 1, r = -x + 2; 2) R = x + 3, r = -6x^2 - 5x + 8;$$

$$3) R = 3x^2 - 5, r = -21x^2 + 17x + 34; 4) R = 2x^2 - 3x + 5, r = 19x - 29. \quad 3. 1) 73; 2) -4;$$

1) 96; 4) 84. 4.  $a=3, b=-1, c=1$ . 5. 1)  $a=2, b=1, c=1$ . 6. 1)  $(x-2)(x+2)(x^2+4)$ ;

2)  $x(x-9)(x+9)$ ; 3)  $5(x-1)(x-2)^2(x-3)$ ; 4)  $(x-3)^2(x-1)(x^2+x+1)$ .

7. 1)  $\frac{3}{x} + \frac{1}{x^2} - \frac{2}{x+1}$ ; 2)  $\frac{2}{x} - \frac{1}{x^2} + \frac{x-4}{x^2+4}$ ; 3)  $\frac{1}{x} + \frac{1}{3(x-1)} - \frac{4}{3(x+2)}$ ;

4)  $\frac{3}{x^2-x+1} - \frac{2}{x^2+x+1}$ . 8. 1)  $\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x^3} - \frac{2}{x+1}$ ; 2)  $\frac{1}{x} - \frac{2}{x-2} + \frac{3}{x+3}$ ;

3)  $\frac{2}{x} - \frac{7}{x^2} + \frac{3}{x-1} - \frac{2}{x+1}$ ; 4)  $-\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x^2-x+1}$ .

### Integrallashning asosiy usullari

1. 1)  $-5 \ln \cos x - 2 \operatorname{arctg} x + \frac{x^5}{5} + C$ ; 2)  $\frac{(x-3)^2}{2} + 2 \ln |x+3| + C$ ; 3)  $-\frac{3}{2\sqrt[3]{x^2}} - e^x - \ln |x| + C$ ;

4)  $3 \operatorname{arctg} x - 2 \arcsin x + C$ ; 5)  $2x + \frac{3 \cdot 2^x}{3^x (\ln 3 - \ln 2)} + C$ ; 6)  $x - \cos x + C$ ; 7)  $e^x + \operatorname{tg} x + C$ ;

8)  $-\operatorname{ctg} x + \cos x + C$ ; 9)  $-\operatorname{ctg} x - x + C$ ; 10)  $-\operatorname{ctg} x + C$ ; 11)  $\frac{1}{10} \operatorname{arctg} \frac{2x}{5} + C$ ;

12)  $\frac{1}{\sqrt{2}} \arcsin \frac{\sqrt{2}(x-1)}{\sqrt{5}} + C$ . 2. 1)  $\frac{1}{4} \operatorname{tg}^2 x + C$ ; 2)  $-\frac{1}{3} \cos^3 x + C$ ; 3)  $\frac{3}{16} \sqrt[3]{\operatorname{arctg}^8 2x} + C$ ;

4)  $\frac{7}{10} \sqrt{\ln^{10}(x+5)} + C$ ; 5)  $e^{\sin x} + C$ ; 6)  $-\frac{1}{3} e^{-x^2} + C$ ; 7)  $-\frac{1}{4 \sin^4 x} + C$ ; 8)  $-2 \cos \sqrt{x} + C$ ;

9)  $\arcsin \frac{e^x}{2} + C$ ; 10)  $-\frac{3}{4} \sqrt{\operatorname{ctg} 4x} + C$ . 3. 1)  $2 \ln(1+e^x) - x + C$ ; 2)  $\frac{1}{6} \ln(x^6+1) + C$ ;

3)  $8 \arcsin \frac{x}{4} + \frac{1}{2} x \sqrt{16-x^2} + C$ ; 4)  $\frac{3}{8} \sqrt[3]{(x^4+4)^2} + C$ ; 5)  $\frac{2}{9} (x^3+3) \sqrt{x^3+3} + C$ ;

6)  $\ln |2 + \sin 2x| + C$ ; 7)  $-\frac{1}{2(\arcsin x)^2} + C$ ; 8)  $2 \ln |x^2+5| - \sqrt{5} \operatorname{arctg} \frac{x}{\sqrt{5}} + C$ ; 9)

$\arcsin \frac{x+2}{3} + C$ ; 10)  $\frac{1}{\sqrt{3}} \ln |3x-1 + \sqrt{9x^2-6x-3}| + C$ ; 11)  $\frac{1}{4} \left( \frac{(2x+7)^{12}}{12} - \frac{7(2x+7)^{11}}{11} \right) + C$ ;

12)  $2 \arcsin \sqrt{x} + C$ ; 13)  $\frac{1}{12} \ln \left| \frac{e^{2x}-3}{e^{2x}+3} \right|$ ; 14)  $\ln x - \ln 2 \cdot |\ln x + 2 \ln 2| + C$ .

4. 1)  $\frac{x^2+1}{2} \operatorname{arctg} x - \frac{x}{2} + C$ ; 2)  $x \arcsin x + \sqrt{1-x^2} + C$ ; 3)  $\frac{x^2}{2} \ln |x| - \frac{x^2}{4} + C$ ;

4)  $e^x(x^2-2x+2) + C$ ; 5)  $\frac{3^x}{\ln^2 3} (x \ln 3 - 1) + C$ ;

6)  $\frac{1}{4} \sin 2x - \frac{1}{2} x \cos 2x + C$ ; 7)  $\frac{1}{2} (x^2-1) \ln |x+1| - \frac{1}{4} x(x-2) + C$ ; 8)  $\frac{x}{2 \cos^2 x} - \frac{1}{2} \operatorname{tg} x + C$ ;

$$9) \frac{x}{2}(\sin \ln |x| - \cos \ln |x|) + C; \quad 10) \frac{1}{5}e^{4x}(\sin 4x - \cos 4x) + C; \quad 11) \operatorname{tg}x(\ln \operatorname{tg}x - 1) + C;$$

$$12) \operatorname{arctg}x(\ln \operatorname{arctg}x - 1) + C. \quad 5) \quad 1) \frac{3}{14}(1+x^2)^2\sqrt[3]{1+x^2} - \frac{3}{8}(1+x^2)\sqrt[3]{1+x^2} + C;$$

$$2) \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C; \quad 3) \frac{e^x}{4}(2 + \sin(2e^x)) + C; \quad 4) -\frac{1}{9}e^{-3x}(3x+1) + C;$$

$$5) \ln |\sin x + \cos x| + C; \quad 6) -\frac{1}{2}\ln |1 - \ln^2 |x|| + C; \quad 7) \frac{1}{5}\ln \left| \frac{2x-3}{x+1} \right| + C;$$

$$8) \frac{1}{2}\operatorname{arctg} \frac{\ln x}{2} + C; \quad 9) x \operatorname{tg}x + \ln |\cos x| + C; \quad 10) \frac{2}{3}\operatorname{arctg} \frac{\sqrt{2x-9}}{3} + C; \quad 11) e^{\operatorname{arctg}x} + C;$$

$$12) \sqrt{3+e^{2x}} + C; \quad 13) \frac{1}{2}x - \frac{1}{6}\sin 3x + C; \quad 14) \frac{1}{2}\operatorname{tg}x^2 - \frac{1}{2}x^2 + C;$$

$$15) \frac{1}{27}x^3(9\ln^2 x - 6\ln x + 2) + C; \quad 16) \frac{2 - \cos x}{\sin x} + C.$$

### *Ratsional funksiyalarni integrallash*

$$1. \quad 1) \ln |x^2 + 3x - 10| + C; \quad 2) \ln \frac{|x+1|}{\sqrt{2x+1}} + C; \quad 3) \frac{1}{2} \ln \frac{(x+2)^4}{|(x+1)(x+3)^3|} + C;$$

$$4) \frac{5}{2} \ln \frac{|x+1|}{|x+5|} + \frac{2}{x+1} + C; \quad 5) \ln \left| \frac{x^3(x-1)}{x+1} \right| + C;$$

$$6) \frac{x}{4} + \ln |x| - \frac{7}{16} \ln |2x-1| - \frac{9}{16} \ln |2x+1| + C;$$

$$7) \frac{2x^2-3}{x} + \ln \left| \frac{x}{x+1} \right| + C; \quad 8) 5x + \frac{1}{2} \ln |x| - \frac{7}{3} \ln |x-1| + \frac{161}{6} \ln |x-4| + C;$$

$$9) x + \frac{1}{3} \ln \left| \frac{(x-2)^5(x-1)^3}{(x+2)^2} \right| + C; \quad 10) -\frac{1}{x} \operatorname{arctg}x + C; \quad 11) \frac{1}{2} \ln \left( \frac{x^2}{x^2+1} \right) + C;$$

$$12) \frac{1}{6} \ln \left( \frac{(x+1)^2}{x^2-x+1} \right) + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + C; \quad 13) \frac{x^3}{3} + x^2 - x + \frac{4}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C;$$

$$14) \frac{1}{4} \left( \frac{2x^6+6x^2}{3} + \ln \left( \frac{x^2-1}{x^2+1} \right) \right) + C; \quad 15) \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \operatorname{arctg}x + C;$$

$$16) \frac{1}{54} \left( \frac{3x}{x^2+9} + \operatorname{arctg} \frac{x}{3} \right) + C; \quad 17) \frac{2x-1}{2(x^2+2x+2)} + \operatorname{arctg}(x+1) + C;$$

$$18) \ln |x-1| + \frac{1}{2} \left( \frac{x}{x^2+1} + \operatorname{arctg}x \right) + C; \quad 19) \frac{1}{8} \operatorname{arctg}(x+2) - \frac{1}{24} \operatorname{arctg} \frac{x+2}{3} + C;$$

$$20) \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} - \frac{1}{2(x+1)} + C; \quad 21) \frac{1}{8} \left( \frac{x(3x^2+5)}{(x^2+1)^2} + 3 \operatorname{arctg} x \right) + C;$$

$$22) \frac{x-9}{8(x^2-2x-5)} + \frac{1}{16} \operatorname{arctg} \frac{x-1}{2} + C; \quad 23) \frac{4x-7}{x^2-3x+3} + \frac{8\sqrt{3}}{3} \operatorname{arctg} \frac{2x-3}{\sqrt{3}} + C;$$

$$24) \ln \left( \frac{x^2+9}{x^2+4} \right) + \operatorname{arctg} \frac{x}{3} + C.$$

### Trigonometrik funksiyalarni integrallash

$$1. 1) \frac{2}{3} \operatorname{arctg} \frac{5 \operatorname{tg} \frac{x}{2} - 4}{3} + C; \quad 2) \frac{1}{8} \left( \operatorname{tg}^2 \frac{x}{2} + 2 \ln \left| \operatorname{tg} \frac{x}{2} \right| \right) + C; \quad 3) \frac{1}{5} \ln \left| 5 \operatorname{tg} \frac{x}{2} + 3 \right| + C;$$

$$4) \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{\operatorname{tg} \frac{x}{2} + 2}{\sqrt{3}} + C; \quad 5) \arccos \frac{\cos x}{\sqrt{3}} + C; \quad 6) \frac{3}{\sin x} - \frac{1}{\sin^3 x} + C; \quad 7) 2 \operatorname{arctg}(\sin x) - \sin x + C;$$

$$8) \frac{1}{4} \ln \left| \frac{1+\operatorname{tg} x}{1-\operatorname{tg} x} \right| + \frac{1}{4} \sin 2x + C; \quad 9) \frac{1}{16} \left( x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C;$$

$$10) \ln |\operatorname{tg} x| + \frac{1}{2} \operatorname{tg}^2 x + C; \quad 11) \frac{1}{10} \ln \left| \frac{\operatorname{tg} x - 1}{\operatorname{tg} x + 1} \right| + C; \quad 12) -\frac{1}{4} (2 \ln |\sin 2x| + \operatorname{ctg}^2 2x) + C;$$

$$13) \sqrt{2} \operatorname{arctg} \left( \frac{\sin x}{\sqrt{2}} \right) - x + C; \quad 14) \frac{1}{42} (3 \sin 7x + 7 \sin 3x) + C;$$

$$15) \frac{1}{2} \left( \frac{1}{3} \sin 3x - \frac{1}{10} \sin 5x - \frac{1}{2} \sin x \right) + C; \quad 16) \frac{x}{4} + \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + C.$$

### Irratsional funksiyalarni integrallash

$$1. 1) 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln(1 + \sqrt[6]{x}) + C; \quad 2) \frac{2}{(1 + \sqrt[3]{x})^2} - \frac{4}{1 + \sqrt[3]{x}} + C;$$

$$3) \frac{2\sqrt{1+x}}{15} \cdot (3(1+x)^2 - 10(1+x) + 9\sqrt{1+x} + 15) + C; \quad 4) 6\sqrt{(x+1)^2} \cdot \left( \frac{2x-3}{20} - \frac{\sqrt{x+1}}{7} \right) + C;$$

$$5) \sqrt[3]{2x-1} - 3\sqrt[6]{2x-1} + 3 \ln(\sqrt[6]{2x-1} + 1) + C; \quad 6) \frac{3}{4} \sqrt[3]{\left( \frac{x+1}{x-1} \right)^2} - \frac{3}{5} \sqrt[5]{\left( \frac{x+1}{x-1} \right)^5} + C;$$

$$7) \ln |2x-3 + 2\sqrt{x^2-3x+2}| + C; \quad 8) \ln |x+1 + \sqrt{x^2+5}| + C; \quad 9) \ln \left| \frac{x-1 + \sqrt{x^2+x+1}}{x+1 + \sqrt{x^2+x+1}} \right| + C;$$

$$10) \frac{1}{2} \ln \left| \frac{x}{4-x+2\sqrt{4-2x-x^2}} \right| + C; \quad 11) \ln \left| \frac{1-x+\sqrt{1-2x-x^2}}{1+\sqrt{1-2x-x^2}} \right| - 2 \operatorname{arctg} \left( \frac{1+\sqrt{1-2x-x^2}}{x} \right) + C;$$

$$12) \ln|1+x+\sqrt{x^2+x+1}| + \frac{2}{x+2+\sqrt{x^2+2x+2}} + C;$$

$$13) \frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C; \quad 14) \frac{x}{2}\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + C;$$

$$15) -2\sqrt{\frac{2-x}{x-1}} + C; \quad 16) -\arcsin\left(\frac{1}{x-1}\right) + C; \quad 17) -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + C;$$

$$18) -2\sqrt{6x-x^2-8} + 9\arcsin(x-3) + C; \quad 19) 23\left(\ln\left|\frac{\sqrt{x}}{1+\sqrt{x}}\right| + \frac{2}{1+\sqrt{x}}\right) + C;$$

$$20) -\frac{\sqrt[3]{(2-x^3)^2}}{4x^2} + C; \quad 21) \sqrt[3]{(1+x^3)^5} \cdot \left(\frac{x^3+1}{8} - \frac{1}{5}\right) + C; \quad 22) \frac{3}{7}(\sqrt[3]{x-3}) \cdot \sqrt[3]{1+\sqrt[3]{x}} + C;$$

$$23) -\frac{\sqrt{1+x^3}}{2x^2} + C; \quad 24) -2\sqrt[3]{\left(1+\frac{1}{\sqrt{x}}\right)^3}$$

### *Aniq integral*

1. 1) 0; 2) 8; 3)  $4\pi$ ; 4) 4.    2. 1)  $I_1 > I_2$ ; 2)  $I_1 > I_2$ ; 3)  $I_1 < I_2$ ; 4)  $I_1 < I_2$ .    3. 1)  $\frac{\pi}{5} \leq I_2 \leq \pi$ ;  
 2)  $4 \leq I_3 \leq 4\sqrt{7}$ ; 3)  $2 \leq I_3 \leq 6$ ; 4)  $\frac{3}{5} \leq I_4 \leq 3$ .    4. 1)  $\frac{\pi}{2}$ ; 2)  $\frac{1}{2}$ ; 3) 8; 4)  $e-2$ .

### *Aniq integralni hisoblash*

1) 9; 2)  $\frac{1}{2}$ ; 3)  $\frac{1}{2}$ ; 4) 1; 5)  $\frac{\pi}{4}$ ; 6)  $\frac{3-\sqrt{3}}{3}$ ; 7)  $\ln\frac{4}{3}$ ; 8)  $\frac{2}{3}$ ; 9)  $\frac{2\sqrt{2}-1}{3}$ ; 10)  $\frac{1}{4}$ ; 11)  $\ln\frac{3}{2}$ ;  
 12)  $\frac{\pi}{36}$ ; 13)  $\frac{\pi\sqrt{3}}{36}$ ; 14)  $\frac{1}{12}(8-5\sqrt{2})$ ; 15)  $\ln\frac{4}{3}$ ; 16)  $1-\frac{\pi}{4}$ ; 17)  $\frac{\pi}{2}-1$ ; 18)  $e-\frac{\pi}{2}$ ; 19) 4;  
 20)  $\frac{1}{5}\left(e^{\frac{\pi}{4}}+2\right)$ ; 21)  $\frac{5e^3-2}{27}$ ; 22)  $\frac{e+1}{4}$ ; 23) 2; 24)  $\frac{e^{\frac{\pi}{2}}-1}{2}$ .

### *Xosmas integrallar*

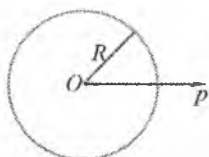
1. 1)  $\frac{\pi}{4}$ ; 2) -4; 3) uzoqlashadi; 4) uzoqlashadi; 5)  $\frac{\pi}{6}$ ; 6)  $\frac{\pi}{4} + \frac{1}{2}\ln 2$ ; 7)  $\frac{1}{2}$ ; 8) 2;  
 9)  $\frac{\pi}{2}$ ; 10)  $\frac{8}{3}$ ; 11)  $14\frac{4}{7}$ ; 12) uzoqlashadi; 13) uzoqlashadi; 14)  $\pi$ .    2. 1)  $\alpha > 1$  da yaqinlashadi va  $0 < \alpha \leq 1$  da uzoqlashadi; 2) yaqinlashadi; 3) yaqinlashadi;  
 4) yaqinlashadi; 5) uzoqlashadi; 6) yaqinlashadi; 7) uzoqlashadi;  
 8) uzoqlashadi; 9) yaqinlashadi; 10) yaqinlashadi;  
 11) absulut yaqinlashadi; 12) absulut yaqinlashadi.



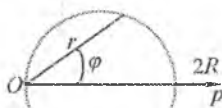
*Aniq integrallarning tatbiqlari*

1. 1) 36; 2)  $\frac{9}{2}$ ; 3)  $1+6\ln\frac{4}{3}$ ; 4)  $e^2+1$ ; 5)  $\frac{9}{2}$ ; 6)  $\frac{15}{2}-8\ln 2$ ; 7)  $\frac{1}{3}$ ; 8)  $\frac{2}{3}$ ; 9)  $12\pi$ ; 10)  $27\pi$ ;  
 11) 9; 12)  $\frac{9\pi}{2}$ ; 13)  $\frac{1}{3}(17\pi+48)$ ; 14)  $\frac{16\pi^3}{3}$ . 2. 1)  $\sqrt{3}+\frac{1}{2}\ln(2+\sqrt{3})$ ; 2)  $\frac{1}{2}\left(e-\frac{1}{e}\right)$ ; 3)  
 $\frac{335}{27}$ ; 4) 2; 5)  $\frac{3}{4}+\frac{1}{2}\ln 2$ ; 6)  $1+\ln\frac{6}{5}$ ; 7)  $4\sqrt{3}$ ; 8)  $\frac{1}{27}(13\sqrt{13}-8)$ ; 9) 16; 10) 48;  
 11)  $4a(2-\sqrt{3})$ ; 12)  $2\pi+3\sqrt{3}$ . 3. 1)  $\frac{56}{3}\pi$ ; 2)  $36\pi$ ; 3)  $\frac{256}{3}\pi$ ; 4)  $\pi(2+\pi)$ . 4.  $\frac{4}{3}\pi R^3$ . 5.  $12\pi$ .  
 6.  $\frac{29}{3}\pi$ . 7.  $36\pi$ . 8. 1)  $\frac{512}{15}\pi$ ; 2)  $\frac{19}{6}\pi$ ; 3)  $\frac{\pi^2}{4}$ ; 4)  $\frac{4}{7}\pi$ ; 5)  $32\pi$ ; 6)  $100\pi$ ; 7)  $40\pi^2$ ; 8)  $\frac{3}{7}\pi$ ;  
 9)  $72\pi$ ; 10)  $\frac{4}{3}\pi R^3$ . 9.  $C(0; R)$ . 10.  $C\left(0; \frac{2a}{5}\right)$ . 11.  $M_x=10\gamma$ ,  $M_y=\frac{15}{2}\gamma$ ,  $C\left(\frac{3}{2}; 2\right)$   
 12.  $M_x=M_y=\frac{4}{3}$ ,  $I_x=I_y=\frac{4}{3}$ ,  $C\left(\frac{2}{3}; \frac{2}{3}\right)$ . 13.  $C\left(0; \frac{8}{5}\right)$ . 14.  $I_x=80\pi$ ,  $I_y=125\pi$ .  
 15.  $C\left(\frac{2}{3}R; \frac{2}{3}R\right)$ . 16.  $C\left(5; \frac{15}{32}\pi\right)$ . 17. 5 sm; 18. 0,32 kGm. 19. 7,68π. 20.  $22\cdot 10^9$  J.  
 21. 150 m. 22. 25 m/s. 23.  $\frac{mg}{l}\left(180+\left(\frac{m}{k}e^{\frac{180}{m}}-1\right)\right)$ . 24. 1000 m. 25.  $gy\frac{h^2(b+2a)}{6}$ .  
 26. 324 T. 27. 175.4 kN.

# Ayrim chiziqning grafiklari va tenglamalari

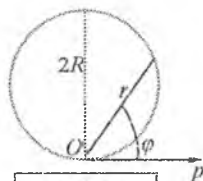


$$r = R$$

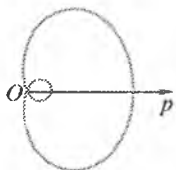


$$r = 2R \cos \varphi$$

R radiusli aylana

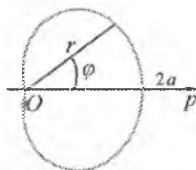


$$r = 2R \sin \varphi$$



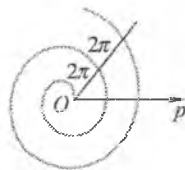
$$r = b + a \cos \varphi \quad (a > b)$$

Paskal chig'anog'i



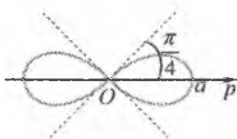
$$r = a(1 + \cos \varphi) \quad (a > 0)$$

Kardioida



$$r = a\varphi \quad (a > 0)$$

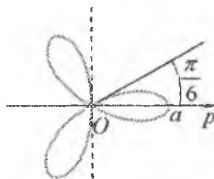
Arximed spirali



$$r = a\sqrt{\cos 2\varphi} \quad (a > 0)$$

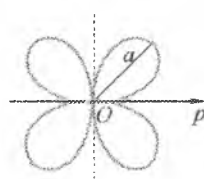
$$(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0$$

Bernulli limniskatasi



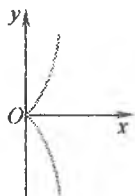
$$r = a \cos 3\varphi \quad (a > 0)$$

Uch yaproqli gul



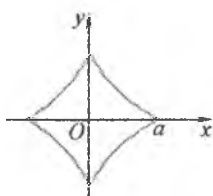
$$r = a \sin 2\varphi \quad (a > 0)$$

To'rt yaproqli gul



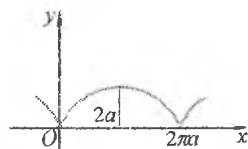
$$y^2 = x^3 \text{ yoki } \begin{cases} x = t^2 \\ y = t^3 \end{cases}$$

Yarimkubik parabola



$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \text{ yoki } \begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases}$$

Astroida



$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad a > 0$$

Sikloida

# MUNDARIJA

SO‘Z BOSHI .....	3
------------------	---

## 1. CHIZIQLI ALGEBRA ELEMENTLARI

1.1. Matritsalar .....	5
1.1.1. Matritsa va uning turlari .....	6
1.1.2. Matritsalar ustida arifmetik amallar .....	8
1.1.3. Mashqlar .....	14
1.2. Determinantlar .....	16
1.2.1. Ikkinchi va uchunchi tartibli determinantlar .....	16
1.2.2. $n$ - tartibli determinant tushunchasi .....	19
1.2.3. Determinantning xossalari .....	20
1.2.4. $n$ - tartibli determinantlarni hisoblash .....	24
1.2.5. Mashqlar .....	27
1.3. Matritsalar ustida almashtirishlar .....	28
1.3.1. Teskari matritsa .....	29
1.3.2. Matritsani LU yoyish .....	36
1.3.3. Matritsaning rangi .....	39
1.3.4. Mashqlar .....	42
1.4. Chiziqli tenglamalar sistemasi .....	45
1.4.1. Asosiy tushunchalar .....	45
1.4.2. Chiziqli tenglamalar sistemasini yechishning Gauss usuli .....	47
1.4.3. $n$ noma'lumli $m$ ta chiziqli tenglamalar sistemasini tekshirish va yechish .....	54
1.4.4. Xosmas tenglamalar sistemasini yechish .....	59
1.4.5. Bir jinsli tenglamalar sistemasi .....	62
1.4.6. Chiziqli tenglamalar sistemasini matematik paketlarda yechish .....	66
1.4.7. Mashqlar .....	70

## 2. VEKTORLI ALGEBRA ELEMENTLARI

2.1. Vektorlar .....	73
2.1.1. Asosiy tushunchalar .....	73
2.1.2. Vektorlar ustida chiziqli amallar .....	75
2.1.3. Vektorning o'qdagi proeksiyasi .....	78

2.1.4. Vektorlarning chiziqli bog'liqligi. Bazis .....	81
2.1.5. Dekart koordinatalar sistemasida vektorlar .....	85
2.1.6. Mashqlar .....	91
2.2. Vektorlarni ko'paytirish .....	93
2.2.1. Ikki vektorning skalyar ko'paytmasi .....	93
2.2.2. Ikki vektorning vektor ko'paytmasi .....	99
2.2.3. Uchta vektorning aralash ko'paytmasi .....	105
2.2.4. Mashqlar .....	112

### 3. ANALITIK GEOMETRIYA

3.1. Tekislikdagi to'g'ri chiziq .....	115
3.1.1. Tekislikdagi chiziq .....	115
3.1.2. Tekislikdagi to'g'ri chiziq tenglamalari .....	117
3.1.3. Tekislikda ikki to'g'ri chiziqning o'zaro joylashishi .....	124
3.1.4. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa .....	130
3.1.5. Mashqlar .....	131
3.2. Ikkinchi tartibli chiziqlar .....	133
3.2.1. Aylana .....	135
3.2.2. Ellips .....	136
3.2.3. Giperbola .....	140
3.2.4. Parabola .....	144
3.2.5. Ikkinchi tartibli chiziqning umumiy tenglamasi .....	146
3.2.6. Mashqlar .....	152
3.3. Qutb koordinatalarida chiziqlar .....	155
3.3.1. Qutb koordinatalari .....	155
3.3.2. Qutb koordinatalar sistemasida chiziqlar .....	157
3.3.3. Mashqlar .....	161
3.4. Tekislik .....	162
3.4.1. Fazoda sirt va chiziq .....	162
3.4.2. Tekislik tenglamalari .....	164
3.4.3. Fazoda ikki tekislikning o'zaro joylashishi .....	171
3.4.4. Nuqtadan tekislikkacha bo'lgan masofa .....	173
3.4.5. Mashqlar .....	175
3.5. Fazodagi to'g'ri chiziq .....	177
3.5.1. Fazodagi to'g'ri chiziq tenglamalari .....	177
3.5.2. Fazoda ikki to'g'ri chiziqning o'zaro joylashishi .....	181

3.5.3. Fazoda to'g'ri chiziq bilan tekislikning o'zaro joylashishi .....	183
3.5.4. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa .....	186
3.5.5. Mashqlar .....	187
3.6. Ikkinchi tartibli sirtlar .....	190
3.6.1. Sfera .....	191
3.6.2. Ellipsoid .....	191
3.6.3. Giperboloidlar .....	193
3.6.4. Konuslar .....	196
3.6.5. Paraboloidlar .....	197
3.6.6. Silindrik sirtlar .....	198
3.6.7. Ikkinchi tartibli sirtlarning to'g'ri chiziqli yasovchilari .....	200
3.6.8. Mashqlar .....	201

#### 4. MATEMATIK ANALIZGA KIRISH

4.1. Haqiqiy sonlar .....	203
4.1.1. To'plam .....	203
4.1.2. Sonli to'plamlar .....	205
4.1.3. Matematik mantiq elementlari .....	210
4.1.4. Mashqlar .....	213
4.2. Sonli ketma-ketliklar .....	214
4.2.1. Sonli ketma-ketliklar .....	214
4.2.2. Cheksiz katta va cheksiz kichik ketma-ketliklar .....	217
4.2.3. Ketma-ketlikning limiti .....	219
4.2.4. Yaqinlashuvchi ketma-ketliklar .....	219
4.2.5. $e$ soni .....	223
4.2.6. Mashqlar .....	224
4.3. Bir o'zgaruvchining funksiyasi .....	226
4.3.1. Funksiya .....	226
4.3.2. Teskari funksiya .....	230
4.3.3. Murakkab funksiya .....	231
4.3.4. Elementar funksiyalar sinfi .....	234
4.3.5. Giperbolik funksiyalar .....	236
4.3.6. Oshkormas va parametrik ko'rinishda berilgan funksiyalar .....	237
4.3.6. Mashqlar .....	238
4.4. Funksiyaning limiti .....	241
4.4.1. Funksiyaning limiti .....	241

4.4.2. Cheksiz kichik funksiyalar .....	249
4.4.3. Mashqlar .....	254
4.5. Funksiyaning uzluksizligi .....	257
4.5.1. Funksiya uzluksizligining ta'riflari .....	257
4.5.2. Uzluksiz funksiyalarning xossalari .....	259
4.5.3. Funksiyaning uzilish nuqtalari .....	264
4.5.4. Tekis uzluksizlik .....	266
4.5.5. Mashqlar .....	267

## **5. BIR O'ZGARUVCHI FUNKSIYASINING DIFFERENSIAL HISOBI**

5.1. Funksiyaning hosilasi va differensialli .....	269
5.1.1. Hosila tushunchasiga olib keluvchi masalalar .....	269
5.1.2. Hosilaning ta'rifi, geometrik va mexanik ma'nolari .....	270
5.1.3. Funksiyaning differensiallanuvchanligi .....	276
5.1.4. Funksiyaning differensialli .....	277
5.1.5. Mashqlar .....	279
5.2. Differensiallash qoidalari va formulalari .....	280
5.2.1. Yig'indi, ayirma, ko'paytma va bo'linmani differensiallash .....	280
5.2.2. Teskari funksiyani differensiallash .....	281
5.2.3. Murakkabi funksiyani differensiallash .....	282
5.2.4. Asosiy elementar funksiyalarning hosilalari .....	283
5.2.5. Differensiallash qoidalari va hosilalar jadvali .....	287
5.2.6. Logoritmik differensiallash .....	289
5.2.7. Parametrik va oshkormas ko'rinishda berilgan funksiyalarni differensiallash .....	290
5.2.8. Yuqori tartibli hosilalar va differensiallar .....	292
5.2.9. Mashqlar .....	298
5.3. Differensial hisobning asosiy teoremlari.....	300
5.3.1. Ferma teoremasi .....	300
5.3.2. Roll teoremasi .....	301
5.3.3. Lagranj teoremasi .....	302
5.3.4. Koshi teoremasi .....	304
5.3.5. Lopital teoremasi .....	305
5.3.6. Teylor teoremasi .....	308
5.3.7. Mashqlar .....	312
5.4. Funksiyalarni hosilalar yordamida tekshirish .....	314

5.4.1. Funksiyaning monotonlik shartlari .....	314
5.4.2. Funksiyaning ekstremumlari .....	316
5.4.3. Kesmada uzluksiz funksiyaning eng katta va eng kichik qiymatlari.....	320
5.4.4. Funksiya grafigining botiqligi, qavariqligi va egilish nuqtalari .....	320
5.4.5. Funksiya grafigining asimptotalari .....	323
5.4.6. Funksiyani tekshirish va grafigini chizishning umumiy sxemasi .....	326
5.4.7. Mashqlar .....	330
5.5. Hosilalarning geometrik tatbiqlari .....	332
5.5.1. Yassi egri chiziq yoyining differensialli .....	332
5.5.2. Yassi egri chiziqning egriligi .....	334
5.5.3. Skalyar argumentning vektor funksiyasi .....	340
5.5.4. Mashqlar .....	346
5.6. Tenglamalarni taqribiy yechish .....	347
5.6.1. Asosiy tushunchalar .....	347
5.6.2. Ildizlarni ajratishning vatartlar va urinmalar usullari .....	349
5.6.3. Mashqlar .....	354

## 6. OLIY ALGEBRA ELEMENTLARI

6.1. Kompleks sonlar .....	355
6.1.1. Kompleks son tushunchasi va tasviri .....	355
6.1.2. Kompleks sonlarning yozilish shakllari .....	358
6.1.3. Kompleks sonlar ustida amallar .....	360
6.1.4. Mashqlar .....	365
6.2. Ko'phadlar .....	367
6.2.1. Ko'phadlar ustida amallar .....	367
6.2.2. Ko'phadning ildizi .....	368
6.2.3. Ko'phadni ko'paytuvchilarga ajratish .....	369
6.2.4. Ratsional kasrlarni sodda kasrlarga yoyish .....	371
6.2.5. Mashqlar .....	374

## 7. BIR O'ZGARUVCHI FUNKSIYASINING INTEGRAL HISOBI

7.1. Aniqmas integral .....	376
7.1.1. Boshlang'ich funksiya va aniqmas integral .....	376
7.1.2. Aniqmas integralning xossalari .....	379

7.1.3. Asosiy integrallar jadvali .....	380
7.1.4. Integrallash usullari .....	382
7.1.5. Mashqlar .....	387
7.2. Ratsional funksiyalarini integrallash .....	389
7.2.1. Sodda kasrlarni integrallash .....	389
7.2.2. Ratsional kasr funksiyalarini integrallash .....	393
7.2.3. Mashqlar .....	395
7.3. Trigonometrik funksiyalarni integrallash .....	395
7.3.1. $\int R(\sin x, \cos x) dx$ ko‘rinishidagi integrallar .....	395
7.3.2. $\int \sin^n x \cos^m x dx$ ko‘rinishidagi integrallar .....	398
7.3.3. $\int \operatorname{tg}^n x dx$ va $\int \operatorname{ctg}^n x dx$ ko‘rinishidagi integrallar .....	399
7.3.4. $\int \sin mx \cos nx dx$ , $\int \sin mx \sin nx dx$ , $\int \cos mx \cos nx dx$ ko‘rinishidagi integrallar .....	400
7.3.5. Mashqlar .....	400
7.4. Irratsional ifodalarni integrallash .....	401
7.4.1. $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \dots\right) dx$ ko‘rinishidagi integrallar .....	401
7.4.2. $\int R(x, \sqrt{ax^2+bx+c}) dx$ ko‘rinishidagi integrallar .....	402
7.4.3. $\int x^m (a+bx^n)^p dx$ binominal differensial integrali .....	407
7.4.4. Elementar funksiyalarda ifodalanmaydigan integrallar .....	408
7.4.5. Mashqlar .....	408
7.5. Aniq integral .....	409
7.5.1. Aniq integral tushunchasiga olib keluvchi masalalar .....	409
7.5.2. Integral yig‘indi va aniq integral .....	412
7.5.3. Aniq integralning geometrik va mexanik ma‘nolari .....	414
7.5.4. Aniq integralning xossalari .....	415
7.5.5. Mashqlar .....	420
7.6. Aniq integralni hisoblash .....	420
7.6.1. Yuqori chegarasi o‘zgaruvchi aniq integral .....	420
7.6.2. Nyuton-Leybnits formulasi .....	422
7.6.3. Aniq integralda o‘zgaruvchini almashtirish .....	423
7.6.4. Aniq integralni bo‘laklab integrallash .....	425
7.6.5. Mashqlar .....	426



7.7. Xosmas integrallar .....	427
7.7.1. Cheksiz chegarali xosmas integrallar .....	427
7.7.2. Chegaralanmagan funksiyalarning xosmas integrallari .....	430
7.7.3. Xosmas integrallarning yaqinlashish alomatlari .....	431
7.7.4. Mashqlar .....	433
7.8. Aniq integralning tadbirlari .....	434
7.8.1. Aniq integralning qo'llanish sxemalari .....	434
7.8.2. Yassi shakl yuzasini hisoblash .....	436
7.8.3. Yassi egri chiziq yoyi uzunligini hisoblash .....	440
7.8.4. Aylanish sirti yuzini hisoblash .....	443
7.8.5. Hajmlarni hisoblash .....	445
7.8.6. Momentlar va og'irlik markazini hisoblash .....	447
7.8.7. Kuchning bajargan ishini hisoblash .....	450
7.8.8. Jismning bosib o'tgan yo'li .....	451
7.8.9. Suyuqlikning vertikal plastinkaga bosimi .....	452
7.8.10. Mashqlar .....	453
7.9. Aniq integralni taqribiy hisoblash .....	457
7.9.1. To'g'ri to'rtburchaklar formulasi .....	457
7.9.2. Trapetsiyalar formulasi .....	458
7.9.3. Simpson formulasi .....	458
7.9.4. Mashqlar .....	462
<b>FOYDALANILGAN ADABIYOTLAR .....</b>	<b>463</b>
<b>JAVOBLAR .....</b>	<b>465</b>
<b>ILOVA .....</b>	<b>484</b>

**Shavkat Rahmatullayevich  
XURRAMOV**

## **OLIV MATEMATIKA**

*Uch jildlik*

**I jild**

*O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim  
vazirligi barcha texnika yo'nalishlari uchun darslik  
sifatida tavsiya etgan*

*Muharrir Dildora Abduraimova  
Badiiy muharrir Maftuna Vaxxobova  
Texnik muharrir Yelena Tolochko  
Musahhah Dildora Abduraimova  
Sahifalovchi Gulchehra Azizova*

Litsenziya raqami AI № 163. 09.11.2009. Bosishga 2018-yil 3-sentyabrda  
ruxsat etildi. Bichimi 60×84<sup>1</sup>/<sub>16</sub>. Ofset qog'oz. Tayms TAD garniturasini  
Shartli bosma tabog'i 28,6. Nashr tabog'i 27,08. Shartnoma № 94–2018.  
Adadi 300 nusxada. Buyurtma № 39.

O'zbekiston Matbuot va axborot agentligining Cho'lpon nomidagi  
nashriyot-matbaa ijodiy uyi tezkor matbaa bo'limida chop etildi.  
100011, Toshkent, Navoiy ko'chasi, 30.  
Telefon: (371) 244-10-45. Faks: (371) 244-58-55.



*Cho'lpon nomidagi  
nashriyot-matbaa ijodiy uyi*

ISBN 978-9943-5379-7-2



9 789943 537972